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# AN ADDENDUM TO *bo*-RESOLUTIONS

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**It has been pointed out to me by W. Lellmann and K. Knapp that there are two difficulties with the paper *bo-resolutions* [M1]. First, the statement of the main theorem claims more than is proved, and second, the proof of the vanishing line is incomplete. In this note a corrected theorem and a discussion of where the error occurs are given. The vanishing line is discussed in complete detail in the third part of this note. Before that discussion occurs, we will show how the several applications of [M1] still follow.**

**§1 contains a correct statement of the main theorem of [M1] and a discussion of the error. §2 shows how the $v_1$-periodicity theorem and the results of [DGM] follow from the revised main theorem. §3 discusses the vanishing line for bo-resolutions.**

**1.** We need to start with a *bo*-resolution which is a tower of spectra

$$\begin{array}{ccccccccc} S^0 & \leftarrow & S_1 & \leftarrow & S_2 & \leftarrow \cdots \leftarrow & S_s & \leftarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ bo & & S_1 \wedge bo & & S_2 \wedge bo & & S_s \wedge bo & \end{array}$$

where

$$S_s \wedge bo \overset{\mathrm{id}\wedge\iota_0}{\leftarrow} S_s \leftarrow S_{s+1}$$

is a cofiber sequence and the map $S_s \overset{\mathrm{id}\wedge\iota_0}{\rightarrow} S_s \wedge bo$ is the composite

$$S_s \overset{\approx}{\rightarrow} S_s \wedge S^0 \overset{\mathrm{id}\wedge\iota_0}{\rightarrow} S_s \wedge bo$$

and $S^0 \overset{\iota_0}{\rightarrow} bo$ is the unit. Such a tower gives rise to a spectral sequence whose $E_1$ term is $E_1^{s,t}(S^0, bo, \pi) = \pi_{t-s}(S_s \wedge bo)$ and whose $E_\infty$-term is an associated graded group to $\pi_*(S^0)$. In [**M1**] and in this note we assume everything is localized at 2. In particular $S^0$, the zero sphere, and $bo$ are both considered as 2-primary spectra.

In [**M1**] the $E_1$ term is calculated. This calculation is done by showing that $H^*(S_s)$, as an $A_1$ module ($A_1 \subset A$ is generated by $Sq^1$ and $Sq^2$), is a direct sum of some irreducible $A_1$ modules $M_{i,s}$ and a free $A_1$ module $W_s$. The Adams spectral sequence to calculate $\pi_*(S_s \wedge bo)$ collapses and so the $\mathrm{Ext}_{A_1}(M_{i,s}, \mathbf{Z}/2)$ calculation determines $\pi_*(S_s \wedge bo)$. Let $\overline{M}_{i,s}$ be the summand of $\pi_*(S_s \wedge bo)$ which projects to $\mathrm{Ext}_{A_1}(M_{i,s}, \mathbf{Z}/2)$ in this Adams spectral sequence. Let $\overline{W}_s$ be the $\mathbf{Z}/2$ vector space $\mathrm{Ext}_{A_1}^{s,0}(W_s, \mathbf{Z}/2)$. Consider the chain complex induced from the *bo*-resolution.

$$\cdots \to \bigoplus_i \overline{M}_{i,s} \to \bigoplus_i \overline{M}_{i,s+1} \to \cdots$$

Theorem 5.11 of **[M1]** correctly calculates this chain complex and shows that the resulting homology consists of classes of Adams filtration 0 in $S_s \wedge bo$ for $s > 1$.

In **[M1]** this was erroneously used to conclude that in $E_2^{s,t}(S^0, bo, \pi)$ only classes of Adams filtration 0 survived for $s > 1$. This neglects the effect of the $\overline{W}_s$. Thus what is proved in **[M1]** is the following corrected part (c) of Theorem 1.1.

(c′) For $s > 1$, $E_\infty^{s,t}(S^0, bo, \pi)$ consists of classes of Adams filtration 0 or 1 in $\pi^*(S_s \wedge bo)$. Also, $E_\infty^{s,t}(S^0, bo, \pi) = 0$ for $6s > t + 14$.

**2.** One of the key applications of the results of **[M1]** is the determination of all homotopy classes which satisfy an "Adams periodicity". Such results are discussed in **[M1]**, but are given a more satisfactory form in **[M2]**. The idea is to consider the homotopy theory $[Y, ]_j$ where $Y^6 = \mathbf{R}P^2 \wedge \mathbf{C}P^2$, $Y^j = \Sigma^{j-6}Y^6$ and $\pi_j(X; Y) = [Y^j, X]$. The spectrum $Y^j$ has a self-map $v_1$: $Y^{j+2} \to Y^j$. An element $\alpha$ in $\pi_j(X; Y)$ is $v_1$ periodic if $Y^{j+2k} \xrightarrow{v_1^k} Y^j \xrightarrow[\alpha]{} X$ is essential for all $k$.

