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HYPERGROUP JOINS AND THEIR DUAL OBJECTS

RICHARD C. VREM

A hypergroup join is a hypergroup formed by the union of a discrete hypergroup with a compact hypergroup. The compact hypergroup is a subhypergroup of the join, but the convolution on the discrete hypergroup is changed in the join. A characterization of compact joins in terms of their dual objects is given which leads to a simpler criteria for their existence. In particular, it is shown that if a compact abelian join has a dual which is a hypergroup, then the dual is also a join. Examples of joins are provided from the study of conjugacy classes of certain semi-direct products of compact groups and a method is described for constructing non-dualizable compact abelian hypergroups.

Introduction. The study of harmonic analysis on topological hypergroups was initiated through the fundamental papers of Dunkl [**1**], Jewett [**4**] and Spector [**6**]. Most of the subsequent work on hypergroups has dealt with the problem of extending known results for topological groups to hypergroups. There has been considerable success in this endeavor. This paper, however, will be concerned with a construction within the category of hypergroups which is not possible within the category of groups. We study the join of two hypergroups, which was introduced by Jewett [**4**, 10.5].

In §1 we define the join of two hypergroups and present some elementary results concerning them. In particular, we show that the join of a compact hypergroup with a discrete hypergroup always possesses a Haar measure. We restrict our attention to compact joins in §2 and characterize them in terms of their dual objects. As a corollary, a much simpler characterization of compact joins is found. Our attention is further restricted in §3 to compact abelian joins, where we show that the dual object of the join can be viewed as the union of the dual objects of the hypergroups making up the join. In fact, if the dual of the join is also a hypergroup, then it too is a join. We conclude the paper with a number of examples in §4. We show how joins can arise naturally from studying the conjugacy class hypergroups of certain semi-direct products of compact groups. Various previously discussed pathologies of hypergroups are shown to occur in this class of examples. Also, a method is given for constructing examples of "non-dualizable" hypergroups. We conclude with a discussion of the family of countably compact hypergroups introduced by Dunkl and Ramirez [**2**].

The notation used will be that of Jewett [4] with these few exceptions: δ_x denotes the point mass at x and $x \to x^\vee$ is the involution on K. If K is compact we denote the collection of finite-dimensional continuous irreducible representations of K (actually $M(K)$) by $K^\wedge$. If $U \in K^\wedge$ then $d(U)$ denotes the dimension of the Hilbert space on which U acts. Also, we will denote the identity and zero operators corresponding to U by I_U and 0_U, respectively. The explanation of other notations can be found in [4] or [7].

1. Joins. In this section we will place no further restrictions on the hypergroups studied except those found in the definition of the join. Following Jewett [4, 10.5] we proceed to define the join of two hypergroups. Suppose H is a compact hypergroup and J is a discrete hypergroup with $H \cap J = \{e\}$, where e is the identity of both hypergroups. Let $K = H \cup J$ have the unique topology for which H and J are closed subspaces of K. Let σ be the normalized Haar measure on H and define the operation $\cdot$ on K as follows:

(i) If $s, t \in H$ then $\delta_s \cdot \delta_t = \delta_s * \delta_t$.

(ii) If $a, b \in J$ and $a \neq b^\vee$ then $\delta_a \cdot \delta_b = \delta_a * \delta_b$.

(iii) If $s \in H$ and $a \in J$ $(a \neq e)$ then $\delta_s \cdot \delta_a = \delta_a \cdot \delta_s = \delta_a$.

(iv) If $a \in J$ and $a \neq e$ and $\delta_{a^\vee} * \delta_a = \Sigma_{b \in J} c_b \delta_b$, then $\delta_{a^\vee} \cdot \delta_a = c_e \sigma + \Sigma_{b \in J - \{e\}} c_b \delta_b$.

We call K the *join* of H and J and write $K = H \vee J$.

It should be noted that if $K = H \vee J$ then $J \vee H$ cannot even be formed unless both hypergroups are finite. In fact, even if both J and H are finite, $H \vee J \neq J \vee H$ unless either H or J is trivial.

Joins can arise quite naturally from studying groups. For example, the hypergroup of conjugacy classes of A_4 has the structure of a join (see [4, 9.10] for details). More examples are provided in §4.

For the remainder of the paper we will adopt the following notation. If $K = H \vee J$ then we will use $*$ to denote the convolution on K (and hence on H), $\cdot$ to denote the convolution on J, and J^* to denote $J - \{e\}$.

