FOR EVERY HAUSDORFF SPACE Y THERE EXISTS A NON-TRIVIAL MOORE SPACE ON WHICH ALL CONTINUOUS FUNCTIONS INTO Y ARE CONSTANT

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Recently H. Herrlich has raised the question whether there exists a single Moore space Y such that every Moore space can be embedded into some topological power of Y. We answer this question negatively by proving the theorem stated in the title.

1. Introduction. It was shown independently by N. C. Heldermann [3] and the second named author [8] that there exists a single developable T_1-space Y such that every developable T_1-space can be embedded into some topological power of Y. In his recent survey article [5] H. Herrlich has raised the problem whether the same is true for Moore spaces. We solve this problem by proving the following:

THEOREM. *For every Hausdorff space Y there exists a non-trivial metacompact Moore space on which all continuous functions into Y are constant.*

This theorem shows that there does not exist *any* Hausdorff space Y such that every Moore space can be embedded into some topological power of Y, in particular there exists no Moore space with this property.

Another consequence of our theorem is that in the special case that Y is the space of reals, one obtains a metacompact Moore space on which every real-valued continuous function is constant. The first example of a Moore space which admits only constant real-valued continuous functions was found by S. Armentrout [1] based on a non-completely regular Moore space constructed by F. B. Jones [6]. Other examples were given by J. N. Younglove [11] and P. Roy (cited in [7]). Although their spaces are not metacompact they do have other nice properties. On the other hand, we believe that the construction presented here is simpler.

Our theorem should be compared with a result of H. Herrlich, who proved in [4] that for every T_1-space Y there exists a non-trivial regular space on which all continuous functions into Y are constant. As in this proof, our argument consists of two steps. In §2 we construct for every

Hausdorff space Y a metacompact Moore space S containing two distinct points, $-\infty$ and $+\infty$, such that $f(-\infty) = f(+\infty)$ for every continuous function f from S into Y. Using this space and a suitable modification of Herrlich's technique, we complete the proof of the theorem in §3. The paper ends with some additional remarks. In particular we show that the method introduced in §2 can be used to give a simple argument for the first step of the proof of Herrlich's theorem.

The notation we use is standard and can be found in [2]. A topological space is *developable* if it has a *development*, i.e. a sequence $\{\mathfrak{A}_n \mid n \in \mathbf{N}\}$ of open covers such that $\{\mathrm{St}(x, \mathfrak{A}_n) \mid n \in \mathbf{N}\}$ is a neighborhood base for every point $x \in X$, where $\mathrm{St}(x, \mathfrak{A}_n) = \bigcup\{U \in \mathfrak{A}_n \mid x \in U\}$. A regular developable space is called a *Moore space*. A Moore space is *metacompact* if it has a development which consists of point-finite open covers. Note that the class of metacompact Moore spaces coincides with the class of regular spaces having a uniform base ([2], 5.4.7).

2. Key lemma. The main step in the proof of our theorem is Lemma 2 below. For its verification we use the following result of Bernstein type, which seems to be a part of the set-theoretical folklore. However, for the sake of completeness, we include a brief argument.

LEMMA 1. *Let λ be an uncountable cardinal. If $\mathcal{G}$ is a family of subsets of a set X such that $|\mathcal{G}| = \lambda$ and $|G| = \lambda$ for each $G \in \mathcal{G}$, then there exists a denumerable partition $\mathcal{Z}$ of X such that $|G \cap Z| = \lambda$ for each $G \in \mathcal{G}$ and $Z \in \mathcal{Z}$.*

