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A CLASS OF SURJECTIVE CONVOLUTION OPERATORS

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Let μ be a distribution with compact support in $\mathbf{R}^n$. In the terminology of Ehrenpreis [2] μ is called invertible for a space of distributions $\mathcal{F}$ in $\mathbf{R}^n$ if $\mu * \mathcal{F} = \mathcal{F}$. Using his characterisation of invertible distributions in terms of the growth of their Fourier transforms, we obtain a class of invertible distributions which properly contains the distributions with finite supports. We consider $\mathcal{F} = \mathcal{E}$ (or $\mathcal{D}'$) and $\mathcal{F} = \mathcal{D}'_F$, but our results for the latter space are only partial.

1. Introduction. We follow the notation of Schwartz [**6**]: by $\mathcal{D}'$ ($\mathcal{D}'_F$) we denote the space of distributions (distributions of finite order) in $\mathbf{R}^n$. $\mathcal{E}$ will denote the space of infinitely differentiable functions in $\mathbf{R}^n$ with the topology of uniform convergence of functions and all their derivatives on compact subsets of $\mathbf{R}^n$. The dual space of $\mathcal{E}$, denoted by $\mathcal{E}'$, consists of distributions with compact support in $\mathbf{R}^n$. For $\mu \in \mathcal{E}'$ we define the Fourier-Laplace transform of μ by

$$\hat{\mu}(\zeta) = \mu(e^{-i\langle\cdot,\zeta\rangle}), \qquad \zeta \in \mathbf{C}^n.$$

Ehrenpreis [**2**] and Hörmander [**3**] have studied the range of convolution operators

$$(1) \qquad u \mapsto \mu * u, \qquad \mu \in \mathcal{E}',$$

in each of the spaces $\mathcal{D}'$, $\mathcal{D}'_F$ and $\mathcal{E}$. We recall their main result: the operator (1) in $\mathcal{E}$ and, equivalently, in $\mathcal{D}'$ (resp. in $\mathcal{D}'_F$) is surjective if and only if $\hat{\mu}$ is slowly decreasing (resp. very slowly decreasing) in the sense of

DEFINITION 1. Let $\mu \in \mathcal{E}'$. $\hat{\mu}$ is called slowly decreasing if there exist constants A, B and m such that

$$\sup_{|\xi-\xi_0|\leq A\log(2+|\xi_0|)} |\hat{\mu}(\xi)| \geq B(1+|\xi_0|)^{-m}$$

for all $\xi_0 \in \mathbf{R}^n$. $\hat{\mu}$ is called very slowly decreasing if there exists a constant m and for each $\varepsilon > 0$ a constant B_ε such that

$$\sup_{|\xi-\xi_0|\leq \varepsilon\log(2+|\xi_0|)} |\hat{\mu}(\xi)| \geq B_\varepsilon(1+|\xi_0|)^{-m}$$

for all $\xi_0 \in \mathbf{R}^n$.

We sketch the proof of this result for the space $\mathcal{E}$ in the Appendix; the given direct proof of the sufficiency of the slowly decreasing condition is due to J. E. Björk (personal communication).

In this note (§§2–4) we prove the following theorems:

THEOREM 1. *Let $\mu = \nu_1 + \nu_2$, where $\nu_1, \nu_2 \in \mathcal{E}'$ have disjoint singular supports and assume that $\hat{\nu}_1$ is slowly decreasing. Then $\hat{\mu}$ is slowly decreasing.*

THEOREM 2. *Let $\mu \in \mathcal{E}'$, let f be real analytic in a neighbourhood of the singular support of μ and assume $(f \cdot \mu)\hat{}$ is slowly decreasing. Then $\hat{\mu}$ is slowly decreasing.*

THEOREM 3. *Let $\mu \in \mathcal{E}'$ be a measure containing an atom (i.e. $\mu\{x_0\} \neq 0$ for some $x_0 \in \mathbf{R}^n$). Then $\hat{\mu}$ is very slowly decreasing.*

REMARK 1. I do not know whether Theorems 1 and 2 remain true with "slowly decreasing" replaced by "very slowly decreasing"; the given proofs show they do if μ is a measure and $\hat{\nu}_1$ (resp. $(f \cdot \mu)\hat{}$) is very slowly decreasing in the sense of Definition 1 with $m = 0$.