Proposition 1.1. *Suppose $K = H \vee J$, where H has normalized Haar measure σ, and J has discrete left Haar measure τ. We define τ^* on K via $\tau^*(x) = 0$ if $x \in H$ and $\tau^*(x) = \tau(x)$ if $x \in J^*$. Then $m = \sigma + \tau^*$ is a left Haar measure on K.*

Proof. Clearly m is supported on K so we need only check that m is left-invariant. Let $f \in C_{00}^{+}(k)$, $x \in K$ and consider

$$(1) \qquad \int_K f_x\,dm = \int_H f(x * t)\,d\sigma(t) + \sum_{S \in J^*} f(x * s)\tau^*(s).$$

If $x \in H$ then the fact that $\delta_x * \delta_s = \delta_s$ for all $s \in H$ shows that (1) is equivalent to $\int_K f_x\,dm = \int_K f\,dm$ as desired. If $x \in J^*$ then (1) can be written

$$\begin{aligned}(2) \quad \int_K f_x\,dm &= f(x) + \sum_{s \in J^*} f(x * s)\tau^*(s) = \sum_{s \in J} f(x * s)\tau(s) \\ &= \sum_{s \in J - \{x^\vee\}} f(x \cdot s)\tau(s) + f(x * x^\vee)\tau(x^\vee) \\ &= \sum_{s \in J} f(x \cdot s)\tau(s) - f(x \cdot x^\vee)\tau(x^\vee) + f(x * x^\vee)\tau(x^\vee).\end{aligned}$$

However,

$$\delta_x \cdot \delta_{x^\vee} = \sum_{t \in J} c_t \delta_t \quad \text{and} \quad \delta_x * \delta_{x^\wedge} = c_e \sigma + \sum_{t \in J^*} c_t \delta_t,$$

so (2) can be written

$$\begin{aligned}\int_K f_x\,dm &= \sum_{s \in J} f(x \cdot s)\tau(s) + \tau(x^\vee)^{-1}\left[\int_H f(t)\,d\sigma(t) - f(e)\right]\tau(x^\vee) \\ &= \sum_{s \in J^*} f(s)\tau(s) + \int_H f(t)\,d\sigma(t) = \int_K f\,dm.\end{aligned}$$

PROPOSITION 1.2. *If $K = H \vee J$ and λ is a Haar measure on H then $\lambda * \delta_y = \delta_y * \lambda = \lambda$ for all $y \in H$ and $\lambda * \delta_x = \delta_x * \lambda = \delta_x$ for all $x \in J^*$.*

Proof. Clearly, if $y \in H$ then $\lambda * \delta_y = \delta_y * \lambda = \lambda$. Now, if $y \in J^*$ and $f \in C_\infty(K)$ then

$$\begin{aligned}\int_K f\,d\delta_y * \lambda &= \int_H \int_K f(t)\,d\delta_y * \delta_b(t)\,d\lambda(b) \\ &= \int_H \int_K f(t)\,d\delta_y(t)\,d\lambda(b) = f(y).\end{aligned}$$

A similar argument holds for $\lambda * \delta_y$.

It is easily shown that J is never a subhypergroup of K unless either H or J is trivial. The next proposition shows that we can view J as a quotient of K.

PROPOSITION 1.3. *If $K = H \vee J$ then $K//H$ is hypergroup isomorphic to J.*

Proof. We have $K//H = \{H * \{x\} * H \colon x \in K\}$. But $\delta_x * \delta_y = \delta_x$ for all $x \in J^*$ and $y \in H$, so we may write $K//H = \{H\} \cup \{\{x\} \colon x \in J^*\}$. If we adopt the notation that $\bar{x} = H * \{x\} * H$, then $K//H = \{\bar{x} \colon x \in J\}$. Thus, we clearly have a bijection $\beta\colon x \to \bar{x}$ between $K//H$ and J. It follows easily from Proposition 1.1 and [**4**, 7.1B] that $K//H$ is discrete. Therefore, it suffices to show that β respects the hypergroup convolution. Suppose we have $x, y \in J$ with $x \neq y^\vee$. Then it follows that

$$\delta_x \cdot \delta_y = \delta_x * \delta_y = \sum_{s \in J} c_s \delta_s$$

and, hence,

$$\sigma * (\delta_x * \delta_y) * \sigma = \sum_{s \in J} c_s \sigma * \delta_s * \sigma = \sum_{s \in J} c_s \delta_{\bar{s}} = \sum_{s \in J} c_s \delta_{\beta(s)}.$$

Furthermore,

$$\delta_x * \delta_{x^\vee} = c_e \sigma + \sum_{s \in J^*} c_s \delta_s,$$

so

$$\sigma * (\delta_x * \delta_{x^\vee}) * \sigma = c_e \sigma + \sum_{s \in J^*} c_s \delta_{\beta(s)} = \sum c_s \delta_{\beta(s)}.$$

2. Compact joins. All hypergroups under consideration in this section will be compact, so if $K = H \vee J$ then J is finite. We begin with two lemmas which will be used in the characterization of the dual object of K.