THEOREM 6.3 **[M1]**. *The only classes in $\pi_*(S^0; Y)$ which are $v_1$-periodic are those detected in $E_1^{0,*}(S^0, bo, \pi)$ and $E_1^{1,*}(S^0, bo, \pi)$.*

*Proof*. The proof given in **[M1]** went as follows. Let $\alpha \in \pi_j(S^0; Y)$ lift to $S_s$.

$$\begin{tikzcd} S_{s-1} \wedge bo \arrow[r] & S_s \arrow[r, "f_s"] \arrow[d] & S_s \wedge bo \\ Y^j \arrow[ur, "\alpha_s"] \arrow[r, "\alpha"'] & S^0 & \end{tikzcd}$$

Then $f_s\alpha_s v_1$ has Adams filtration 1 and so could be modified by a map $Y^{j+2} \to S_{s-1} \wedge bo \to S_s \to S_s \wedge bo$ to be zero. The correction says that this modified map could be detected by $\overline{W}_s$, and so $\alpha_s v_1^2$ lifts to $S_{s+1}$. In order to recover the result without using the vanishing line we need the following.

PROPOSITION 2. *The space $S_s \wedge bo$ is homotopically equivalent to a product of $K(\mathbf{Z})$'s and $K(\mathbf{Z}/2)$'s through dimension $5s - 1$.*

*Proof.* This is a restatement of a portion of the calculation in §3 of [**M1**]. In particular the edge for $S_s$ is given by $B(1)^{\wedge s}$. Now $B(1) \wedge B(1) \wedge bo = \Sigma^6 bo^2$, as is explicitly calculated on p. 373 of [**M1**]. Now $\Sigma^6 bo^{(2)}$ is homotopy equivalent to $K(\mathbf{Z}, 6) \vee K(\mathbf{Z}/2, 8)$ through dimension 9. The balance of the cases follows similarly.

We now return to the proof of 6.3. Given a map $\alpha\colon Y^j \to S^0$ which lifts to $\alpha_s\colon Y^j \to S_s$ for $s > 1$, we can lift $\alpha_s v_1^{2k}$ to $S_{s+k}$. If $k = j - 5s + 1$ then $\alpha_s v_1^{2k}$ lifts to $S_{j-4s}$. $S_{j-4s} \wedge bo$ is a wedge of Eilenberg-MacLane spaces through $5j - 20s + 4$ and $Y^{j+4j-20s+4} \to S_{j-4s}$ lifts once for each application of $v_1$. Thus if $\bar{k} = 2j - 8s$, then $\alpha v_1^{2k+\bar{k}}$ factors through $S_{s+k+\bar{k}}$, which is a point through dimension $j + 4k + 2\bar{k}$.

This result, then, validates the application of [**M1**] used in [**M2**]. A very similar argument applies to recover the results in [**DGM**]. A more detailed discussion will appear later.

3. In this section we will establish the vanishing line for *bo* resolutions. In particular we will prove:

THEOREM 3. *For all* $s$, $E_2^{s,t}(S^0, bo, \pi) = 0$ *for* $6s > t + 14$.

This is just the second part of part (c) of Theorem 1.1 of [**M1**] and (c′), the correction given in §1 of this paper. The program will be to first prove the result for the $\mathbf{Z}/2$ Moore space $M$, and then to get Theorem 3 from this case. We will introduce some additional notation which uses the *bo*-resolution tower already introduced.

Let $X$ be any space. Then for each $s$ and $t$ we can form a chain complex $C_{\cdot}^{s,t}(X)$ with

$$C_j^{s,t+j}(X) = \operatorname{Ext}_A^{s,t}\big(H^*(S_j \wedge bo \wedge X), \mathbf{Z}/2\big).$$

The differential $d_j\colon C_j^{s,t} \to C_{j+1}^{s,t}$ is induced by the map

$$\Omega S_j \wedge bo \to S_{j+1} \to S_{j+1} \wedge bo,$$

which is the $d_1$ for the *bo*-resolution.

Let $A \to X \to Y$ be a cofiber sequence whose cohomology exact sequence also splits. We get long exact sequences

$$\cdots \to C_j^{s,t}(A) \to C_j^{s,t}(X) \to C_j^{s,t}(Y) \to C_j^{s+1,t}(A) \to \cdots$$

and, hence, cannot expect long exact sequences in $H_*(C_{\cdot}^{s,t}(\,))$ theory. With some additional hypotheses, though, we do get long exact sequences.

Let $\bar{A} \to X \to Y$ be a cofiber sequence such that
(1) the cohomology exact sequence splits, and
(2) $H^*(\bar{A})$ is free over $A_1$.

