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# HEREDITARY CONES, ORDER IDEALS AND HALF-NORMS

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**Let $\mathfrak{B}$ be an ordered normed space with positive cone $\mathfrak{B}_+$ and let $N$ be the canonical half-norm of $\mathfrak{B}_+$, i.e.,**

$$N(a) = \inf\{\|a + b\|;\, b \in \mathfrak{B}_+\}.$$

**Then, for any hereditary subcone $\mathcal{C}$ of $\mathfrak{B}_+$, the positive bipolar $\mathcal{C}^{\perp\perp}$ coincides with the $N$-closure $\mathcal{C}^N$ of $\mathcal{C}$, i.e.,**

$$\mathcal{C}^N = \{a \in \mathfrak{B}_+;\, N(a - a_n) \to 0 \text{ for some } a_n \in \mathcal{C}\}.$$

**Similar facts are proved for order ideals and these results are used to derive a result of Størmer on archimedean order ideals.**

**0. Introduction.** Let $\mathfrak{B}$ be a normed space. A half-norm on $\mathfrak{B}$ is a real-valued function $N$ satisfying the following conditions:

(1) $0 \le N(x) \le k\|x\|$ for some $k > 0$;
(2) $N(x + y) \le N(x) + N(y)$;
(3) $N(\lambda x) = \lambda N(x)$ for $\lambda \ge 0$;
(4) $N(x) \vee N(-x) = 0$ if, and only if, $x = 0$.

For a half-norm $N$ on $\mathfrak{B}$, the set $\{x \in \mathfrak{B};\, N(-x) = 0\}$ is a closed proper cone which defines an order relation on $\mathfrak{B}$. Conversely, when $\mathfrak{B}$ is an ordered normed space with positive cone $\mathfrak{B}_+$, which is closed and proper,

$$N(x) = \inf\{\|x + y\|;\, y \in \mathfrak{B}_+\}$$

is a half-norm on $\mathfrak{B}$. This half-norm is called the *canonical half-norm* [**1**] associated with the positive cone $\mathfrak{B}_+$. It is also called the *order half-norm* [**6**] associated with $\mathfrak{B}_+$ because it can also be characterized by

$$N(x) = \inf\{\lambda \ge 0;\, x \le \lambda u \text{ for some } u \in \mathfrak{B}_1\}$$

where $\mathfrak{B}_1$ is the unit ball of $\mathfrak{B}$.

Half-norms were explicitly introduced by Arendt, Chernoff and Kato [**1**], although particular examples occurred in the works of Grosberg and Krein [**4**], Kadison [**5**], and many subsequent authors on order-unit and base-norm spaces. Half-norms are particularly useful for studying positive semigroups [**1**], [**2**], [**9**]. For further developments, see [**6**], [**7**].

The aim of this paper is to clarify the positive bipolar property considered in [**2**]. It turns out that the positive bipolar $\mathcal{C}^{\perp\perp}$ of a hereditary subcone $\mathcal{C}$ of $\mathcal{B}_+$ coincides with the $N$-closure $\mathcal{C}^N$ of $\mathcal{C}$. A similar fact is valid also for order ideals. Hereditary subcones and order ideals are crucial notions in the study of the hermitian part of a $C^*$-algebra.

**1. Hereditary cones and order ideals.** Let $\mathcal{B}$ be a normed space ordered by a closed proper cone $\mathcal{B}_+$, which we call the positive cone of $\mathcal{B}$. A subcone $\mathcal{C}$ of $\mathcal{B}_+$ is said to be *hereditary* if $0 \le a \le c$ and $c \in \mathcal{C}$ imply $a \in \mathcal{C}$. The following lemma is easily proved.

LEMMA 1. *A subcone $\mathcal{C}$ of $\mathcal{B}_+$ is hereditary if, and only if $\mathcal{C} = \mathcal{B}_+ \cap (\mathcal{C} - \mathcal{B}_+)$.*

Hereditary cones are closely related to order ideals. An *order ideal* in $\mathcal{B}$ is a linear subspace $\mathcal{J}$ of $\mathcal{B}$ such that $a \le x \le b$ and $a, b \in \mathcal{J}$ imply $x \in \mathcal{J}$. In other words, a linear subspace $\mathcal{J}$ of $\mathcal{B}$ is an order ideal if, and only if,

$$\mathcal{J} = (\mathcal{J} + \mathcal{B}_+) \cap (\mathcal{J} - \mathcal{B}_+).$$

An order ideal is said to be *positively generated* if $\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_+$ for $\mathcal{J}_+ = \mathcal{J} \cap \mathcal{B}_+$.

