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ON STRONGLY DECOMPOSABLE OPERATORS

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A strongly decomposable operator, as defined by C. Apostol, is a bounded linear operator T which, for every spectral maximal space Y, induces two decomposable operators: the restriction $T \mid Y$ and the coinduced T/Y on the quotient space X/Y. In this paper, we give some necessary and sufficient conditions for an operator to be strongly decomposable.

Throughout the paper, T is a bounded linear operator acting on an abstract Banach space X over the field $\mathbf{C}$ of complex numbers. T^* denotes the conjugate of T on the dual space X^*. For a set S, S^c is the complement, $\bar{S}$ is the closure, $\bar{S}^w$ is the weak*-closure in X^*, $S^\perp$ is the annihilator of $S \subset X$ in X^*, ${}^\perp S$ is the annihilator of $S \subset X^*$ in X and Int S represents the interior of S. We write $\sigma(T)$ for the spectrum, $\rho(T)$ for the resolvent set of T and $R(\cdot\,; T)$ for the resolvent operator. If T is endowed with the single valued extension property (SVEP), then for any $x \in X$, $\sigma_T(x)$ denotes the local spectrum. For $S \subset \mathbf{C}$, we shall extensively use the spectral manifold

$$X_T(S) = \{x \in X: \sigma_T(x) \subset S\}.$$

We say that T satisfies condition α, if

(a) T has the SVEP, and (b) $X_T(F)$ is closed for every closed $F \subset \mathbf{C}$.

Two special types of subspaces, invariant under the given operator T, enter the theory of decomposable operators: (1) spectral maximal spaces [7]; (2) analytically invariant subspaces [9].

1. PROPOSITION. *Let Y be a spectral maximal space of T.*

(i) [9, *Proposition* 1] *If T has the SVEP then, for any $x \in X$,*

$$\sigma_T(x) = [\sigma_T(x) \cap \sigma(T \mid Y)] \cup \sigma_{\hat{T}}(\hat{x}), \qquad \hat{x} = x + Y, \hat{T} = T/Y. \tag{1}$$

(ii) [2, *Lemma* 1.4]. *If T is decomposable, then*

$$\sigma(T/Y) = \overline{\sigma(T) - \sigma(T \mid Y)}. \tag{2}$$

(iii) [7, *Theorem* 2.3]. *If T satisfies condition α, then $Y = X_T[\sigma(T \mid Y)]$.*

(iv) [3, *Proposition* I.3.2]. *If $Z \subset Y$ is a spectral maximal space of T, then Y/Z is a spectral maximal space of T/Z.*

(v) [7, *Lemma* 2.1]. *If* T *is decomposable and* $G \subset \mathbf{C}$ *is open, then* $\sigma(T) \cap G \neq \varnothing$ *implies that* $X_T(\overline{G}) \neq \{0\}$.

(vi) [7, *Theorem* 2.3]. *If* T *satisfies condition* α, *then for every closed* $F \subset \mathbf{C}$, $X_T(F)$ *is a spectral maximal space of* T *and*

$$\sigma[T \mid X_T(F)] \subset F. \tag{3}$$

(vii) [12, *Corollary* 1(c)]. *For* T *decomposable and for any closed* $F \subset \mathbf{C}$,

$$\sigma[T/X_T(F)] \subset (\operatorname{Int} F)^{c}.$$

(viii) [8, *Theorem* 1]. *If* T *is decomposable then, for every closed* $F \subset \mathbf{C}$, $X_T(F^c)^{\perp}$ *is a spectral maximal space of* T^* *and* $X_T(F^c)^{\perp} = X^*_{T^*}(F)$.

(ix) [9, *Theorem* 2]. *If* T *has the SVEP, then* Y *is analytically invariant under* T.

REMARK. More generally than in the original versions, properties (iii) and (vi) hold without the restriction of T being decomposable.

2. PROPOSITION. *Let* Y *be an analytically invariant subspace under* T. *Then*

(i) [9, *Theorem* 1]. T/Y *has the SVEP* (*the converse property is also true*).

(ii) [4, *Lemma* 3.4]. *If* T *has the SVEP then, for every* $y \in Y$,

$$\sigma_{T|Y}(y) = \sigma_T(y).$$

(iii) [9, *Theorem* 3]. *If* T *is decomposable then, for every open* $G \subset \mathbf{C}$, $\overline{X_T(G)}$ *is analytically invariant under* T.

