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A REMARK ON A PAPER OF KUHFITTIG

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It is observed that an iterative scheme introduced by Kuhfittig satisfies a property which allows us to obtain sufficient conditions for its strong and weak convergence to a common fixed point of a finite family of nonexpansive mappings, via other known results. In particular we generalize the results obtained by Kuhfittig.

Let E be a strictly convex Banach space and let C be a closed, convex subset of E. We consider a finite family of nonexpansive mappings $\{T_i: i = 1, 2, \ldots, k\}$, with $\bigcap_{i=1}^{k} \mathcal{F}(T_i) \neq \varnothing$, where $\mathcal{F}(T_i)$ is the fixed point set of T_i. In **[5]** Kuhfittig introduced the following functions U_i by putting

$$U_i(x) = \left[(1-\alpha)I + \alpha T_i U_{i-1}\right](x), \qquad x \in C,$$

where $\alpha \in]0, 1[$, $U_0 = I$, I identity on E, and he studied convergence of the sequence $\{U_k^n(x_0)\}$, $x_0 \in C$, to a point of $\bigcap_{i=1}^{k} \mathcal{F}(T_i)$.

Purpose of this note is to obtain convergence results for such a sequence.

Our results are based on the following property of $\{U_k^n(x_0)\}$: the sequence $\{x_n\}$, $x_n = U_k^n(x_0)$, $x_0 \in C$, $n \in N$, is a particular case of the following iterative procedure considered by Ishikawa (see **[4]**) for a nonexpansive mapping F

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n F(x_n), \qquad n \in N,$$

$\alpha_n \in]0, b] \subset]0, 1[$. In our case $F = T_k U_{k-1}$. Then, following Ishikawa, we have

$$\lim_n \|x_n - T_k U_{k-1}(x_n)\| = 0, \qquad x_0 \in C, \tag{1}$$

$$\lim_n \left\|U_k^n(x_0) - U_k^{n+1}(x_0)\right\| = 0, \qquad x_0 \in C. \tag{2}$$

Moreover, like in **[5]**, Theorem 1, we have the following

LEMMA. *Let E, C, T_i, U_i, α be as above. If $y \in C$ is such that $y = T_k U_{k-1}(y)$, then y is a common fixed point of $\{T_i: i = 1, 2, \ldots, k\}$.*

Now, we are ready to show our theorems. First is an improvement of Theorem 1 of [5]

THEOREM 1. *Let E, C, T_i, U_i, α be as above. We suppose that $j \in \{1, 2, \ldots, k\}$ exists for which T_j is α-condensing, i.e. $\alpha(T(A)) < \alpha(A)$, for each bounded $A \subset C$, $\alpha(A) > 0$, where α denotes the Kuratowski measure of non compactness.*

Then, $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{k} \mathfrak{F}(T_i)$.

Proof. It is easy to show that $T_k U_{k-1}$ is α-condensing and so $\{x_n\}$ converges strongly to a fixed point of $T_k U_{k-1}$, by a result of [3]. The Lemma completes the proof.

THEOREM 2. *Let E, C, T_i, U_i, α be as above. If E is uniformly convex and $C = -C$, then $\{U_k^n(x_0)\}$ converges strongly to a common fixed point of $\{T_i\colon i = 1, 2, \ldots, k\}$, if any T_i is odd.*

Proof. We observe that U_k is odd and asymptotically regular, by (2). Baillon, Bruck and Reich [1] have shown that $\{U_k^n(x_0)\}$ converges strongly to a fixed point of U_k, i.e. a fixed point of $T_k U_{k-1}$, under these assumptions. Like in Theorem 1, we conclude the proof.

The following Theorem 3 generalizes a result by Kuhfittig ([5], Theorem 2) if (i) below holds.

THEOREM 3. *Let E, C, T_i, U_i, α be as above. We suppose that C is boundedly weakly compact and one of the following conditions is satisfied*:

(i) *E satisfies Opial's condition.*

(ii) *E is uniformly convex and it has a Frechét differentiable norm.*

Then, $\{U_k^n(x_0)\}$ converges weakly to a point of $\bigcap_{i=1}^{k} \mathfrak{F}(T_i)$.

Proof. If (i) is true, using (1) we show that $\{x_n\}$ converges weakly to a fixed point of $T_k U_{k-1}$, by a result of [3]. Then, the Lemma concludes the proof.

Moreover, if (ii) is satisfied, we have to use results by Bruck [2] in order to show that $\{U_k^n(x_0)\}$ converges weakly to a fixed point of U_k, i.e. a fixed point of $T_k U_{k-1}$. Using the Lemma, we have our thesis.

References

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