LEMMA 2. *If $\mathcal{J}$ is a positively generated order ideal, then $\mathcal{J}_+$ is an hereditary subcone of $\mathcal{B}_+$. If $\mathcal{C}$ is an hereditary subcone of $\mathcal{B}_+$, its linear span $L(\mathcal{C})$ is a positively generated order ideal.*

*Proof.* Let $\mathcal{J}$ be a positively generated order ideal. Then,

$$\begin{aligned}\mathcal{J}_+ &= \mathcal{J} \cap \mathcal{B}_+ = (\mathcal{J} + \mathcal{B}_+) \cap (\mathcal{J} - \mathcal{B}_+) \cap \mathcal{B}_+ \\ &\supset (\mathcal{J}_+ + \mathcal{B}_+) \cap (\mathcal{J}_+ - \mathcal{B}_+) \cap \mathcal{B}_+ \\ &= (\mathcal{J}_+ - \mathcal{B}_+) \cap \mathcal{B}_+ \supset \mathcal{J}_+ .\end{aligned}$$

Hence, by Lemma 1, $\mathcal{J}_+$ is hereditary. Conversely, if $\mathcal{C}$ is hereditary, we have

$$\mathcal{C} = L(\mathcal{C}) \cap \mathcal{B}_+ .$$

It is obvious from this equality that $L(\mathcal{C})$ is positively generated with $L(\mathcal{C})_+ = \mathcal{C}$. Furthermore, since $\mathcal{C} + \mathcal{B}_+ = \mathcal{B}_+$,

$$\begin{aligned}(L(\mathcal{C}) + \mathcal{B}_+) &\cap (L(\mathcal{C}) - \mathcal{B}_+)\\ &= (\mathcal{C} - \mathcal{C} + \mathcal{B}_+) \cap (\mathcal{C} - \mathcal{C} - \mathcal{B}_+) = (\mathcal{B}_+ - \mathcal{C}) \cap (\mathcal{C} - \mathcal{B}_+)\\ &\subset \mathcal{B}_+ \cap (\mathcal{C} - \mathcal{B}_+) - \mathcal{C} = L(\mathcal{C}).\end{aligned}$$

Therefore, $L(\mathcal{C})$ is also an order ideal.

**2.** ***N*-closures.** Let $\mathcal{B}$ be an ordered normed space with positive cone $\mathcal{B}_+$ and let $N$ be the canonical half-norm associated with $\mathcal{B}_+$. For a subset $\mathcal{M}$ of $\mathcal{B}$, we define the $N$-closure $\mathcal{M}^N$ of $\mathcal{M}$ by

$$\mathcal{M}^N = \{a \in \mathcal{B}_+ ; N(a - a_n) \to 0 \text{ for some } a_n \in \mathcal{M}\}.$$

By definition, $\mathcal{M}^N$ is a subset of $\mathcal{B}_+$.

LEMMA 3. (1) $\mathcal{M} \cap \mathcal{B}_+ \subset \mathcal{M}^N$;

(2) $\mathcal{M}^N$ *is closed*;

(3) *If* $\mathcal{C}$ *is an hereditary subcone of* $\mathcal{B}_+$, $\mathcal{C}^N$ *is also an hereditary subcone of* $\mathcal{B}_+$, $\mathcal{C}^N = L(\mathcal{C})^N$, *and*

$$\mathcal{C}^N = \mathcal{B}_+ \cap \overline{(\mathcal{C} - \mathcal{B}_+)};$$

(4) *If* $\mathcal{J}$ *is an order ideal in* $\mathcal{B}$, $\mathcal{J}^N$ *is also an order ideal and*

$$\mathcal{J}^N = \mathcal{B}_+ \cap \overline{(\mathcal{J} - \mathcal{B}_+)}.$$

*Proof.* (1) is obvious. To prove (2), assume that $a_n \in \mathcal{M}^N$ and $\|a_n - a\| \to 0$ for some $a \in \mathcal{B}$. Then, one can choose $b_n \in \mathcal{M}$ such that $N(a_n - b_n) < 1/n$. Since $N$ is canonical, one always has $0 \le N(x) \le \|x\|$ for every $x \in \mathcal{B}$. Hence,

$$N(a - b_n) \le N(a - a_n) + N(a_n - b_n) \le \|a - a_n\| + N(a_n - b_n) \to 0.$$

Therefore, $a \in \mathcal{M}^N$.