REMARK 2. Measures with non-empty singular support but without an atom may fail to be invertible as the following elementary example shows: Let $n = 1$, let φ be a test function equal to 1 near $x = 0$ and put $\mu = \varphi \cdot \log|\cdot|$; then μ is invertible if and only if $\varphi \cdot Vp(1/x)$ is, but $(\varphi \cdot Vp(1/x))\hat{}(\xi) = \int_{-\infty}^{\xi} \hat{\varphi}(\xi')\, d\xi'$ is not slowly decreasing.

As a corollary to the theorems, we describe in §5 a class of invertible (for $\mathcal{E}$) distributions which properly contains the distributions with finite supports (see Ehrenpreis [**1**] and Hörmander [3], Theorem 4.4).

Finally I would like to thank Professor J. E. Björk for the generous advice I was fortunate to profit from during the work on this paper.

2. Proof of Theorem 1. It is no restriction to assume μ is a measure with total mass not greater than 1 (otherwise regularise μ by convoluting it with a suitable invertible distribution, see Ehrenpreis [**2**]).

Since by adding a test function one does not affect the invertibility of μ we may also assume that "singular support" in the theorem has been replaced by "support".

Let φ be a test function such that

(2) $\varphi = 1$ on a neighbourhood of $\operatorname{supp} \nu_1$ and $\operatorname{supp} \nu_2 \cap \operatorname{supp} \varphi = \varnothing$.

By assumption $(\varphi \cdot \mu)^{\hat{}}$ is slowly decreasing: for any $\xi_0 \in \mathbf{R}^n$ there exists $\xi_1 \in \mathbf{R}^n$ such that

$$|\xi_1 - \xi_0| \leq A \log(2 + |\xi_0|) \quad \text{and} \quad B(1 + |\xi_0|)^{-m} \leq \left| \int \hat{\varphi}(\xi) \hat{\mu}(\xi_1 - \xi)\, d\xi \right|$$

with some constants A, B, m and we shall assume $B = 1$.

For any $R > 0$ we may estimate the part of the integral over the ball $|\xi| \leq R$ by

$$\|\hat{\varphi}\|_{L^1} \cdot \sup\{|\hat{\mu}(\xi)| : |\xi - \xi_0| \leq R + A \log(2 + |\xi_0|)\}$$

and the remaining part by $\int_{|\xi| \geq R} |\hat{\varphi}(\xi)|\, d\xi$; we obtain

$$(3) \quad (1 + |\xi_0|)^{-m} \leq \|\hat{\varphi}\|_{L^1} \cdot \sup_{|\xi - \xi_0| \leq R + A \log(2 + |\xi_0|)} |\hat{\mu}(\xi)| + \int_{|\xi| \geq R} |\hat{\varphi}(\xi)| d\xi.$$

We may now pass to infimum over all φ satisfying (2). To do this we need

LEMMA 1. *Let Φ be any test function with property* (2). *Denote by $\mathfrak{F}$ the set of all test functions φ which satisfy* (2) *and are such that $\|\hat{\varphi}\|_{L^1} \leq \|\hat{\Phi}\|_{L^1}$. Then there exist constants C_1, $C_2 > 0$ such that, for any $R > 0$,*

$$\inf_{\varphi \in \mathfrak{F}} \int_{|\xi| \geq R} |\hat{\varphi}(\xi)| d\xi \leq C_1 e^{-C_2 R}.$$

By Lemma 1 with $R = NA \log(2 + |\xi_0|)$, the constant N to be determined shortly, it follows from (3) that

$$(1 + |\xi_0|)^{-m} \leq \|\hat{\Phi}\|_{L^1} \cdot \sup_{|\xi - \xi_0| \leq (N+1) A \log(2 + |\xi_0|)} |\hat{\mu}(\xi)| + C_1 (2 + |\xi_0|)^{-C_2 NA}$$

implying

$$\sup_{|\xi - \xi_0| \leq (m/C_2 + A) \log(2 + |\xi_0|)} |\hat{\mu}(\xi)| \geq B(1 + |\xi_0|)^{-m}$$

for a suitable constant B if N was chosen so that $C_2 NA > m$.