LEMMA 2.1. *Let K be an arbitrary compact hypergroup with J^* a finite subset of K with the following properties*:

(i) $K - J^* = H$ *a subhypergroup of* K.

(ii) $\operatorname{supp}(\delta_x * \delta_y) \subseteq J^*$ *for all* $x, y \in J^*$ *with* $x \neq y^\vee$.

(iii) $\delta_z * \delta_{z^\vee}|_H = k_z \sigma$ *for* $z \in J^*$, *where* $k_z > 0$ *and* σ *is normalized Haar measure on* H.

Then $J = J^* \cup \{e\}$ can be made into a discrete hypergroup as follows:

(a) *Involution* $z \to z^\vee$ *as on* K.

(b) *Define* $\cdot$ *on* J *via*

$$\delta_x \cdot \delta_y = \delta_x * \delta_y|_J \quad \text{if } x, y \in J,\ x \neq y^\vee,$$

$$\delta_{z^\vee} \cdot \delta_z = \mu_z(J)^{-1} \mu_z|_J$$

where

$$\mu_z = \delta_z * \delta_{z^\vee} - (\delta_z * \delta_{z^\vee}(\{e\}) - k_z)\delta_e.$$

Proof. We have J is finite and $\|\delta_x * \delta_y\|_{M(J)} = 1$, so by [**4**, 2.4B] it follows that $\cdot$ has a unique extension to a positive continuous bilinear mapping from $M(J) \times M(J)$ to $M(J)$. It is immediate from the definition of $\cdot$ and condition (ii) that $\delta_x * \delta_y$ is a probability measure with compact support on J for each $x, y \in J$. We need only check that $e \in \operatorname{supp}(\delta_x \cdot \delta_y)$ if and only if $x = y^\vee$. Clearly, if $e \in \operatorname{supp}(\delta_x * \delta_y)$ then $x = y^\vee$ because $\delta_x \cdot \delta_y = \delta_x * \delta_y$ whenever $x \neq y^\vee$. Also,

$$\delta_z \cdot \delta_{z^\vee}(\{e\}) = \mu_z(J)^{-1} k_z > 0,$$

which implies $e \in \operatorname{supp}(\delta_z * \delta_{z^\wedge})$. Thus $(J, \cdot)$ is a discrete hypergroup.

LEMMA 2.2. *If $K = H \vee J$ is a compact hypergroup with normalized measure m and H has normalized Haar measure σ with neither H nor J trivial, then there exists a proper subset P of $K^\wedge$ with $\{1\} \subsetneq P$ such that*

$$\sigma^\wedge(U) = \begin{cases} I_U & \text{for all } U \in P, \\ 0_U & \text{for all } U \in K^\wedge - P. \end{cases}$$

Proof. If $U \in K^\wedge$ then $\hat{\sigma}(U) = (\sigma * \sigma)^\wedge(U) = [\sigma(U)]^2$ and $\hat{\sigma}(U)^* = (\sigma^\vee)^\wedge(U) = \hat{\sigma}(U)$, so $\hat{\sigma}$ is a projection operator on H_U. Thus, there is an orthonormal basis $\{\zeta_1 \cdots \zeta_{d(U)}\}$ for H_U and an integer $l_U \in \{0, 1, \ldots, d(U)\}$ such that

$$\hat{\sigma}(U)(\zeta_j) = \begin{cases} \zeta_j & \text{if } j \leq l_U, \\ 0 & \text{if } j > l_U. \end{cases}$$

Using Proposition 1.2 it is immediate that $U_x U_\sigma = U_\sigma U_x$ for all $x \in K$, so by Schur's Lemma (see, for example, [**5**, 6.3]), $U_\sigma = kI_U$ for some constant k. This forces either $l_U = 0$ or $l_U = d(U)$. If $l_U = 0$ for all $U \in K^\wedge - \{1\}$ then $\sigma^\wedge = m^\wedge$, and if $l_U = d(U)$ for all $U \in K^\wedge$ then $\sigma^\wedge = \delta_e^\wedge$, in both cases a contradiction by the uniqueness of the Fourier-Stieltjes transform [**7**, 3.2].

The following theorem characterizes compact joins in terms of their duals.