3. THEOREM. *The following assertions are equivalent*:

(i) T *is strongly decomposable*;

(ii) (a) T *satisfies condition* α;

(b) *for every spectral maximal space* Y *of* T *and any* $x \in X$,

$$\sigma_{\hat{T}}(\hat{x}) = \overline{\sigma_T(x) - \sigma(T \mid Y)}, \qquad \hat{T} = T/Y, \hat{x} = x + Y; \tag{4}$$

(c) *for every special maximal space* Y *of* T *and any open* $G \subset \mathbf{C}$, $G \cap \sigma(T \mid Y) \neq \varnothing$ *implies that* $X_T[\overline{G} \cap \sigma(T \mid Y)] \neq \{0\}$.

Proof. (i) $\Rightarrow$ (ii). (a) is evident. (b). (1) implies

$$\sigma_{\hat{T}}(\hat{x}) \supset \sigma_T(x) - [\sigma_T(x) \cap \sigma(T \mid Y)] = \sigma_T(x) - \sigma(T \mid Y)$$

and hence

$$\sigma_{\hat{T}}(\hat{x}) \supset \overline{\sigma_T(x) - \sigma(T \mid Y)}.$$

To obtain the opposite inclusion, for $x \in X$, put

$$F = \sigma_T(x) \cup \sigma(T \mid Y) \tag{5}$$

and for the decomposable $T \mid X_T(F)$ use (2) and (3) as follows:

$$\begin{aligned}\sigma[\hat{T} \mid X_T(F)/Y] &= \overline{\sigma[T \mid X_T(F)] - \sigma(T \mid Y)} \subset \overline{F - \sigma(T \mid Y)} \\ &= \overline{\sigma_T(x) - \sigma(T \mid Y)}.\end{aligned}$$

By (5), $x \in X_T(F)$ and hence $\hat{x} = x + Y \in X_T(F)/Y$. Consequently,

$$\sigma_{\hat{T}}(\hat{x}) \subset \sigma[\hat{T} \mid X_T(F)/Y] \subset \overline{\sigma_T(x) - \sigma(T \mid Y)}$$

and this establishes (4).

Since $T \mid Y$ is decomposable, (c) is a consequence of Proposition 1 (v).

(ii) ⇒ (i): Let Y be a spectral maximal space of T. By (a) and Proposition 1 (iii), Y has a representation $Y = X_T[\sigma(T \mid Y)]$.

Let $G \subset \mathbf{C}$ be open and put $Z = X_T(\bar{G})$. We shall prove inclusion

$$\overline{G \cap \sigma(T \mid Y)} \subset \sigma(T \mid Y \cap Z). \tag{6}$$

If $G \cap \sigma(T \mid Y) = \varnothing$, then (6) is evident. Therefore, assume

$$G \cap \sigma(T \mid Y) \neq \varnothing.$$

Let $\lambda_0 \in G \cap \sigma(T \mid Y)$ and let $\delta_0 \subset G$ be a neighborhood of λ_0. Then, since $\delta_0 \cap (T \mid Y) \neq \varnothing$, (c) implies that $X_T[\bar{\delta}_0 \cap \sigma(T \mid Y)] \neq \{0\}$ and hence

$$\sigma(T \mid X_T[\bar{\delta}_0 \cap \sigma(T \mid Y)]) \neq \varnothing.$$

Let $\lambda_1 \in \sigma(T \mid X_T[\bar{\delta}_0 \cap \sigma(T \mid Y)])$. Then $\lambda_1 \in \bar{\delta}_0$ and it follows from

$$X_T[\bar{\delta}_0 \cap \sigma(T \mid Y)] \subset X_T[\bar{G} \cap \sigma(T \mid Y)] = X_T[\sigma(T \mid Y)] \cap Z = Y \cap Z$$