Proof of Lemma 1. Suitably chosen positive constants occurring in the proof will all be denoted by C. We shall assume with no loss of generality that $R \geq 1$.

Since for all test functions φ and all $N = 0, 1, \ldots,$

$$|\xi_j|^N \cdot |\hat{\varphi}(\xi)| \leq \|D_j^N \varphi\|_{L^1}, \qquad 1 \leq j \leq n,$$

and since $|\xi|^N \leq C^N \cdot \Sigma|\xi_j|^N$, $\xi \in \mathbf{R}^n$, we easily see that

$$\int_{|\xi|\geq R} |\hat{\varphi}| \leq C^N R^{n-N} \sum_j \|D_j^N \varphi\|_{L^1}, \qquad N > n.$$

For each such N let φ_N be a function in $\mathcal{F}$ with the property

$$\|D_j^N \varphi_N\|_{L^1} \leq C^{N+1} \cdot N!, \qquad 1 \leq j \leq n;$$

for example, for a non-negative test function ψ with sufficiently small support and $\int \psi = 1$, put $\varphi_N = \psi_{(N)} * \cdots * \psi_{(N)} * \Phi$, where $\psi_{(N)}(x) = N^n \psi(Nx)$ occurs in the convolution N times.

Then

$$\text{(4)} \qquad \inf_{\varphi \in \mathcal{F}} \int_{|\xi|\geq R} |\hat{\varphi}| \leq \int_{|\xi|\geq R} |\hat{\varphi}_N| \leq R^n \cdot C \cdot \left(\frac{C}{R}\right)^N \cdot N!$$

for $N > n$ and, since $R \geq 1$, also for $N = 0, 1, \ldots, n$.

Now, for each N, take the inverse of (4), multiply it by 2^{-N} and then sum over all $N \geq 0$; we obtain

$$\inf_{\varphi \in \mathcal{F}} \int_{|\xi|\geq R} |\hat{\varphi}| \leq R^n \cdot C \cdot e^{-CR},$$

which is clearly bounded by $C_1 e^{-C_2 R}$ for some constants C_1, $C_2 > 0$.

3. Proof of Theorem 2. The proof of Theorem 1 applies almost verbatim with condition (2) replaced by

(2)′ $\varphi = 1$ on a neighbourhood of $\operatorname{supp} \mu$ and f real analytic on $\operatorname{supp} \varphi$,

and then φ and φ_N in Lemma 1 replaced by $f \cdot \varphi$ and $f \cdot \varphi_N$, respectively.

4. Proof of Theorem 3. We may clearly assume $x_0 = 0$.

Let φ be a non-negative test function with support contained in the unit ball in $\mathbf{R}^n_\xi$ and $\int \varphi = 1$.

For $R > 0$ put $\varphi_R(\xi) = R^{-n}\varphi(R^{-1}\xi)$; observe that the equalities $\hat{\varphi}_R(x) = \hat{\varphi}(Rx)$ and $\hat{\varphi}(0) = 1$ imply that the functions $\hat{\varphi}_R$ converge pointwise to $\chi_{\{0\}}$ (= the characteristic function of the set $\{0\}$) as $R \to \infty$.