THEOREM 2.3. *Suppose K is a compact hypergroup with nontrivial subsets H and J with the properties that $H \cup J = K$ and $H \cap J = \{e\}$. Then $K = H \vee J$ if and only if there exists $\{1\} \underset{\neq}{\subset} P \underset{\neq}{\subset} K^{\wedge}$ such that $U_{|H} = I_U$ for all $U \in P$ and $U_{|J^*} = 0_U$ for all $U \in K^{\wedge} - P$ (where $J^* = J - \{e\}$).*

Proof. Suppose P exists as described. We first note that if $U \in K^{\wedge}$ then $I(U) \equiv \{x \in K: U_x = I_U\}$ is a (closed) subhypergroup of K. This follows easily from the fact that $\delta_x * \delta_y$ is a probability measure and the observation that

$$I_U = \int_K U_t \, d\delta_x * \delta_y(t)$$

for all $x, y \in I(U)$. Furthermore, we claim

$$H = \bigcap_{U \in P} I(U). \tag{1}$$

Clearly, we have $H \subseteq \bigcap_{U \in P} I(U)$. If $H \neq \bigcap_{U \in P} I(U)$ then there exists $x \in J^*$ with $U_x = I_U$ for all $U \in P$. Therefore, we can write

$$U_x = \begin{cases} I_U & \text{if } U \in P, \\ 0_U & \text{if } U \in K^{\wedge} - P. \end{cases}$$

Thus $U_{\delta_x * \delta_x} = U_x U_x = U_x$ for all $U \in K^{\wedge}$ and, hence, $(\delta_x * \delta_x)^{\wedge} = \delta_x^{\wedge}$. The uniqueness of the Fourier-Stieltjes transform [7, 3.2] shows that $\delta_x * \delta_x = \delta_x$. Similarly, $\delta_{x^\vee} = \delta_x$. But this is a contradiction since

$$e \in \operatorname{supp}(\delta_x * \delta_{x^\vee}) = \operatorname{supp}(\delta_x * \delta_x) = \{x\} \subseteq K - H.$$

This establishes (1).

The fact that each $I(U)$ is a subhypergroup of K, together with (1), gives that H is a subhypergroup of K. Now, if $x \in H$, $y \in J^*$ then, for each $U \in K^{\wedge}$,

$$U_{\delta_x * \delta_y} = U_x U_y = U_y, \qquad U_{\delta_y * \delta_x} = U_y,$$

for if $U \in P$ then $U_x = I_U$, and if $U \in K^{\wedge} - P$ then $U_y = 0_U$. Again, the uniqueness of the transform shows that $\delta_x * \delta_y = \delta_y * \delta_x = \delta_y$ for all $x \in H, y \in J^*$.

Now suppose $y, z \in J$ and consider $t \in \operatorname{supp}(\delta_y * \delta_z)$. Thus, for $t \in H$ we have $z \in \operatorname{supp}(\delta_{y^\vee} * \delta_t) = \operatorname{supp}(\delta_{y^\vee})$ and hence $z = y^{\vee}$. Therefore, if $y \neq z^{\vee}$ then $\operatorname{supp}(\delta_y * \delta_z) \subseteq J^*$. If $z \in J$ then $z^{\vee} \in J$, since H is a hypergroup and $e \in \operatorname{supp}(\delta_z * \delta_{z^\wedge})$. If $t \in H$ then (using the notation of [4, 3.2])

$$\{t\} = \{t\} * \{e\} \subseteq \{t\} * \{z\} * \{z^\vee\} = \{z\} * \{z^\vee\}.$$

Thus $H \subseteq \operatorname{supp}(\delta_z * \delta_{z^\vee})$ for all $z \in J$. In fact,

$$\delta_t * (\delta_z * \delta_{z^\vee}) = \delta_z * \delta_{z^\vee},$$

so $\delta_z * \delta_{z^\wedge}{}_{|H} = k\sigma$, where σ is normalized Haar measure on H.

We have $P \subsetneq K^\wedge$, so choose $U \in K^\wedge - P$. Clearly,

$$J^* \subseteq \{x \in K \colon U_x = 0_U\}$$

and, hence, $H = K - J^*$ has nonempty interior, which implies H is open by [4, 10.2A]. For each $x \in J^*$ we have $\{x\} * H = \{x\}$ is open, which implies J^* is discrete. Thus both J^* and J are finite.

Finally, we define $\cdot$ on J as in Lemma 2.1 and use Lemma 2.1 to conclude $(J, \cdot)$ is a discrete hypergroup and $K = H \vee J$.