that $\lambda_1 \in \bar{\delta}_0 \cap \sigma(T \mid Y \cap Z)$. Thus,

$$\bar{\delta}_0 \cap \sigma(T \mid Y \cap Z) \neq \varnothing$$

and since δ_0 is an arbitrary neighborhood of λ_0, we must have $\lambda_0 \in \sigma(T \mid Y \cap Z)$. By the definition of λ_0, inclusion (6) holds. Finally, we shall conclude the proof by showing that $T \mid Y$ is decomposable. The subspace $W = Y \cap Z$ is a spectral maximal space of T. By denoting $\tilde{T} = T/W$ and for $x \in Y$, $\tilde{x} = x + W$, with the help of condition (b) and inclusion (6), we obtain successively

$$\sigma_{\tilde{T}}(\tilde{x}) = \overline{\sigma_T(x) - \sigma(T|W)} \subset \overline{\sigma_T(x) - [G \cap \sigma(T|Y)]} \tag{7}$$
$$\subset \overline{\sigma(T|Y) - [G \cap \sigma(T|Y)]} = \overline{\sigma(T|Y) - G} \subset G^c.$$

Since Y is a spectral maximal space of T and W is a spectral maximal space of $T|Y$, Proposition 1 (iv) implies Y/W is a spectral maximal space of T/W. Then, with the help of (7) and [**13**, Theorem 1.1 (g)], we obtain

$$\sigma[\hat{T}|(Y/W)] = \bigcup_{\tilde{x} \in Y/W} \sigma_{\tilde{T}}(\tilde{x}) \subset G^c.$$

Consequently, $T|Y$ is decomposable by [**5**, Theorem 12] and [**1**] (or [**11**]), (see also [**10**]). □

If one slightly strengthens condition (b) in Theorem 3, then (c) becomes redundant.

4. THEOREM. *The following assertions are equivalent*:
(I) *T is strongly decomposable*;
(II) (A) *T satisfies condition α*;
(B) *for every closed $F \subset \mathbf{C}$, and each $x \in X$,*

$$\sigma_{\hat{T}}(\hat{x}) = \overline{\sigma_T(x) - F} \tag{8}$$

where $\hat{T} = T/X_T(F)$, $\hat{x} = x + X_T(F)$.
(III) (A) *T satisfies condition α*;
(C) *For every pair F_1, F_2 of closed sets in* $\mathbf{C}$,

$$\sigma[(T/Y_2)|X_T(F_1 \cup F_2)/Y_2] \subset F_1, \quad \textit{where } Y_2 = X_T(F_2). \tag{9}$$

Proof. (I) ⇒ (III). Let F_1, F_2 be closed in $\mathbf{C}$. Since T is strongly decomposable, $T|X_T(F_1 \cup F_2)$ is decomposable. Let G_1, G_2 be open sets in $\mathbf{C}$ such that $F_1 \cup F_2 \subset G_1 \cup G_2$, $F_1 \subset G_1$ and $\overline{G}_2 \cap F_1 = \varnothing$. For $x \in X_T(F_1 \cup F_2)$, we have a representation

$$x = x_1 + x_2 \quad \text{with } x_i \in X_T(F_1 \cup F_2) \cap X_T(\overline{G}_i), i = 1, 2.$$

It follows from

$$\sigma_T(x_2) \subset (F_1 \cup F_2) \cap \overline{G}_2 = F_2 \cap \overline{G}_2 \subset F_2$$

that $x_2 \in X_T(F_2) = Y_2$.

Let $\lambda_0 \notin \overline{G}_1$. Then $\lambda_0 \in \rho(T|X_T[(F_1 \cup F_2) \cap \overline{G}_1])$ and hence there is $y \in X_T[(F_1 \cup F_2) \cap \overline{G}_1]$ verifying

$$(\lambda_0 - T)y = x_1.$$

By the natural homomorphism $X \to X/Y_2$, we obtain

$$(\lambda_0 - T/Y_2)\hat{y} = \hat{x}_1 = \hat{x},$$

and hence $\lambda_0 - (T/Y_2) \mid X_T(F_1 \cup F_2)/Y_2$ is surjective. Since T/Y_2 has the SVEP by Proposition 1 (vi), (ix) and Proposition 2 (i), we have $\lambda_0 \in \rho[(T/Y_2) \mid X_T(F_1 \cup F_2)/Y_2]$ by [**6**, Theorem 2]. By the definition of λ_0, we have

$$\sigma[(T/Y_2) \mid X_T(F_1 \cup F_2)/Y_2] \subset \bar{G}_1$$

and since $G_1 \supset F_1$ is arbitrary, inclusion (9) holds.