Proof. For every $G \in \mathcal{G}$ there exists a partition $\mathcal{F}_G$ of G into λ sets of cardinality λ. The set $\mathcal{F} = \bigcup\{\mathcal{F}_G \mid G \in \mathcal{G}\}$ can be written as $\mathcal{F} = \{F_\gamma \mid 0 \leq \gamma < \lambda\}$. Using transfinite induction one can choose, for every $\gamma < \lambda$, a denumerable set $Z_\gamma = \{Z_{\gamma,i} \mid i < \omega\} \subset F_\gamma - \bigcup\{Z_\alpha \mid 0 \leq \alpha < \gamma\}$. If $Z_i = \{Z_{\gamma,i} \mid 0 \leq \gamma < \lambda\}$ for each $i \geq 2$ and $Z_1 = X - \bigcup\{Z_i \mid i \geq 2\}$, then $\mathcal{Z} = \{Z_i \mid i < \omega\}$ is the desired partition of X. $\sqcap$

LEMMA 2. *For every Hausdorff space Y there exists a metacompact Moore space S containing two distinct points $-\infty$ and $+\infty$ such that $f(-\infty) = f(+\infty)$ for every continuous function f from S into Y.*

Proof. Let Y be an arbitrary Hausdorff space and let κ be a cardinal such that $|Y| \leq \kappa$ and $\kappa^\omega = 2^\kappa$. Note, for example, that the cardinal $|Y| + 2^{|Y|} + 2^{2^{|Y|}} + \cdots$ satisfies the hypotheses on κ (see [10], Ch. 8.15).

Moreover, let $X = (X, d)$ be a metric space of weight κ such that every non-empty open subset of X has cardinality 2^κ, e.g. the countable power of a discrete space of cardinality κ with the usual product metric. Denote by $\mathcal{G}$ the collection of all subsets of X which have cardinality 2^κ and are intersections of $\leq \kappa$ F_σ-sets in X. Since $\mathcal{G}$ is of cardinality 2^κ, it follows from Lemma 1 that there exists a partition $\{Z_i \mid i \in \mathbf{Z}\}$ of X such that $|G \cap Z_i| = 2^\kappa$ for each integer i and for each G in $\mathcal{G}$.

The space S is obtained by defining the following topology on $\{-\infty\} \cup X \times X \cup \{+\infty\}$, where $-\infty$ and $+\infty$ are two distinct points not contained in $X \times X$. All points in $X \times X$ with distinct coordinates are isolated. For a point x in Z_i a basic neighborhood system of (x, x) in S consists of all sets

$$U_n(x, x) = \{(x, z) \mid z \in Z_i \cup Z_{i+1}, d(x, z) < 1/n\}$$
$$\cup \{(z, x) \mid z \in Z_i \cup Z_{i+1}, d(x, z) < 1/n\},$$

where $n \in \mathbf{N}$. Finally, basic neighborhoods of $-\infty$ and $+\infty$ are of the form

$$U_n(-\infty) = \{-\infty\} \cup \bigcup \{U_n(x, x) \mid x \in \bigcup \{Z_i \mid i \leq -n\}\}$$

and

$$U_n(+\infty) = \{+\infty\} \cup \bigcup \{U_n(x, x) \mid x \in \bigcup \{Z_i \mid i \geq n\}\},$$

respectively. Since basic neighborhoods of points from $X \times X$ are clopen in S and $\mathrm{cl}_S\, U_{n+1}(-\infty) \subset U_n(-\infty)$, $\mathrm{cl}_S\, U_{n+1}(+\infty) \subset U_n(+\infty)$ for each $n \in \mathbf{N}$, the space S is regular. Moreover, S is a metacompact Moore space — the open covers

$$\mathcal{U}_n = \{U_n(-\infty), U_n(+\infty)\} \cup \{U_n(x, x) \mid x \in X\}$$
$$\cup \{\{(y, z)\} \mid (y, z) \notin \bigcup \{U_n(x, x) \mid x \in X\}\}$$

are point-finite and form a development of S.