Conversely, suppose $K = H \vee J$ with σ normalized Haar measure on H. Then by Lemma 2.2 there exists $\{1\} \subsetneq P \subsetneq K^\wedge$ such that

$$\hat{\sigma} = \begin{cases} I_U & \text{on } P, \\ 0_U & \text{on } K^\wedge - P. \end{cases}$$

If $x \in H$ then clearly $\delta_x * \sigma = \sigma$, in which case $\hat{\delta}_x \hat{\sigma} = \hat{\sigma}$. Therefore, $\hat{\delta}_x(U) = I_U = U_{x^\vee}$ for all $x \in H$. Proposition 1.2 shows that $\delta_y * \sigma = \delta_y$ for all $y \in J^*$ and, hence, $\hat{\delta}_y(U) = 0_U = U_{y^\vee}$ for all $U \in K^\wedge - P$. This completes the proof.

The preceding theorem allows us to provide an easier characterization of compact joins.

COROLLARY 2.4. *Suppose K is a compact hypergroup with nontrivial subsets H and J with the property that $H \cup J = K$ and $H \cap J = \{e\}$. Furthermore, we assume each $U \in K^\wedge$ has the property that either $U_a = I_U$ for all $a \in H$ or $(I_U - U_a)$ is invertible for some $a \in H$. Then $K = H \vee J$ if and only if $\delta_a * \delta_s = \delta_s$ for all $a \in H$ and $s \in J^*$.*

Proof. Necessity is obvious. We assume $\delta_a * \delta_s = \delta_s$ for all $a \in H$ and $s \in J^*$. Let $P = \{U \in K^\wedge : U_a = I_U \text{ for all } a \in H\}$. If $U \in K^\wedge - P$, then by our assumption there exists $a \in H$ such that $(I_U - U_a)$ is invertible. If $s \in J^*$ then $U_s = U_a U_s$ or $(I_U - U_a) = 0_U$, which implies $U_s = 0_U$. This completes the proof.

It should be remarked here that the additional hypothesis added to the statement of the corollary trivially holds for any compact abelian hypergroup.

3. Compact abelian joins. In this section the hypergroups under discussion will be compact and abelian. We begin with a lemma.

LEMMA 3.1. *Suppose $K = H \vee J$ is a compact abelian hypergroup. Then each character $\chi \in H^\wedge$ extends to a character $\chi^* \in K^\wedge$ via*

$$\chi^*(t) = \begin{cases} \chi(t) & \text{if } t \in H, \\ 0 & \text{if } t \in J^*. \end{cases} \tag{1}$$

Also, for each character $\psi \in (J, \cdot)^\wedge$ there corresponds a character $\psi^* \in K^\wedge$ given by

$$\psi^*(t) = \begin{cases} \psi(t) & \text{if } t \in J, \\ 1 & \text{if } t \in H. \end{cases} \tag{2}$$

Proof. Suppose $\chi \in H^\wedge$ and define χ^* on K as in (1). Now χ^* is clearly continuous and hermitian. We need to show χ^* is multiplicative. It is obviously multiplicative on H. If $s \in H$ and $t \in J$ then $\chi^*(s * t) = \chi^*(t) = 0 = \chi^*(t)\chi^*(s)$. For $s,\ t \in J^*$, $s \neq t^\vee$, we have $\operatorname{supp}(\delta_s * \delta_t) \subseteq J^*$, so again χ^* is multiplicative. If $t \in J^*$ with $\delta_t \cdot \delta_{t^\vee} = \Sigma_{z \in J} c_z \delta_z$ then

$$\begin{aligned} \chi^*(t * t^\vee) &= \int_K \chi^*(s)\, d\delta_t * \delta_{t^\vee}(s) = c_e \int_H \chi(s)\, d\sigma(s) + \sum_{z \in J^*} c_z \chi^*(s) \\ &= 0 = \chi^*(t)\chi^*(s) \end{aligned}$$

by the orthogonality conditions on H (see [7, 2.6]). Thus $\chi^* \in K^\wedge$.

Next, we assume $\psi \in (J, \cdot)^\vee$ and define ψ^* as in (2). It is evident that ψ^* is continuous and hermitian so we again need to check that it is multiplicative. We verify only the case $\psi^*(t * t^\vee)$ where $t \in J^*$. Suppose $\delta_t * \delta_{t^\vee} = \Sigma_{z \in J} c_z \delta_z$ and consider

$$\begin{aligned} \psi^*(t * t^\vee) &= \int_K \psi^*(s)\, d\delta_t * \delta_{t^\vee}(s) = c_e \int_H \psi^*(s)\, d\sigma(s) + \sum_{z \in J^*} c_z \psi(z) \\ &= c_e + \sum_{z \in J^*} c_z \psi(z) = \sum_{z \in J} c_z \psi(z) \\ &= \psi(t)\psi(t^\vee) = \psi^*(t)\psi^*(t^\vee). \end{aligned}$$

We conclude that $\psi^* \in K^\wedge$.