(III) ⇒ (II): Let $x \in X$ and $F \subset \mathbf{C}$ be closed. For $F_1 = \overline{\sigma_T(x) - F}$ and $Y = X_T(F)$, (9) implies

$$\sigma[(T/Y) \mid X_T(F_1 \cup F)/Y] \subset F_1 = \overline{\sigma_T(x) - F}.$$

It follows from the definition of F_1 that $x \in X_T(F_1 \cup F)$. Consequently, for $\hat{x} = x + Y$ and $\hat{T} = T/Y$, we have

$$\sigma_{\hat{T}}(\hat{x}) \subset \sigma[\hat{T} \mid X_T(F_1 \cup F)/Y] \subset \overline{\sigma_T(x) - F}.$$

On the other hand, it follows from Proposition 1 (i) that

$$\sigma_{\hat{T}}(\hat{x}) \supset \overline{\sigma_T(x) - \sigma(T \mid Y)} \supset \overline{\sigma_T(x) - F}$$

and hence (8) holds.

(II) ⇒ (I). In view of Theorem 3, we only have to prove that, for every open G and spectral maximal space $Y = X_T[\sigma(T \mid Y)]$,

$$G \cap \sigma(T \mid Y) \neq \varnothing \tag{10}$$

implies that $X_T[\bar{G} \cap \sigma(T \mid Y)] \neq \{0\}$. Choose an open G verifying (10), denote $Z = X_T[\bar{G} \cap \sigma(T \mid Y)]$ and for $x \in X$, let $\tilde{x} = x + Z$. If $Z = \{0\}$, then

$$\sigma_{\tilde{T}}(\tilde{x}) = \sigma_T(x), \qquad \tilde{T} = T/Z. \tag{11}$$

In view of (11), by hypothesis, we have

$$\begin{aligned}\sigma_T(x) = \sigma_{\tilde{T}}(\tilde{x}) &= \overline{\sigma_T(x) - [\bar{G} \cap \sigma(T \mid Y)]} \\ &= \overline{[\sigma_T(x) - \bar{G}]} \cup \overline{[\sigma_T(x) - \sigma(T \mid Y)]}.\end{aligned}$$

Let $x \in Y$. Since $\sigma_T(x) \subset \sigma(T \mid Y)$, we have

$$\sigma_T(x) = \overline{\sigma_T(x) - \bar{G}}$$

and hence

$$\sigma_T(x) \cap G = \varnothing.$$

Now, with the help of [**13**, Theorem 1.1 (g)], Proposition 1 (v), (ix) and Proposition 2 (ii), we obtain

$$\sigma(T|Y) \cap G = \left[\bigcup_{x\in Y} \sigma_{T|Y}(x)\right] \cap G = \left[\bigcup_{x\in Y} \sigma_T(x)\right] \cap G$$
$$= \bigcup_{x\in Y} [\sigma_T(x) \cap G] = \varnothing .$$

But this contradicts hypothesis (10). Therefore, $Z = X_T[\bar{G} \cap \sigma(T|Y)] \neq \{0\}$. □

Next, we shall obtain a characterization of a strongly decomposable operator in terms of the conjugate operator. First, we need some preparation.

5. LEMMA. *Given T, let Y and Z be invariant subspaces of X with $Z \subset Y$. Then*

$$(T/Z)^* | (Y/Z)^{\perp} \cong T^* | Y^{\perp} . \tag{12}$$

Proof. The mapping $X/Z \to X/Y$ is a continuous surjective homomorphism with kernel Y/Z. Therefore, the quotient spaces $(X/Z)/(Y/Z)$ and X/Y are isomorphic. Given $x \in X$, we use the following notations for the equivalent classes containing x in the corresponding quotient spaces: $\hat{x} \in X/Y$, $\tilde{x} \in X/Z$, $\tilde{\tilde{x}} \in (X/Z)/(Y/Z)$. Note that $u \in \hat{x}$ iff $u - x \in Y$ iff $(u-x)^{\sim} \in Y/Z$ iff $\tilde{u} \in \tilde{\tilde{x}}$. Since

$$\inf_{v\in\tilde{u}} \|v\| \leq \|u\|,$$

we have

$$\|\tilde{\tilde{x}}\| = \inf_{\tilde{u}\in\tilde{\tilde{x}}} \|\tilde{u}\| = \inf_{\tilde{u}\in\tilde{\tilde{x}}} \inf_{v\in\tilde{u}} \|v\| \leq \inf_{u\in\hat{x}} \|u\| = \|\hat{x}\|. \tag{13}$$