Now let f be an arbitrary continuous function from S into Y. To prove that $f(-\infty) = f(+\infty)$ it suffices to show that $f(+\infty) \in V$ for every open neighborhood V of $f(-\infty)$ in Y. To this end consider an arbitrary open set V in Y containing $f(-\infty)$. For each $y \in Y$ denote by $A(y)$ the set of all points $x \in X$ such that $f(x, x) = y$. By the continuity of f there exists an $i_0 \in \mathbf{Z}$ such that $\{A(y) \mid y \in V\}$ covers Z_{i_0}. Consequently, since $|V| \leq \kappa$ and $|Z_{i_0}| = 2^\kappa$, there must exist a $y_0 \in V$ such that $|A(y_0) \cap Z_{i_0}| = 2^\kappa$. We claim that $f(+\infty) = y_0$. Note that this is proved once we have

shown that every neighborhood of $+\infty$ meets $f^{-1}(y_0)$, which, however, is an immediate consequence of the following:

Claim. $|A(y_0) \cap Z_i| = 2^\kappa$ for every $i \geq i_0$.

To prove the claim we proceed by induction. Assume $i \geq i_0$ and $|A(y_0) \cap Z_i| = 2^\kappa$. To see that $|A(y_0) \cap Z_{i+1}| = 2^\kappa$ it suffices to construct a set G in $\mathcal{G}$ such that $G \cap Z_{i+1} \subset A(y_0) \cap Z_{i+1}$. For this purpose we define for every point y in Y distinct from y_0 an F_σ-set $F(y)$ in X containing $A(y_0) \cap Z_i$ and disjoint from $A(y) \cap Z_{i+1}$. Observe that $G = \bigcap \{F(y) \mid y \in Y, y \neq y_0\}$ then has the desired properties.

Let y be an arbitrary point in Y distinct from y_0. There exists an open neighborhood W of y_0 in Y which does not contain y in its closure. For each $n \in \mathbf{N}$, denote by A_n the set of all points $x \in A(y_0) \cap Z_i$ such that $f(U_n(x, x)) \subset W$. Since $F(y) = \bigcup \{\operatorname{cl}_X A_n \mid n \in \mathbf{N}\}$ is an F_σ-set in X containing $A(y_0) \cap Z_i$, it only remains to prove that $\operatorname{cl}_X A_n$ and $A(y) \cap Z_{i+1}$ are disjoint for each $n \in \mathbf{N}$. To this end consider an arbitrary $n \in \mathbf{N}$ and a point $z \in A(y) \cap Z_{i+1}$. Since $f(z, z)$ does not belong to the closure of W in Y, there exists an $m \geq n$ such that $f(U_m(z, z)) \cap W = \varnothing$. It follows that $d(x, z) \geq 1/m$ for each $x \in A_n$, for otherwise there would exist an $x \in A_n$ such that $U_m(x, x) \cap U_m(z, z) \neq \varnothing$, which is impossible. Hence $z \notin \operatorname{cl}_X A_n$, which completes the proof. $\square$

3. Proof of the theorem. Let Y be an arbitrary Hausdorff space. By virtue of Lemma 2 there exists a metacompact Moore space S containing two distinct points, $-\infty$ and $+\infty$ such that $f(-\infty) = f(+\infty)$ for every continuous function f from S into Y. Using an ingenious construction of H. Herrlich [4], one can now obtain from S a regular space X on which every continuous function into Y is constant. However, since we want the resulting space to be a metacompact Moore space, we have to modify this construction slightly.

Our first aim is to construct for every Moore space M a Moore space $S(M)$ containing M as a closed subspace such that every continuous function from $S(M)$ into Y is constant on M. Additionally $S(M)$ will be metacompact if and only if M is metacompact.