We now introduce the following notation:

$$(\hat{H})^* = \{\chi^* : \chi \in H\}, \qquad (\hat{J})^* = \{\psi^* : \psi \in (J, \cdot)^{\wedge}\}.$$

THEOREM 3.2. *If $K = H \vee J$ is a compact abelian hypergroup then $\hat{K} = (\hat{H})^* \cup (\hat{J})^*$.*

Proof. Lemma 3.1 shows that $(\hat{H})^* \cup (\hat{J})^* \subseteq \hat{K}$. Suppose ψ is a character on K. In the notation of Theorem 2.3 we have either $\psi \in P$ or $\psi \in \hat{K} - P$. If ψ is in P then $\psi_{|H} \equiv 1$. Thus we need to show $\psi_{|J} \equiv \tilde{\psi}$ is a character on $(J, \cdot)$. It is clearly multiplicative for $s, t \in J^*$ with $s \neq t^{\vee}$. If $t \in J^*$ with $\delta_t \cdot \delta_{t^\vee} = \Sigma_{z \in J} c_z \delta_z$, then

$$\begin{aligned}\tilde{\psi}(t \cdot t^{\vee}) &= \sum_{z \in J} c_z \tilde{\psi}(z) = c_e \psi(e) + \sum_{z \in J^*} c_z \psi(z)\\ &= c_e \int_H \psi(t)\, d\sigma(t) + \sum_{z \in J^*} c_z \psi(z)\\ &= \psi(z * \check{z}) = \psi(z)\psi(\check{z}) = \tilde{\psi}(z)\tilde{\psi}(\check{z}).\end{aligned}$$

Hence $\psi = (\tilde{\psi})^*$.

If $\psi \in \hat{K} = P$ then $\psi_{|J^*} = 0$. This time we set $\psi_{|H} \equiv \tilde{\psi}$. It is straightforward that $\tilde{\psi}$ is hermitian and multiplicative. Furthermore, if S is open in $\mathbf{C}$ then $(\tilde{\psi})^{-1}(S) = \psi^{-1}(S) \cap H$, which is open in H, so $(\tilde{\psi})^* = \psi$.

The next theorem shows that $\hat{K}$ can be written as a join if both $\hat{H}$ and $\hat{J}$ are hypergroups.

THEOREM 3.3. *Suppose $K = H \vee J$ is a compact abelian hypergroup with $\hat{H}$ and $\hat{J}$ hypergroups. Then $\hat{K} \approx \hat{J} \vee \hat{H}$.*

Proof. The isomorphism here is the obvious one, namely $\psi \to \psi^*$. Clearly, $\hat{J}$ is finite (compact) and $\hat{H}$ is discrete. The theorem will follow once we establish the following results:

(i) If $\chi, \psi \in \hat{J}$ with $\chi\psi = \Sigma_{\tau \in \hat{J}} a_\tau \tau$, then $\chi^* \psi^* = \Sigma_{\tau \in \hat{J}} a_\tau \tau^*$.

(ii) If $\chi, \psi \in \hat{H}$ with $\chi \neq \bar{\psi}$ and $\chi\psi = \Sigma_{\zeta \in \hat{H} - \{1\}} c_\zeta \zeta$, then

$$\chi^* \psi^* = \sum_{\zeta \in \hat{H} - \{1\}} c_\zeta \zeta^*.$$

(iii) If $\chi \in \hat{J}$, $\psi \in \hat{H} - \{1\}$, then $\chi^* \psi^* = \psi^*$.

(iv) If $\psi \in \hat{H} - \{1\}$ with $\bar{\psi}\psi = \Sigma_{\zeta \in \hat{H}} c_\zeta \zeta$, then $\delta_{\bar{\psi}^*} * \delta_{\psi^*} = c_1 \eta + \Sigma_{\zeta \in \hat{H} - \{1\}} c_\zeta \zeta^*$, where η is normalized Harr measure on $\hat{J}$.

Note that η on $\hat{J}$ is given by

$$\eta = \alpha^{-1}\left[\sum_{\psi\in\hat{J}} \left(\delta_{\bar{\psi}} * \delta_{\psi}(1)\right)^{-1}\delta_{\psi}\right],$$

where $\alpha = \Sigma_{\psi\in\hat{J}}(\delta_{\bar{\psi}} * \delta_{\psi}(1))^{-1}$.