On the other hand, for every $u \in \hat{x}$, $\tilde{u} = u + Z \subset u + Y = \hat{x}$ and hence $\tilde{u} \subset \hat{x}$. Thus,

$$\inf_{v\in\tilde{u}} \|v\| \geq \|\hat{x}\|$$

and hence

$$\|\tilde{\tilde{x}}\| = \inf_{\tilde{u}\in\tilde{\tilde{x}}} \inf_{v\in\tilde{u}} \|v\| \geq \|\hat{x}\|. \tag{14}$$

Then, by (13) and (14), $\|\tilde{\tilde{x}}\| = \|\hat{x}\|$. Thus, it follows from the isometrical isomorphisms

$$(X/Y)^* \cong Y^{\perp}, \qquad [(X/Z)/(Y/Z)^* \cong (Y/Z)]^{\perp}$$

that the unitary equivalence (12) holds. □

6. LEMMA. *If T is decomposable then, for every open $G \subset \mathbf{C}$,*

$$X_T(G^c)^{\perp} = \overline{X^*_{T^*}(G)}^{w}. \tag{15}$$

Proof. Let T be decomposable. By [**14**], for every closed $F \subset \mathbf{C}$,

$$JX_T(F) = JX \cap X^{**}_{T^{**}}(F) \tag{16}$$

where J is the natural imbedding of X into X^{**}. By Proposition 1 (viii) and the fact that T decomposable implies T^* decomposable,

$$X^{**}_{T^{**}}(F) = X^*_{T^*}(F^c)^{\perp}. \tag{17}$$

Relations (16) and (17) imply

$$X_T(F) = {}^{\perp}X^*_{T^*}(F^c)$$

and hence, for $F = G^c$, (15) follows. □

7. LEMMA. *If T^* is decomposable then, for every open $G \subset \mathbf{C}$, $\overline{X^*_{T^*}(G)}^{w}$ (i.e. the weak*-closure of $X^*_{T^*}(G)$) is analytically invariant under T^*.*

Proof. Let $f^*: D \to X^*$ be analytic on an open $D \subset \mathbf{C}$ and verify condition

$$(\lambda - T^*)f^*(\lambda) \in \overline{X^*_{T^*}(G)}^{w} \quad \text{on } D.$$

We may assume D is connected. Put $F = G^c$, $Y = X_T(F)$, use Lemma 6, Proposition 1 (vii) and obtain successively

$$\sigma\left[T^* \mid \overline{X^*_{T^*}(G)}^{w}\right] = \sigma(T \mid Y^{\perp}) = \sigma[(T/Y)^*] = \sigma(T/Y) \subset (\operatorname{Int} F)^c = \overline{G}.$$

First, assume $D \subset \overline{G}$. Then $D \subset G \subset \rho(T \mid Y)$ and, for every $x \in Y$, $\lambda \in D$, we have

$$\begin{aligned}\langle x, f^*(\lambda)\rangle &= \langle (\lambda - T)R(\lambda; T \mid Y)x, f^*(\lambda)\rangle \\ &= \langle R(\lambda; T \mid Y)x, (\lambda - T^*)f^*(\lambda)\rangle = 0.\end{aligned}$$

Since $x \in Y$ is arbitrary, $f^*(\lambda) \in Y^{\perp} = \overline{X^*_{T^*}(G)}^{w}$ on D.