By a direct calculus we see that

$$\text{(5)} \qquad \lim_{R\to\infty} \int \varphi_R(\xi')\hat{\mu}(\xi - \xi')\, d\xi' = \mu\{0\}$$

uniformly in $\xi \in \mathbf{R}^n$:

$$\varphi_R * \hat{\mu}(\xi) - \mu\{0\} = \hat{\varphi}_R \cdot \mu(e^{-\langle \cdot, \xi \rangle}) - \mu(\chi_{\{0\}})$$
$$= \mu\left(\hat{\varphi}_R \cdot e^{-i\langle \cdot, \xi \rangle} - \chi_{\{0\}} \cdot e^{-i\langle \cdot, \xi \rangle}\right),$$

and this is bounded by

$$\int |\hat{\varphi}_R - \chi_{\{0\}}| d|\mu|,$$

which is clearly convergent to zero as $R \to \infty$.

It now follows from (5) that, for some $R > 0$,

$$\sup_{|\xi - \xi_0| \le R} |\hat{\mu}(\xi)| \ge \left| \int \varphi_R(\xi) \hat{\mu}(\xi_0 - \xi)\, d\xi \right| \ge \tfrac{1}{2} |\mu\{0\}|$$

for all $\xi_0 \in \mathbf{R}^n$.

5. A class of invertible distributions.

THEOREM 4. *Let $\mu \in \mathcal{E}'$ be a measure with an atom, let $\nu \in \mathcal{E}'$ have singular support disjoint from that of μ and let P be a non-zero polynomial. Then $P \cdot \hat{\mu} + \hat{\nu}$ is slowly decreasing.*

Proof. By Theorems 1 and 3 all we need to prove is that non-zero polynomials are (very) slowly decreasing: for any $\varepsilon > 0$ the function

$$\mathbf{R}^n \ni \xi_0 \mapsto \int_{|\xi| \le \varepsilon} |P(\xi_0 + \xi)|^2\, d\xi$$

is a polynomial with no real zeroes, hence it is bounded away from zero. Therefore, for some $B_\varepsilon, C_\varepsilon > 0$,

$$\sup_{|\xi - \xi_0| \le \varepsilon} |P(\xi)| \ge C_\varepsilon \cdot \left(\int_{|\xi| \le \varepsilon} |P(\xi_0 + \xi)|^2\, d\xi \right)^{1/2} \ge B_\varepsilon.$$

Appendix. We briefly sketch the proof of the following result of Ehrenpreis [**2**].

The mapping

$$\text{(A1)} \qquad \mathcal{E} \ni u \mapsto \check{\mu} * u \in \mathcal{E}$$

is surjective if and only if $\hat{\mu}$ is slowly decreasing.

Since the adjoint of (A1),

$$\text{(A2)} \qquad \mathcal{E}' \ni \nu \mapsto \mu * \nu \in \mathcal{E}',$$

is injective, $\check{\mu} * \mathcal{E}$ is dense in $\mathcal{E}$; it is equal to $\mathcal{E}$ if and only if $\mu * \mathcal{E}'$ is weak* closed (see, for example, Kelley and Namioka [**4**], Theorem 2.19). By reflexivity of $\mathcal{E}$ the weak* closure of $\mu * \mathcal{E}'$ is equal to its weak closure and therefore also to its strong closure, the strong topology of $\mathcal{E}'$ being locally convex. Malgrange [**5**], Corollary on p. 310, proved that $\mu * \mathcal{E}'$ is strongly closed if and only if $\hat{\mu}$ has the following division property:

(A3) if $\nu \in \mathcal{E}'$ and $\hat{\nu}/\hat{\mu}$ is entire, then $\nu = \mu * \gamma$ for some $\gamma \in \mathcal{E}'$.

We now show that (A3) holds if and only if $\hat{\mu}$ is slowly decreasing.