Parts (i) and (ii) are immediate from the definition of the bijection *. To establish (iii) we consider $\chi \in \hat{J}$, $\psi \in \hat{H} - \{1\}$. If $y \in J^*$ then $\chi^*\psi^*(y) = 0 = \psi^*(y)$, and if $y \in H$ then $\chi^*\psi^*(y) = 1\psi^*(x) = \psi^*(x)$. Thus $\chi^*\psi^* = \psi^*$. In order to establish (iv) it suffices to show for each $\psi \in \hat{H} - \{1\}$:

$$\bar{\psi}^*\psi^* = c_1\alpha^{-1}\sum_{\chi\in\hat{J}}\left(\delta_{\bar{\chi}} * \delta_{\chi}(1)\right)^{-1}\chi^* + \sum_{\zeta\in\hat{H}-\{1\}} c_\zeta\zeta^*. \tag{3}$$

If $y \in J^*$ then $\bar{\psi}^*\psi^*(y) = 0$. To show the right-hand side of (3) takes on the same value, we first note that $\zeta^*(y) = 0$ for each $\zeta \in \hat{H} - \{1\}$. Also,

$$\sum_{\chi\in\hat{J}}\left(\delta_{\bar{\chi}} * \delta_{\chi}(1)\right)^{-1}\chi^*(y) = \sum_{\chi\in\hat{J}}\left(\delta_{\bar{\chi}} * \delta_{\chi}(1)\right)^{-1}\chi(y) = 0$$

by an application of the orthogonality relations on $\hat{J}$. Here we used the fact that the map $\tilde{y}$: $\hat{J} \to \mathbf{C}$ via $\tilde{y}(\chi) = \chi(y)$ is a character on $\hat{J}$ [**4**, 12.4B]. Finally, if $y \in H$ then the right-hand side of (3) can be written

$$\begin{aligned} c_1\alpha^{-1}\sum_{\chi\in\hat{J}}\left(\delta_{\bar{\chi}} * \delta_{\chi}(1)\right)^{-1} &+ \sum_{\zeta\in\hat{H}-\{1\}} c_\zeta\zeta(y) \\ &= c_1 + \sum_{\zeta\in\hat{H}-\{1\}} c_\zeta\zeta(y) = \sum_{\zeta\in\hat{H}} c_\zeta\zeta(y) = \bar{\psi}(y)\psi(y) \\ &= \bar{\psi}^*(y)\psi^*(y), \end{aligned}$$

which establishes the theorem.

COROLLARY 3.4. *If $K \vee J$ is a compact abelian hypergroup then $\hat{K}$ is a hypergroup if and only if $\hat{H}$ and $\hat{J}$ are hypergroups.*

Proof. Sufficiency is contained in Theorem 3.3. To show necessity we consider ψ, $\zeta \in \hat{J}$. Thus $\psi^*\zeta^*$ is positive definite which implies $\psi\zeta$ is positive definite. An application of [**4**, 12.4] shows that $\hat{J}$ is a hypergroup. A similar argument holds for $\hat{H}$.

4. Examples. We begin with an example of how joins can arise quite naturally from topological groups.

EXAMPLE 4.1. In this example we use the notation of [**3**, 2.6]. Suppose $G = W \circledS S$ (semi-direct product) where W is a compact group and S is a finite group. Then there is a homomorphism $s \to \tau_s$ which carries S onto a group of automorphisms of W and multiplication is given by

$$(w, s)(w', s') = (w(\tau_s(w')), ss').$$

Furthermore, we may identify W with a normal subgroup of G and S with a subgroup of G such that $G/W \approx S = \{e, s_1, \ldots, s_n\}$. Indeed, the elements of G/W are of the form $\{(w, s): w \in W\} \equiv Ws$. We shall further assume that W acts transitively (under conjugation) within each of the cosets Ws $(s \neq e)$. That is, given (w, s), (w', s) in Ws there exists an $x \in W$ such that

$$(x, e)^{-1}(w, s)(x, e) = (w', s).$$

Clearly, G cannot act transitively between cosets. Under these hypothesis, the hypergroup of conjugacy classes of G, written G_I, consists of the conjugacy classes of W, written W_I, along with the cosets $Ws_1, \ldots, Ws_n$. We will show that G_I has the structure of a compact abelian join. Let $H = W_I \equiv \{[w]: w \in W\}$ and $J = \{e, Ws_1, \ldots, Ws_n\}$. By Corollary 2.4 and the remark following it we need only show that $\delta_{[w]} * \delta_{Ws} = \delta_{Ws}$ for all $[w] \in H$ and $W_s \in J$. But

$$\delta_{[w]} * \delta_{Ws} = \int_G \delta_{[t^{-1}wts]}\, dm(t) = \delta_{Ws}$$

by the normality of W.