To prove (3), let $\mathcal{C}$ be an hereditary subcone of $\mathcal{B}_+$. If $0 \le a \le c$ and $c \in \mathcal{C}^N$, we can choose $c_n \in \mathcal{C}$ such that $N(c - c_n) \to 0$. Since $N$ is canonical, we always have $N(x) \le N(y)$ if $x \le y$. Hence,

$$0 \le N(a - c_n) \le N(c - c_n) \to 0.$$

Therefore, $a \in \mathcal{C}^N$, i.e., $\mathcal{C}^N$ is hereditary. Next, let $a \in L(\mathcal{C})^N$ and choose $a_n \in L(\mathcal{C})$ such that $N(a - a_n) \to 0$. Since $L(\mathcal{C}) = \mathcal{C} - \mathcal{C}$ by Lemma 2, there exist $c_n, d_n \in \mathcal{C}$ such that $a_n = c_n - d_n$. Then, $a_n \le c_n$ and, hence, $N(a - c_n) \to 0$. This shows $a \in \mathcal{C}^N$. Therefore, $\mathcal{C}^N = L(\mathcal{C})^N$. To prove the equality $\mathcal{C}^N = \mathcal{B}_+ \cap \overline{(\mathcal{C} - \mathcal{B}_+)}$, let $a \in \mathcal{C}^N$ and choose $c_n \in \mathcal{C}$ such that $N(a - c_n) \to 0$. By the definition of the canonical half-norm, there exist $b_n \in \mathcal{B}_+$ such that $\|a - c_n + b_n\| \to 0$. Therefore, $a \in \mathcal{B}_+ \cap \overline{(\mathcal{C} - \mathcal{B}_+)}$. Conversely, if $a \in \mathcal{B}_+ \cap \overline{(\mathcal{C} - \mathcal{B}_+)}$, then $a \in \mathcal{B}_+$ and there exist $c_n \in \mathcal{C}$ and $b_n \in \mathcal{B}$ such that $\|a - c_n + b_n\| \to 0$. Then, $N(a - c_n) \to 0$ and, hence, $a \in \mathcal{C}^N$.

Finally, to prove (4), let $\mathcal{J}$ be an order ideal and $a \leq x \leq b$ for $a$, $b \in \mathcal{J}^N$. Then, for $b_n \in \mathcal{J}$ such that $N(b - b_n) \to 0$, we have $0 \leq N(x - b_n) \leq N(b - b_n) \to 0$. Hence $x \in \mathcal{J}^N$. The remaining equality can be proved in the same manner as in (3).

When $\mathcal{C}$ is an hereditary subcone of $\mathcal{B}_+$, we have

$$\mathcal{C} = \mathcal{B}_+ \cap (\mathcal{C} - \mathcal{B}_+),$$

by Lemma 1. Then

$$\bar{\mathcal{C}} = \overline{\mathcal{B}_+ \cap (\mathcal{C} - \mathcal{B}_+)}.$$

If $\bar{\mathcal{C}}$ is hereditary, again by Lemma 1,

$$\bar{\mathcal{C}} = \mathcal{B}_+ \cap (\bar{\mathcal{C}} - \mathcal{B}_+),$$

whereas by Lemma 3,

$$\mathcal{C}^N = \mathcal{B}_+ \cap \overline{(\mathcal{C} - \mathcal{B}_+)},$$

and, in general, we only have

$$\overline{\mathcal{B}_+ \cap (\mathcal{C} - \mathcal{B}_+)} \subset \mathcal{B}_+ \cap (\bar{\mathcal{C}} - \mathcal{B}_+) \subset \mathcal{B}_+ \cap \overline{(\mathcal{C} - \mathcal{B}_+)}.$$

Similar relations hold for order ideals. To obtain relations $\bar{\mathcal{C}} = \mathcal{C}^N$ and $\bar{\mathcal{J}} \cap \mathcal{B}_+ = \mathcal{J}^N$, we introduce new types of hereditary cones and order ideals.