PROPOSITION 3.1. *For cofiber sequences satisfying the above hypothesis, we get a long exact sequence*

$$\to H_j\big(C_\cdot^{s,t}(\bar{A})\big) \to H_j\big(C_\cdot^{s,t}(X)\big) \to H_j\big(C_\cdot^{s,t}(Y)\big) \\ \to H_{j+1}\big(C_\cdot^{s,t}(\bar{A})\big) \to \cdots$$

*for each $s$ and $t$.*

*Proof.* Our hypotheses imply $C_\cdot^{s,t}(\bar{A}) = 0$ for $s > 0$. This gives us short exact sequences

$$0 \to C_\cdot^{s,t}(\bar{A}) \to C_\cdot^{s,t}(X) \to C_\cdot^{s,t}(Y) \to 0$$

for each $s$ and $t$ and standard arguments give the desired result.

Let $X \to Y \to \bar{A}$ be a cofiber sequence such that
(1) the cohomology exact sequence splits into short exact sequences, and
(2) $\tilde{H}(\bar{A})$ is free over $A_1$.

PROPOSITION 3.2. *For cofiber sequences such as above we get a long exact sequence*

$$\to H_j\big(C_\cdot^{s,t}(X)\big) \to H_j\big(C_\cdot^{s,t}(Y)\big) \to H_j\big(C_\cdot^{s,t}(\bar{A})\big) \\ \to H_{j+1}\big(C_\cdot^{s,t}(X)\big) \to \cdots$$

*for each $s$ and $t$.*

*Proof.* For $s > 1$ the argument is as above. For $s = 0$ and 1 we have

$$0 \to C_j^{0,t}(X) \to C_j^{0,t}(Y) \to C_j^{0,t}(\bar{A}) \xrightarrow{\delta} C_j^{1,t}(X) \to C_j^{1,t}(Y) \to 0.$$

Consider $A \wedge bo \to Y \wedge bo \xrightarrow{f} \bar{A} \wedge bo$. The space $\bar{A} \wedge bo$ is a wedge of $K(\mathbf{Z}/2)'2$ and $f^*$ is a monomorphism. The result of Margolis [**Ma**] shows there is a map $g\colon \bar{A} \wedge bo \to Y \wedge bo$ such that $fg \simeq \mathrm{id}$. This implies $\delta = 0$ and proves the proposition.

Let $X = (\Sigma^4 B(1))^{\wedge 4} \wedge M$ and let $f(j)$ be defined by $H_j(C_\cdot^{0,t}(M)) = 0$ for $t < f(j)$. Let $\bar{A}_1$ be some space whose cohomology is free on a single class in dimension 0 as a module over $A_1$.

PROPOSITION 3.3. (a) *There is a cofiber sequence* $\Sigma^{16}\overline{A}_1 \to X \to X/\Sigma^{16}A_1$ *whose cohomologyy exact sequence splits.*

(b) *There is a mapping* $\Sigma^{24}M \to X/\Sigma^{16}\overline{A}_1$ *whose cofiber has* $A_1$ *free cohomology.*

*Proof.* The calculations of **[M1]** show that $X$ is stably $A_1$ equivalent to $\Sigma^{24}M$. It is also easy to see that through dimension 21, $X$ is a wedge of Eilenberg-MacLane spaces, and so by possibly changing the $\overline{A}_1$ there is no obstruction to mapping $\Sigma^{16}\overline{A}$ into $X$. Clearly, $H^*(X/\Sigma^{16}A_1)$ is still stably $A_1$ isomorphic to $\Sigma^{24}M$. It is now an easy calculation to verify that the map $\Sigma^{24}M \to X/\Sigma^{16}\overline{A}$ exists realizing this isomorphism.

PROPOSITION 3.4. *For* $s > 0$, $H_j(C_.^{s,t}(X)) \simeq H_j(C_.^{s,t}(\Sigma^{24}M))$. *If* $t - 16 < \min(6j - 6, f(j) + 8)$ *then* $H_j(C_.^{0,t}(X)) = 0$.

COROLLARY 3.5. $H_j(C_.^{0,t}(M)) = 0$ *for* $t < 6j - 14$.

This is the vanishing line for the $\mathbf{Z}/2$ Moore space.

*Proof of Corollary.* Clearly there is a map $(\Sigma^3 B(1))^{\wedge 4} \wedge M \to S_4 \wedge M$, and it is easy to see that this induces a stable $A_1$ equivalence through dimension 27. Thus the proposition implies $H_j(C_.^{0,t}(X)) = H_{j+4}(C_.^{0,t}(M)) = 0$ for $t - 16 < \min[6j - 6, f(j) + 8]$. Now $f(1) = 4$, $f(2) = 8$, $f(3) = 12$. The above recursion formula gives $f(4) \geq 10$, $f(5) \geq 16$, $f(6) \geq 22$, $f(7) \geq 28$ and, in general, $f(j) \geq 6j - 14$.