Let M be an arbitrary Moore space and let $\{\mathcal{V}_n \mid n \in \mathbf{N}\}$ be a development of M such that $\mathcal{V}_{n+1}$ refines $\mathcal{V}_n$ for each $n \in \mathbf{N}$. The underlying set of $S(M)$ is $\{p\} \cup (S' \times M) \cup M$, where $S' = S - \{-\infty, +\infty\}$ and p is a point neither contained in $S' \times M$ nor in M. For each $n \in \mathbf{N}$ we define a cover $S(\mathcal{V}_n)$ of $S(M)$ as follows. Let $\{\mathcal{U}_n \mid n \in \mathbf{N}\}$

be a development of S such that $\mathfrak{U}_{n+1}$ refines $\mathfrak{U}_n$ and $\mathfrak{U}_n$ is point-finite for each $n \in \mathbf{N}$. Moreover let $\{U_n^- \mid n \in \mathbf{N}\}$ and $\{U_n^+ \mid n \in \mathbf{N}\}$ be sequences of open subsets of S' such that $\{U_n^- \cup \{-\infty\} \mid n \in \mathbf{N}\}$ and $\{U_n^+ \cup \{+\infty\} \mid n \in \mathbf{N}\}$ are neighborhood bases in S of $-\infty$ and $+\infty$, respectively, and $\mathrm{cl}_{S'} U_{n+1}^- \subset U_n^-$, $\mathrm{cl}_{S'} U_{n+1}^+ \subset U_n^+$ for each $n \in \mathbf{N}$. Then $S(\mathcal{V}_n)$ consists of $\{p\} \cup (U_n \times M)$, all sets $U \times \{z\}$, where $U \in \mathfrak{U}_n$ and $z \in M$, and all sets $S(V) = (U_n \times V) \cup V$, where $V \in \mathcal{V}_n$. One can easily check that taking $\cup \{S(\mathcal{V}_n) \mid n \in \mathbf{N}\}$ as a base yields a regular space $S(M)$ for which $\{S(\mathcal{V}_n) \mid n \in \mathbf{N}\}$ is a development. Obviously, M is contained as a closed subspace in $S(M)$. For further reference we note some additional properties of $S(M)$:

(i) $S(V) \cap M = V$ for each $V \in \mathcal{V}_n$;

(ii) $S(V) \cap S(W) = \varnothing$ whenever $V, W \in \cup \{\mathcal{V}_n \mid n \in \mathbf{N}\}$ and $V \cap W = \varnothing$;

(iii) $S(W) \subset S(V)$ whenever $V, W \in \cup \{\mathcal{V}_n \mid n \in \mathbf{N}\}$ and $W \subset V$;

(iv) $\mathrm{cl}_{S(M)} S(W) \subset S(V)$ whenever $\mathrm{cl}_M W \subset V$, $W \in \mathcal{V}_m$, $V \in \mathcal{V}_n$ and $n < m$;

(v) $S(\mathcal{V}_{n+1})$ refines $S(\mathcal{V}_n)$ for each $n \in \mathbf{N}$;

(vi) $S(\mathcal{V}_n)$ is point-finite if and only if $\mathcal{V}_n$ is point-finite.

Note that the topology of $S(M)$ is defined in such a way that for every $z \in M$ there is an obvious homeomorphism from S onto the subspace $\{p\} \cup (S' \times \{z\}) \cup \{z\}$ of $S(M)$ mapping $-\infty$ onto p and $+\infty$ onto z. Therefore $f(z) = f(p)$ for every continuous function f from $S(M)$ into Y, i.e. every such function is constant on M.

Starting with an arbitrary Moore space M_1 having a development $\{\mathcal{V}_n^1 \mid n \in \mathbf{N}\}$ with every $\mathcal{V}_n^1$ point-finite and $\mathcal{V}_{n+1}^1$ refining $\mathcal{V}_n^1$, we can now define a sequence $M_1 \subset M_2 \subset M_3 \subset \cdots$ of Moore spaces according to the rule $M_{k+1} = S(M_k)$. In the following we consider for every M_k the development $\{\mathcal{V}_n^k \mid n \in \mathbf{N}\}$ which is inductively obtained from $\{\mathcal{V}_n^1 \mid n \in \mathbf{N}\}$ by $\mathcal{V}_n^{k+1} = S(\mathcal{V}_n^k)$. Our aim is to define on $X = \cup \{M_k \mid k \in \mathbf{N}\}$ a topology which turns X into a metacompact Moore space on which every continuous function into Y is constant. For this purpose we introduce the following notation.