A subcone $\mathcal{C}$ of $\mathcal{B}_+$ is said to be *topologically hereditary* if the following condition is satisfied: if $a_n \leq c_n$, $c_n \in \mathcal{C}$, $a \in \mathcal{B}_+$ and $\|a - a_n\| \to 0$, then $a \in \bar{\mathcal{C}}$. An order ideal $\mathcal{J}$ is called a *topological order ideal* if the following condition is satisfied: if $x_n \leq a_n$, $a_n \in \mathcal{J}$, $a \in \mathcal{B}_+$ and $\|a - x_n\| \to 0$, then $a \in \bar{\mathcal{J}}$. An example of a topologically hereditary cone is $\mathcal{C}^N$ for a hereditary cone $\mathcal{C}$. It will be shown in Lemma 10, that archimedean order ideals are topological order ideals.

LEMMA 4. (1) *Let $\mathcal{C}$ be an hereditary subcone of $\mathcal{B}_+$. Then $\mathcal{C}$ is topologically hereditary if, and only if, $\bar{\mathcal{C}} = \mathcal{C}^N$.*

(2) *Let $\mathcal{J}$ be an order ideal. Then $\mathcal{J}$ is a topological order ideal if, and only if, $\bar{\mathcal{J}} \cap \mathcal{B}_+ = \mathcal{J}^N$.*

*Proof*. To prove (1), let $\mathcal{C}$ be an hereditary subcone of $\mathcal{B}_+$. If $\mathcal{C}$ is topologically hereditary and $a \in \mathcal{C}^N$, by Lemma 3, there exist $c_n \in \mathcal{C}$ and $b_n \in \mathcal{B}_+$ such that $\|a - c_n + b_n\| \to 0$. Then, for $a_n = c_n - b_n$, we have $a_n \leq c_n$ and $\|a - a_n\| \to 0$. Hence, $a \in \bar{\mathcal{C}}$. Therefore, $\mathcal{C}^N \subset \bar{\mathcal{C}}$. But $\bar{\mathcal{C}} \subset \mathcal{C}^N$ follows from (1) and (2) of Lemma 3. Conversely, assume $\bar{\mathcal{C}} = \mathcal{C}^N$, $a_n \leq c_n$, $c_n \in \mathcal{C}$, $a \in \mathcal{B}_+$ and $\|a - a_n\| \to 0$. Then, since

$$0 \leq N(a - c_n) \leq N(a - a_n) \leq \|a - a_n\| \to 0,$$

we have $a \in \mathcal{C}^N = \bar{\mathcal{C}}$.

Property (2) can be proved in the same manner, using (4) in Lemma 3.

There are two cases where some points of $\mathcal{C}^N$ automatically belong to $\bar{\mathcal{C}}$.

PROPOSITION 5. *Let $\mathcal{M}$ be an hereditary subcone of $\mathcal{B}_+$, or an order ideal in $\mathcal{B}$. If $\mathcal{B}_+$ contains an interior point $u$, $u \in \mathcal{M}^N$ implies $u \in \overline{\mathcal{M}}$.*

*Proof*. If $u \in \mathcal{M}^N$, there exist $a_n \in \mathcal{M}$ and $x_n \in \mathcal{B}$ such that $x_n \leq a_n$ and $\|u - x_n\| \to 0$. Since $u$ is an interior point of $\mathcal{B}_+$, we may suppose $x_n \in \mathcal{B}_+$. Then, by assumption, $x_n \in \mathcal{M}$ and, hence, $u \in \overline{\mathcal{M}}$.