EXAMPLE 4.2. Suppose W is any compact group and form $G = W \circledS \mathbf{Z}_2$, where $\mathbf{Z}_2 = \{e, x\}$ and the action is given by $\tau_e(w) = w$ and $\tau_x(w) = w^{-1}$. In this case G/W consists of two cosets W and Wx. In order for W to act transitively on Wx we must be able to find a $t \in W$ for each $a \in W$ such that

$$(t^{-1}, e)(a, x)(t, e) = (e, x).$$

Applying the multiplication on G this is equivalent to the equation $(t^{-1}at, x) = (e, x)$ and, hence, $a = t^2$. Thus we need $W = W^{(2)} = \{w^2: w \in W\}$. Therefore, if W is a compact 2-divisible group then G_I is a compact abelian join. Note that G_I has at least one isolated point, namely x. For a finite example of this type we need only look at those dihedral groups D_{2n} where n is odd. It should be noted here that $(D_{2n})_I$ is not a join when n is even.

We next indicate how joins can be used to construct "pathological" examples.

EXAMPLE 4.3. Suppose $G = W \circledS \mathbf{Z}_2$ where W is compact and 2-divisible. If m is normalized Haar measure on G_I then $\hat{m} \in c_0(\hat{K})$, but m is not a continuous measure (see [**8**] for other examples of this type). Also, a slight modification of [**8**, 2.6] will show that G_I also provides an example of a compact abelian hypergroup where the space of continuous measures on K does not form an ideal in $M(K)$.

One of the most disappointing features of compact abelian hypergroups is that their dual objects are not always hypergroups. The following example provides a technique for constructing many such pathologies.

EXAMPLE 4.4. Let H be any compact abelian hypergroup and J any finite abelian hypergroup whose dual, $\hat{J}$, is not a hypergroup (see [**4**, 9.1C] or [**1**, 4.6] for 3 element examples whose duals are not hypergroups). It follows from Corollary 3.4 that $K = H \vee J$ has the property that $\hat{K}$ is not a hypergroup.

The last example deals with the family of countable compact hypergroups introduced by Dunkl and Ramirez [**2**].

EXAMPLE 4.5. Let $H_a = \{0, 1, 2, \ldots, \infty\}$, where $0 < a \le \frac{1}{2}$. Dunkl and Ramirez showed in [**2**] that H_a can be given the structure of a countable compact abelian hypergroup. Indeed, convolution is defined by

$$\delta_n * \delta_m = \delta_m * \delta_n = \delta_m \quad \text{if } m < n,$$

$$\delta_n * \delta_n(\{t\}) = \begin{cases} 0 & \text{if } t < n, \\ \dfrac{1-2a}{1-a} & \text{if } t = n, \\ a^k & \text{if } t = n + k > n. \end{cases}$$

If we let $H = \{1, 2, \ldots, \infty\}$ and $J_0 = \{0, \infty\}$, then Corollary 2.4 shows that $H_a = H \vee J_0$. In this case, the convolution $\cdot$ on J_0 is given by

$$\delta_0 \cdot \delta_0 = \frac{1}{1-a}\delta_\infty + \frac{1-2a}{1-a}\delta_0.$$

Clearly, H is a hypergroup isomorphic to H_a and, hence, can also be written as a join. Indeed, if we let $H_k = \{k, \infty\}$ with

$$\delta_k * \delta_k = \frac{1}{1-a}\delta_\infty + \frac{1-2a}{1-a}\delta_k$$

and inductively form the discrete hypergroups $J_n = J_{n-1} \vee H_n$ whose convolution is given by

$$\delta_r * \delta_s = \delta_s * \delta_r = \delta_r \qquad \text{if } r < s,$$

$$\delta_r * \delta_r(\{t\}) = \begin{cases} 0 & \text{if } t < r, \\ \dfrac{1-2a}{1-a} & \text{if } t = r, \\ a^k & \text{if } t = r + k \leq n, \\ \dfrac{a^{n-r+1}}{1-a} & \text{if } t = \infty, \end{cases}$$

then H_a can be viewed as the projective limit of $\{J_n\}$. I.e., H_a is the subhypergroup of the complete direct product hypergroup $\prod_{n=0}^{\infty} J_n$ (with the product topology) consisting of sequences $\{x_0, x_1, \ldots\}$ such that $\Pi_n x_n = x_{n-1}$, where Π_n is the projection of H_n onto H_{n-1}.

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HUMBOLDT STATE UNIVERSITY
ARCATA, CA 95521