If $V \in \mathcal{V}_n^k$ for some $n, k \in \mathbf{N}$ define $T(V) = \cup \{S^i(V) \mid i \geq 0\}$, where $S^0(V) = V$, $S^1(V) = S(V)$, $S^2(V) = S(S(V)), \ldots$. It can be easily seen that for each $n \in \mathbf{N}$ the collection $\mathcal{V}_n = \{T(V) \mid V \in \cup \{\mathcal{V}_n^k \mid k \in \mathbf{N}\}\}$ is a cover of X. From (v) it follows that $\mathcal{V}_{n+1}$ refines $\mathcal{V}_n$ for each $n \in \mathbf{N}$. For each $x \in X$ denote by $k(x)$ the smallest $k \in \mathbf{N}$ such that $x \in M_k$. The crucial property of the covers $\mathcal{V}_n$ is the following:

(vii) If $T(V) \in \mathcal{V}_n$ and $x \in T(V)$, then there exists a $W \in \mathcal{V}_n^{k(x)}$ containing x such that $T(V) = T(W)$.

It can be easily verified using (i) above. As an immediate consequence of (vii) one obtains

(viii) $\operatorname{St}(x, \mathcal{V}_n) = \bigcup \{T(V) \mid V \in \mathcal{V}_n^{k(x)}, \ x \in V\}$ for each $x \in X$ and $n \in \mathbf{N}$.

Using (ii), (iii) and (viii) it is not difficult to check that $\bigcup \{\mathcal{V}_n \mid n \in \mathbf{N}\}$ is a base for a topology which turns X into a developable T_1-space. Finally it follows from (iv) that X is a Moore space which is metacompact because of (vi) and (vii).

Now let f be an arbitrary continuous function from X into Y. To see that f is constant consider two distinct points x and x' in X. There exists a $k \in \mathbf{N}$ such that x and x' are both contained in M_k. Since it follows from (i) that M_{k+1} is a subspace of X, the restriction of f to M_{k+1} is a continuous function from M_{k+1} into Y which is constant on M_k because $M_{k+1} = S(M_k)$. Hence $f(x) = f(x')$, which completes the proof. $\qquad \square$

4. Remarks. 1. For every non-empty class $\mathfrak{E}$ of topological spaces there exists a smallest class $EH(\mathfrak{E})$ of topological spaces which contains $\mathfrak{E}$ and is closed with respect to the formation of arbitrary products and subspaces. $EH(\mathfrak{E})$ is called the *epireflective hull* of $\mathfrak{E}$. It consists of all those spaces which can be embedded into products of spaces belonging to $\mathfrak{E}$. A class $\mathfrak{A}$ of topological spaces is called *simple* if there exists a single space E such that $\mathfrak{A} = EH(\{E\})$. Originally Herrlich's question, mentioned in the introduction, was stated in these terms, namely, whether the epireflective hull of the class of all Moore spaces is simple (see [5], Problem 21). Our theorem implies that any class of Hausdorff spaces which contains all metacompact Moore spaces is not simple.

2. A weaker notion than simplicity is that of semi-simplicity of a class of spaces: A class $\mathfrak{A}$ is called *semi-simple* (e.g. see [5], Ch. 3.2.8) if there exists a *minimal* class $\mathfrak{E}$ of spaces such that $\mathfrak{A} = EH(\mathfrak{E})$, where minimality of $\mathfrak{E}$ means that $EH(\mathfrak{E} - \{E\}) \neq \mathfrak{A}$ for every space E in $\mathfrak{E}$. In view of the preceding remark one might wonder whether the epireflective hull of all Moore spaces is semi-simple. We give a brief argument which shows that it is not.