PROPOSITION 6. *Let $\mathcal{B}$ be a normed lattice. Then, every hereditary subcone of $\mathcal{B}_+$ is topologically hereditary, and every positively generated order ideal is a topological order ideal.*

*Proof*. Let $\mathcal{M}$ be an hereditary subcone, or a positively generated order ideal, and suppose $x_n \leq a_n$, $a_n \in \mathcal{M}$, $a \in \mathcal{B}_+$, and $\|a - x_n\| \to 0$. Then, $0 \leq x_n^+ \leq a_n^+$ and $\|a - x_n^+\| \to 0$, where $x_n^+$ and $a_n^+$ denote the positive parts of $x_n$ and $a_n$. In both cases, $a_n^+ \in \mathcal{M}$ and, hence $x_n^+ \in \mathcal{M}$. Then we have $a \in \overline{\mathcal{M}}$.

REMARK. When $\mathcal{J}$ is a positively generated order ideal in a vector lattice $\mathcal{B}$, every $a \in \mathcal{J}$ has a decomposition $a = a_1 - a_2$ with $a_i \in \mathcal{J} \cap \mathcal{B}_+$. Then $a \leq a^+ \leq a_1$, which implies $a^+ \in \mathcal{J}$. This fact was used in the above proof and it also means that $\mathcal{J}$ is a sublattice.

**3. Positive bipolars.** Let $\mathcal{B}$ be an ordered normed space with positive cone $\mathcal{B}_+$ and let $N$ be the canonical half-norm associated with $\mathcal{B}_+$. Let $\mathcal{B}^*$ be the dual of $\mathcal{B}$ and denote by $f(x)$ the value of $f \in \mathcal{B}^*$ at $x \in \mathcal{B}$. Then $f \in \mathcal{B}^*$ is defined to be positive if $f(x) \geq 0$ for all $x \in \mathcal{B}_+$. The *dual cone* $\mathcal{B}^*_+$ consists of all positive elements of $\mathcal{B}^*$.

We recall a result, proved in [7], Lemma 2, which states that $f \in \mathfrak{B}^*$ is positive if, and only if,

$$f(x) \leq \|f\| N(x) \quad \text{for all } x \in \mathfrak{B}.$$

We start with a proposition which illustrates the relevance of the $N$-closure.

PROPOSITION 7. *Let $\mathfrak{M}$ be a linear subspace of $\mathfrak{B}$ and let $a \in \mathfrak{B}_+$. Then, there exists $f \in \mathfrak{B}_+^*$ such that*

$$f(a) > 0 \quad \textit{and} \quad f(x) = 0 \qquad \textit{for all } x \in \mathfrak{M}$$

*if, and only if, $a \notin \mathfrak{M}^N$.*

*Proof.* If such an $f \in \mathfrak{B}_+^*$ exists, then, for any $b \in \mathfrak{M}$,

$$0 < f(a) = f(a - b) \leq \|f\| N(a - b),$$

which implies $a \notin \mathfrak{M}^N$. Conversely, assume $a \notin \mathfrak{M}^N$ and set

$$\alpha = \inf_{b \in \mathfrak{M}} N(a - b) > 0.$$

Let $\mathfrak{M}_a$ be the linear subspace spanned by $a$ and $\mathfrak{M}$, and define a linear functional $g$ on $\mathfrak{M}_a$ by

$$g(\xi a + b) = \alpha \xi$$

for all $\xi \in \mathbf{R}$ and $b \in \mathfrak{M}$. Then one has

$$g(c) \leq N(c) \quad \text{for all } c \in \mathfrak{M}_a,$$

because, if $c = \xi a + b$ with $\xi \leq 0$,

$$g(c) = \alpha \xi \leq 0 \leq N(\xi a + b),$$

and, if $c = \xi a + b$ with $\xi > 0$,

$$g(c) = \alpha \xi \leq \xi N(a + b/\xi) = N(\xi a + b).$$

Hence $g$ can be extended to an $f \in \mathfrak{B}^*$ such that

$$f(c) \leq N(c) \quad \text{for all } c \in \mathfrak{B}.$$

Therefore, $f$ is positive and, by construction, $f(a) > 0$ and $f(c) = 0$ for all $c \in \mathfrak{M}$.