Next, assume $D \not\subset \overline{G}$. Then, for $\lambda \in D - \overline{G}$, the resolvent operator $R[\lambda; T^* \mid \overline{X^*_{T^*}(G)}^{w}]$ is defined, and for $h^*(\lambda) = (\lambda - T^*)f^*(\lambda)$ we have

$$(\lambda - T^*)\left\{f^*(\lambda) - R\left[\lambda; T^* \mid \overline{X^*_{T^*}(G)}^{w}\right]h^*(\lambda)\right\} = 0.$$

Since T^* has the SVEP,

$$f^*(\lambda) = R\left[\lambda; T^* \mid \overline{X^*_{T^*}(G)}^{w}\right]h^*(\lambda) \in \overline{X^*_{T^*}(G)}^{w}$$

on $D - \overline{G}$, and $f^*(\lambda) \in \overline{X^*_{T^*}(G)}^{w}$ on D, by analytic continuation. □

8. THEOREM. *The bounded operator T (resp. T^*) is strongly decomposable iff*:

(i) *T (resp. T^*) has the SVEP and for open $G \subset \mathbf{C}$, $T^* \mid \overline{X^*_{T^*}(G)}^{\mathrm{w}}$ (resp. $T \mid \overline{X_T(G)}$) is decomposable*;

(ii) *for every pair G, H of open sets in $\mathbf{C}$,*

$$(18) \quad \overline{X^*_{T^*}(G \cap H)}^{\mathrm{w}} = \overline{Y^*_{T^*|Y^*}(H)}^{\mathrm{w}} \quad \left(\textit{resp.}\ \overline{X_T(G \cap H)} = \overline{Y_{T|Y}(H)}\right),$$

where $Y^* = \overline{X^*_{T^*}(G)}^{\mathrm{w}}$ (*resp.* $Y = \overline{X_T(G)}$).

Proof. We confine the proof to the operator T, the proof concerning T^* being similar.

(only if): Assume T is strongly decomposable. Let $G \subset \mathbf{C}$ be open, $F = G^{\mathrm{c}}$ and $Z = X_T(F)$. The operator $(T/Z) \mid (X/Z)$ is decomposable. Then, by Lemma 6, $X_T(F)^{\perp} = \overline{X^*_{T^*}(G)}^{\mathrm{w}}$ and hence

$$(19) \qquad (X/Z)^* \cong \overline{X^*_{T^*}(G)}^{\mathrm{w}}.$$

By [**8**, Theorem 2] and [**12**], $T^* \mid \overline{X^*_{T^*}(G)}^{\mathrm{w}}$ is decomposable. Apply Lemma 5 to a closed $F_1 \supset F$, and obtain

$$(20) \qquad [X_T(F_1)/Z]^{\perp} \cong X_T(F_1)^{\perp}.$$

Denote $\tilde{T} = T/Z$, $\tilde{X} = X/Z$. Before embarking on the proof of (ii), we shall show that

$$(21) \qquad \tilde{X}_{\tilde{T}}(\overline{F_1 - F}) = X_T(F_1)/Z.$$

In fact, if $\tilde{x} \in \tilde{X}_{\tilde{T}}(\overline{F_1 - F})$, then $\sigma_{\tilde{T}}(\tilde{x}) \subset \overline{F_1 - F}$ and hence, for every $x \in \tilde{x}$,

$$\sigma_T(x) \subset (\overline{F_1 - F}) \cup F = F_1.$$

Therefore, $\tilde{x} \in \tilde{X}_{\tilde{T}}(\overline{F_1 - F})$ implies $x \in X_T(F_1)$ and hence $\tilde{x} \in X_T(F_1)/Z$. Conversely, if $\tilde{x} \in X_T(F_1)/Z = X_T(\overline{F_1 - F} \cup F)/Z$, then Theorem 4 (III, C) implies

$$\sigma_{\tilde{T}}(\tilde{x}) \subset \sigma\left[\tilde{T} \mid X_T(\overline{F_1 - F} \cup F)/Z\right] \subset \overline{F_1 - F}$$

and hence $\tilde{x} \in \tilde{X}_{\tilde{T}}(\overline{F_1 - F})$. Thus (21) is proved.

Now we are in a position to prove (ii). To simplify notation, put $X^{\cdot} = (\tilde{X})^*$ and $T^{\cdot} = (\tilde{T})^*$. Let H be open and let $F_1 = G^{\mathrm{c}} \cup H^{\mathrm{c}}$. Then $F_1 \supset F$ and $\overline{F_1 - F} \subset H^{\mathrm{c}}$. By Lemma 6, Lemma 5, (20), (21) and (19), we obtain successively:

$$\begin{aligned} \overline{X^*_{T^*}(G \cap H)}^{\mathrm{w}} &= X_T(F_1)^{\perp} \cong [X_T(F_1)/Z]^{\perp} = \tilde{X}_{\tilde{T}}(\overline{F_1 - F})^{\perp} \supset [\tilde{X}_{\tilde{T}}(H^{\mathrm{c}})]^{\perp} \\ &= \overline{X^{\cdot}_{T^{\cdot}}(H)}^{\mathrm{w}} = \overline{Y^*_{T^*|Y^*}(H)}^{\mathrm{w}}. \end{aligned}$$