Note that this formula is rather crude and the estimate could be easily improved. I do not know what the sharpest vanishing line actually is.

*Proof of* 3.4. Apply 3.1 to part (a) of 3.3 and 3.2 to part (b) of 3.3. Thus $H_j(C_.^{s,t}(X)) \simeq H_j(C_.^{s,t}(\Sigma^{24}M))$ for $s > 0$ and $H_j(C_.^{0,t}(X))$ fits into a long exact sequence

$$\cdots \to H_j\big(C_.^{0,t}(\Sigma^{16}A_1)\big) \to H_j\big(C_.^{0,t}(\Sigma^{16}\overline{A}_1)\big) \to H_j\big(C_.^{0,t}(X)\big) \to H_j\big(C_.^{0,t}(\Sigma^{24}M)\big) \to \cdots.$$

Now $H_j(C_.^{0,t}(\Sigma^{16}A_1)) = 0$ if $t - 16 < 6j - 6$ and $H_j(C_.^{0,t}(\Sigma^{24}M)) = 0$ if $t - 24 < f(j)$. This gives the proposition:

We will use Corollary 3.5 to prove Theorem 3. Consider the cofiber sequence $S^0 \to S^0 \cup_{2\iota}^{e^1}, \to S'$. As above, we do not get short exact sequences in the groups $C_.^{0,t}(\,)$. The vanishing line is concerned with that portion of the resolution where, for $s > 0$ $\mathrm{Ext}_{A_1}^{s,t}(H^*(S_j), \mathbf{Z}/2)$ is zero or in the image of multiplication by $h_0$. We will use this fact to get the short exact sequence we need.

The results of [**M1**] give the following, where $a(j)$ is the function:

$$\begin{array}{rcccc} a(j) = 0 & -2 & -2 & -1 \\ j(4) \equiv 0 & 1 & 2 & 3 \end{array}$$

PROPOSITION 3.6. *If $t < 6j + a(j)$, then*:
(i) *if $a \in \mathrm{Ext}_{A_1}^{0,t}(H^*S_j, \mathbf{Z}/2)$, then either $h_0^i a \neq 0$ for all $i$ or $h_0 a = 0$;*
(ii) *if $a \in \mathrm{Ext}_{A_1}^{s,t}(H^*S_j, \mathbf{Z}/2)$, $a \neq 0$, then $a = h_0^s a'$ for some $a'$.*

Let $\mathrm{TExt}_{A_1}^{s,t}(H^*S_j, Z/2)$ be the subvector space spanned by classes $a$ such that $h_0^l a = 0$ for some 1. One might think of this as the "torsion subgroup".

PROPOSITION 3.7. *If $t < 6j + a(j)$, then*

$$C_j^{0,t}(M) = C_j^{0,t}(S^0) \oplus \mathrm{TExt}_{A_1}^{0,t}\left(H^*S^1 \wedge S_j, \mathbf{Z}/2\right).$$

*Proof*. The connecting homomorphism in the long exact sequence in the Ext groups in multiplication by $h_0$. Since in our range the subgroup TExt is exactly the kernel of multiplication by $h_0$ the proposition follows.

Let $\bar{C}_j^{s,t}(S^0) = \mathrm{TExt}_{A_1}^{s,t}(H^*S_j, \mathbf{Z}/2)$.

PROPOSITION 3.8. *For $t < 6j + a(j) - 4$, $H_*(\bar{C}_\cdot^{0,t}) = H_*(C_\cdot^{0,t})$ and $H_*(C_\cdot^{s,s+t}) = 0 = H_*(\bar{C}_\cdot^{s,s+t})$ for $s > 0$.*

*Proof*. In this range of dimensions the $\mathbf{Z}/2[h_0]$ towers from acyclic complexes. Also note that $\bar{C}_\cdot^{s,s+t} = 0$.

These propositions imply

PROPOSITION 3.9. *If $t < 6j + a(j) - 4$, then there is an exact sequence*

$$H_j\left(\bar{C}_\cdot^{0,t}(S^0)\right) \to H_j\left(C_\cdot^{0,t}(M)\right) \to H_j\left(C_\cdot^{0,t}(S^1)\right)$$
$$\to H_{j+1}\left(\bar{C}_\cdot^{0,t}(S^0)\right) \to \cdots.$$

It is easy to prove the theorem from this result. Indeed, the result implies that if $H_{j'}(C_\cdot^{0,t}(M)) = 0$ for all $j' \geq j$, then $H_{j'+1}(C_\cdot^{0,t}(S^1)) \simeq H_{j'}(C_\cdot^{0,t}(S^0))$ for all $j' > j$. If $j' > t/4$, then $H_{j'}(C_\cdot^{0,t}(S^0)) = 0$. This completes the proof.

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