An immediate consequence is a characterization of the positive bipolars $\mathcal{C}^{\perp\perp}$ of an hereditary subcone $\mathcal{C}$, and $\mathfrak{J}^{\perp\perp}$ of an order ideal $\mathfrak{J}$. The definitions of these polars are as follows:

$$\mathcal{C}^{\perp} = \{f \in \mathcal{B}_+^* \,; f(c) = 0 \text{ for every } c \in \mathcal{C}\},$$

$$\mathcal{C}^{\perp\perp} = \{a \in \mathcal{B}_+ \,; f(a) = 0 \text{ for every } f \in \mathcal{C}^{\perp}\},$$

$$\mathcal{J}^{\perp} = \{f \in \mathcal{B}_+^* \,; f(x) = 0 \text{ for every } x \in \mathcal{J}\},$$

$$\mathcal{J}^{\perp\perp} = \{a \in \mathcal{B}; f(a) = 0 \text{ for every } f \in \mathcal{J}^{\perp}\}.$$

THEOREM 8. (1) *If $\mathcal{C}$ is an hereditary subcone of $\mathcal{B}_+$, then $\mathcal{C}^{\perp\perp} = \mathcal{C}^N$.*
(2) *If $\mathcal{J}$ is an order ideal, $\mathcal{J}^{\perp\perp} \cap \mathcal{B}_+ = \mathcal{J}^N$.*

*Proof.* When $\mathcal{C}$ is an hereditary subcone of $\mathcal{B}_+$, $\mathcal{C}^{\perp\perp} \subset \mathcal{C}^N$ follows from Proposition 7. Conversely, if $a \in \mathcal{C}^N$ and $f \in \mathcal{C}^{\perp}$, there exist $c_n \in \mathcal{C}$ such that $N(a - c_n) \to 0$ and

$$0 \leq f(a) = f(a - c_n) \leq \|f\| N(a - c_n) \to 0.$$

Therefore, $\mathcal{C}^N \subset \mathcal{C}^{\perp\perp}$. The second part can be proved in the same manner.

Now we can characterize the subcones $\mathcal{C}$ which satisfy the equality $\mathcal{C} = \mathcal{C}^{\perp\perp}$.

COROLLARY 9. *Let $\mathcal{C}$ be a subcone of $\mathcal{C}_+$. Then $\mathcal{C} = \mathcal{C}^{\perp\perp}$ if, and only if, $\mathcal{C}$ is closed and topologically hereditary.*

*Proof.* By Lemma 4, if $\mathcal{C}$ is closed and topologically hereditary, we have $\mathcal{C} = \mathcal{C}^N$. Conversely, if $\mathcal{C} = \mathcal{C}^{\perp\perp}$, then $\mathcal{C}$ is hereditary and $\mathcal{C} = \mathcal{C}^N$. Hence, by Lemma 4, $\mathcal{C} = \bar{\mathcal{C}}$ and $\mathcal{C}$ is topologically hereditary.

In [2] it was proved that if $\mathcal{B}$ is the hermitian part of a $C^*$-algebra, $\mathcal{B}_+$ the positive elements of the algebra, and $\mathcal{C}$ an hereditary subcone of $\mathcal{B}_+$, then one has the bipolar property $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$. A similar property was established for separable, countably order-complete, Banach lattices. But this latter result can be improved by the foregoing.

COROLLARY 10. *Let $\mathcal{B}$ be a normed lattice and $\mathcal{C}$ an hereditary subcone of $\mathcal{B}_+$. Then $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$ and hence $\bar{\mathcal{C}}$ is hereditary.*

*Proof.* It follows from Proposition 6 that $\mathcal{C}$ is topologically hereditary and hence $\bar{\mathcal{C}} = \mathcal{C}^N$ by Lemma 4. But $\mathcal{C}^N = \mathcal{C}^{\perp\perp}$ by Theorem 8 and hence $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$.

Of course in the $\mathcal{C}^*$-algebra case this argument could be reversed. If $\mathcal{C}$ is an hereditary subcone of $\mathcal{B}_+$ then $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$ by [2] and hence $\bar{\mathcal{C}} = \mathcal{C}^N$. Thus each hereditary subcone is topologically hereditary.

The case of order ideals is discussed in the next section.

**4. Archimedean order ideals.** As an application of the above results, we derive the positive bipolar property for archimedean order ideals (see **[8]**).