For the last equality, we used the equivalence

$$T^{\cdot} = [T/X_T(F)]^* \cong T^* \mid \overline{X^*_{T^*}(G)}^{\mathrm{w}} = T^* \mid Y^*.$$

To obtain the opposite inclusion, note that if $x^* \in X^*_{T^*}(G \cap H)$, then

$$\sigma_{T^*}(x^*) = \subset G \cap H \subset G$$

and hence $x^* \in X^*_{T^*}(G) \subset Y^*$. Since Y^* is analytically invariant under T^* (Lemma 7), in view of Proposition 2 (ii), we obtain

$$\sigma_{T^*|Y^*}(x^*) = \sigma_{T^*}(x^*) \subset H$$

and hence

$$x^* \in Y^*_{T^*|Y^*}(H) \subset \overline{Y^*_{T^*|Y^*}(H)}^{\mathrm{w}}.$$

Thus

$$\overline{X^*_{T^*}(G \cap H)}^{\mathrm{w}} \subset \overline{Y^*_{T^*|Y^*}(H)}^{\mathrm{w}}.$$

(if): Assume conditions (i) and (ii) are satisfied. Let F, $F_1 \subset \mathbf{C}$ be closed. Since $X^*_{T^*}(\mathbf{C}) = X^*$, we conclude that T^* is decomposable and hence T is decomposable by [**14**, Corollary 2.8]. Therefore, $Z = X_T(F)$ is closed. Also $T^* \mid \overline{X^*_{T^*}(F^c)}^{\mathrm{w}}$ is decomposable. Then, by Lemma 6,

$$T^* \mid \overline{X^*_{T^*}(F^c)}^{\mathrm{w}} = T^* \mid X_T(F)^{\perp} \cong T^{\cdot},$$

where $\tilde{T} = T/Z$ and $T^{\cdot} = (\tilde{T})^*$. Thus $T^{\cdot}$ is decomposable and hence $\tilde{T}$ is decomposable. Therefore, letting $\tilde{X} = X/Z$, $\tilde{X}_{\tilde{T}}(F_1)$ is closed and

$$\sigma[\tilde{T} \mid \tilde{X}_{\tilde{T}}(F_1)] \subset F_1. \tag{22}$$

Put $G = F^c$, $H = F_1^c$ and $Y^* = \overline{X^*_{T^*}(G)}^{\mathrm{w}}$. It follows from Lemma 6 that

$$T^* \mid X_T(F \cup F_1)^{\perp} = T^* \mid \overline{X^*_{T^*}(G \cap H)}^{\mathrm{w}},$$

$$T^{\cdot} \mid \tilde{X}_{\tilde{T}}(F_1)^{\perp} \cong T^* \mid \overline{Y^*_{T^*|Y^*}(H)}^{\mathrm{w}}.$$

Then (18) implies

$$T^{\cdot} \mid \tilde{X}_{\tilde{T}}(F_1)^{\perp} \cong T^* \mid X_T(F \cup F_1)^{\perp}. \tag{23}$$

By Lemma 5 we have

$$T^{\cdot} \mid [X_T(F \cup F_1)/Z]^{\perp} \cong T^* \mid X_T(F \cup F_1)^{\perp}. \tag{24}$$

Consequently, with the help of (24), (23) and (22), we obtain

$$\begin{aligned}\sigma[\tilde{T} \mid X_T(F \cup F_1)/Z] &= \sigma\{T^{\cdot} \mid [X_T(F \cup F_1)/Z]^{\perp}\} = \sigma[T^{\cdot} \mid \tilde{X}_{\tilde{T}}(F_1)^{\perp}] \\ &= \sigma[\tilde{T} \mid \tilde{X}_{\tilde{T}}(F_1)] \subset F_1.\end{aligned}$$

Thus, conditions (III) of Theorem 4 are satisfied and hence T is strongly decomposable. □

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