Let $\mathfrak{A}$ be the epireflective hull of all Moore spaces and consider a class $\mathfrak{E}$ of spaces such that $\mathfrak{A} = EH(\mathfrak{E})$. We claim that $\mathfrak{A} = EH(\mathfrak{E} - \{E\})$ for each $E \in \mathfrak{E}$. To this end consider an arbitrary $E \in \mathfrak{E}$. There exists a family $\{M_i \mid i \in I\}$ of Moore spaces such that E can be embedded into the product of this family. The disjoint union M of all spaces M_i is a Moore space such that $E \in EH(\{M\})$. Therefore it suffices to prove that $M \in EH(\mathfrak{E} - \{E\})$. From the proof of our theorem it follows that there exists

a Moore space X containing M as a subspace such that all continuous functions from X into E are constant. It follows that $X \in EH(\mathfrak{E} - \{E\})$ and, consequently, that $M \in EH(\mathfrak{E} - \{E\})$.

3. It is worth mentioning that such a familiar T_1-space like the reals with co-finite topology has the property that every Moore space, in fact every developable T_1-space, can be embedded into some of its powers. To verify this fact it suffices to recall that every developable T_1-space can be embedded into a product of second countable developable T_1-spaces [**3**, **8**], and that every T_1-space of cardinality $\leq c$ can be embedded into some power of the reals with co-finite topology (see [**2**], Problem 2.7.8(b)).

4. We like to indicate how the construction used to prove Lemma 2 can be modified to obtain a simple proof of the theorem of H. Herrlich and A. Ramer (see [**2**], Problems 2.7.17 and 2.7.18) that *for every T_1-space Y there exists a regular space S containing two distinct points $-\infty$ and $+\infty$ such that $f(-\infty) = f(+\infty)$ for every continuous function f from S into Y.*

In fact, if κ is an infinite cardinal such that $|Y| \leq \kappa$, let $\{Z_i \mid i \in \mathbf{Z}\}$ be a family of disjoint sets of cardinality κ^+ and consider the set $S = \{-\infty\} \cup X \times X \cup \{+\infty\}$, where $X = \cup \{Z_i \mid i \in \mathbf{Z}\}$. A regular topology on S is defined as follows. All points in $X \times X$ with distinct coordinates are isolated. For a point $x \in Z_i$ a basic neighborhood system of (x, x) in S consists of all sets

$$U_F(x, x) = \{(x, x)\} \cup \left(\left(\{x\} \times X \cup Z_{i+1} \times \{x\}\right) - F\right),$$

where F is an arbitrary finite subset of $X \times X$. Finally, basic neighborhoods of $-\infty$ and $+\infty$ of the form

$$U_n(-\infty) = \{-\infty\}$$
$$\cup \cup \left\{U_F(x, x) \mid x \in \cup \{Z_i \mid i \leq -n\}, F \subset X \times X \text{ finite}\right\}$$

and

$$U_n(+\infty) = \{+\infty\}$$
$$\cup \cup \left\{U_F(x, x) \mid x \in \cup \{Z_i \mid i \geq n\}, F \subset X \times X \text{ finite}\right\},$$

respectively. If f is a continuous function from S into Y and V is a neighbourhood of $f(-\infty)$, there exists an i_0 in $\mathbf{Z}$ such that f maps $\{(x, x) \mid x \in Z_{i_0}\}$ into V. Since $|V| \leq \kappa$ and $|Z_{i_0}| = \kappa^+$, there exists a point y_0 in V and a denumerable subset A_{i_0} of Z_{i_0} such that $\{(x, x) \mid x \in A_{i_0}\} \subset f^{-1}(y_0)$. Inductively one can now show that $f^{-1}(y_0)$ contains infinitely many points of $\{(x, x) \mid x \in A_i\}$ for every $i \geq i_0$. We omit the details, for the argument is similar to that used in [9]. It follows that $f(+\infty) = y_0 \in V$, consequently $f(-\infty) = f(+\infty)$.

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