According to Kadison **[5]**, an archimedean ordered vector space is an ordered vector space $\mathfrak{B}$ with an order unit $u$ whose order is archimedean, i.e., if $x \leq \lambda u$ for all $\lambda > 0$, then $x \leq 0$.

When $\mathfrak{B}$ is an archimedean ordered vector space, its positive cone $\mathfrak{B}_+$ is a proper cone. We set

$$N(x) = \inf\{\lambda > 0;\ x \leq \lambda u\}$$

and

$$\|x\| = N(x) \vee N(-x).$$

Under the assumption that the order is archimedean, this defines a norm and $\mathfrak{B}_+$ is closed with respect to the corresponding norm topology. Furthermore, $N$ is the canonical half-norm of $\mathfrak{B}_+$. (For more details about the canonical half-norms defined by order units, see **[6]**.) Therefore, an archimedean ordered vector space in the sense of Kadison is an ordered normed space equipped with the canonical half-norm defined by an order unit.

An *archimedean order ideal* is a closed, positively generated, order ideal $\mathfrak{J}$ such that the quotient $\mathfrak{B}/\mathfrak{J}$ is archimedean with respect to the order defined by the cone $\theta(\mathfrak{B}_+)$, where $\theta: \mathfrak{B} \mapsto \mathfrak{B}/\mathfrak{J}$ is the canonical map. When $\mathfrak{A}$ is a $\mathcal{C}^*$-algebra, $\mathfrak{J}$ is an archimedean ideal if, and only if, $\mathfrak{J}$ is the hermitian part of a two sided ideal of $\mathfrak{A}$.

When $\mathfrak{A}$ is an archimedean order ideal in $\mathfrak{B}$, since $\mathfrak{B}/\mathfrak{J}$ is archimedean, the positive cone $\theta(\mathfrak{B}_+)$ is closed with respect to the quotient norm:

$$\|\theta(x)\| = \inf\{\|y\|;\ y \in \theta(x)\}.$$

Evidently, $\theta$: $\mathfrak{B} \mapsto \mathfrak{B}/\mathfrak{J}$ is continuous with respect to this norm topology.

LEMMA 11. *Archimedean order ideals are topological order ideals.*

*Proof.* Let $\mathfrak{J}$ be an archimedean order ideal and suppose $x_n \leq a_n$, $a_n \in \mathfrak{J}$, $a \in \mathfrak{B}_+$ and $\|a - x_n\| \to 0$. Then, $\theta(x_n) \leq \theta(a_n) = 0$ and $\|\theta(a) - \theta(x_n)\| \to 0$. Hence, $\theta(a) \leq 0$. On the other hand, $\theta(a) \geq 0$ because $a \geq 0$. Therefore, $\theta(a) = 0$, i.e., $a \in \mathfrak{J}$.

We are now ready to reproduce a result in **[8]**, Theorem 2.4 from the previous results.

PROPOSITION 12. *For an archimedean order ideal* $\mathcal{J}$, $\mathcal{J} = \mathcal{J}^{\perp\perp}$.

*Proof.* By Lemma 11, $\mathcal{J}$ is a closed topological order ideal. Hence, by Lemma 4, $\mathcal{J} \cap \mathfrak{B}_+ = \mathcal{J}^N$. Therefore, by Theorem 8, $\mathcal{J}_+ = \mathcal{J} \cap \mathfrak{B}_+ = \mathcal{J}^{\perp\perp} \cap \mathfrak{B}_+$. Let $a \in \mathcal{J}^{\perp\perp}$. Then, by a version of a result of Effros [3], Theorem 2.4, and Størmer [8], Lemma 2.3, there exists $b \in \mathfrak{B}_+$ such that $b \geq a$ and $b \in \mathcal{J}^{\perp\perp}$. Since $\mathcal{J}$ is positively generated, $\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_+$ and $\mathcal{J}^{\perp} = \mathcal{J}_+^{\perp}$. Now, since $a = b - (b - a)$ and $b$, $b - a \in \mathcal{J}^{\perp\perp} \cap \mathfrak{B}_+ = \mathcal{J}^N = \mathcal{J} \cap \mathfrak{B}_+ = \mathcal{J}_+$ by Lemma 4 and Theorem 8, we conclude that $a \in \mathcal{J}$.

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