If $\hat{\mu}$ is slowly decreasing then, without losing generality, we may assume that for every $\xi_0 \in \mathbf{R}^n$ there exists $\xi_1 \in \mathbf{R}^n$ such that

$$|\xi_1 - \xi_0| \leq A \log\bigl(2 + |\xi_0|\bigr) \quad \text{and} \quad |\hat{\mu}(\xi_1)| \geq 1.$$

Let $\nu \in \mathcal{E}'$ and assume $\hat{\nu}/\hat{\mu}$ is entire. For $\tau \in \mathbf{C}$ put $\varphi(\tau) = \hat{\mu}(\xi_1 + 2\tau(\xi_0 - \xi_1))$ and $\psi(\tau) = \hat{\nu}(\xi_1 + 2\tau(\xi_0 - \xi_1))$. By Harnack's inequality

$$\text{(A4)} \qquad \log^+\left|\frac{\hat{\nu}}{\hat{\mu}}(\xi_0)\right| = \log^+\left|\frac{\psi}{\varphi}\left(\frac{1}{2}\right)\right| \leq 3 \cdot \int_{|\tau|=1} \log^+\left|\frac{\psi}{\varphi}\right|.$$

By subadditivity of $\log^+$ and the equality $\log|\varphi| = \log^+|\varphi| - \log^+|1/\varphi|$, we may estimate the integral in (A4) first by

$$\int_{|\tau|=1} (\log^+|\psi| + \log^+|\varphi|) - \int_{|\tau|=1} \log|\varphi|,$$

and then only by

$$\text{(A5)} \qquad \int_{|\tau|=1} \bigl(\log^+|\psi| + \log^+|\varphi|\bigr)$$

because, by the assumption,

$$\int_{|\tau|=1} \log|\varphi| \geq \log|\varphi(0)| = \log|\hat{\mu}(\xi_1)| \geq 0.$$

Since the points on the circle $|\tau| = 1$ lie at a distance at most $2A\log(2 + |\xi_0|)$ from the real space $\mathbf{R}^n_\xi$ and we have an estimate on $\hat{\mu}$ and $\hat{\nu}$ in terms of the exponential of that distance, the integral (A5) is not greater than $\log C + N\log(1 + |\xi_0|)$ for some constants C, N. Thus

$$\left|\frac{\hat{\nu}}{\hat{\mu}}(\xi_0)\right| \leq C\bigl(1 + |\xi_0|\bigr)^N, \qquad \xi_0 \in \mathbf{R}^n,$$

proving that $\hat{\nu}/\hat{\mu}$ has polynomially bounded growth on $\mathbf{R}^n_\xi$ and therefore, being necessarily of exponential type (see Malgrange [**5**]), is a Fourier-Laplace transform of some $\gamma \in \mathcal{E}'$.

Conversely, if $\hat{\mu}$ is not slowly decreasing, then there exists a sequence $\xi_j \in \mathbf{R}^n, j = 1, 2, \ldots$, such that

$$|\hat{\mu}(\xi)| < |\xi_j|^{-j} \quad \text{when } |\xi - \xi_j| \leq j \log|\xi_j|$$

and we may assume $|\xi_j| \to \infty$ suitably quickly. It is now possible to construct an entire function g which itself is not a Fourier-Laplace transform of any $\gamma \in \mathcal{E}'$, but becomes one when multiplied by $\hat{\mu}$. We indicate the idea: for each j we let φ_j be a test function with support in a fixed set k such that $\hat{\varphi}_j(\xi)$ is about the size of $|\xi_j|^j$ when $\xi = \xi_j$, but is conveniently small when $|\xi - \xi_j| \geq j \log |\xi_j|$. The function $g = \Sigma \hat{\varphi}_i$ is of exponential type but not polynomially bounded on $\mathbf{R}^n_\xi$. At the same time $\hat{\mu} \cdot g = \Sigma \hat{\mu}\hat{\rho}_j$ is polynomially bounded on $\mathbf{R}^n_\xi$ because $\hat{\mu}$ is small where $\hat{\varphi}_j$ is big. For the details of the construction we refer to Ehrenpreis [**2**] and Hörmander [**3**].

Added in proof. I wish to thank Olaf von Grudzinski for bringing my attention to the papers [7], [**8**] of L. Hörmander and in particular to the fact that Theorem 2 of this note (hence also Theorem 1) is a consequence of Theorem 3 in [**8**] and Lemma 5.4 in [7]. It may be remarked, however, that the proof presented here is independent of the much more advanced methods of [7].

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