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POISSON PROCESS OVER σ-FINITE MARKOV CHAINS

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There is a well-known construction which associates with each σ-finite measure space $(X, \mathbb{S}, \mu)$ a certain stochastic process $\{N(F)$: $F \in \mathbb{S}, \mu(F) < \infty\}$ called the Poisson process over $(X, \mathbb{S}, \mu)$. Any μ-preserving bimeasurable map τ on X "lifts" to a probability preserving map T, characterized by $N(F) \circ T = N(\tau^{-1}F)$. We show the following: If τ is the shift arising from a Markov chain preserving a σ-finite measure with stochastic matrix $(p_{i,j})_{i,j\in\mathbf{N}}$. Then T is a Bernoulli shift iff $p_{i,j}^{(n)} \to 0$ $\forall i, j \in \mathbf{N}$ as $n \to \infty$. If, in addition, τ has a recurrent state or if it is transient and $(\mathbb{S}, \mu)$ is not completely atomic, then T has infinite entropy. The analogous results are valid for ν-step Markov chains preserving a σ-finite measure ($\nu > 1$).

Introduction. We will examine the ergodic properties of dynamical systems arising by the use of the Poisson process as described in the following result (see [**8**]).

THEOREM 0. *Let $(X, \mathbb{S}, \mu)$ be a σ-finite (infinite) measure space. There exists a unique probability space $(\Omega, \mathcal{Q}, p)$ together with a countably additive set function N defined on sets $F \in \mathbb{S}$ with $\mu(F) < \infty$, satisfying:*

(i) *$N(F)$ is a Poisson random variable with mean $\mu(F)$.*

(ii) *If (F_i) is a sequence of pairwise disjoint sets* (mod μ) *then the sequence $(N(F_i))$ is independent.*

(iii) *$\mathcal{Q}$ is generated by the class $\{N(F)$: $F \in \mathbb{S}, \mu(F) < \infty\}$.*

Throughout this paper τ will denote an invertible measure-preserving transformation, i.e. an automorphism, acting on the Lebesgue space $(X, \mathbb{S}, \mu)$, and it will also be assumed that there is no finite τ-invariant measure equivalent to μ. τ gives rise to an automorphism T on $(\Omega, \mathcal{Q}, p)$ satisfying $N(F) \circ T = N(\tau^{-1}F)$. We call $((\Omega, \mathcal{Q}, p), T)$ the Poisson dynamical system with base $((X, \mathbb{S}, \mu), \tau)$.

The following result is shown by F. A. Marchat [7].

THEOREM 1. (a) *τ has no invariant sets of positive finite measure iff T is ergodic iff T is weak mixing.*

(b) *τ satisfies the mixing condition: $\mu(F \cap \tau^{-n}G) \to 0$ as $n \to \infty$ whenever $F, G \in \mathbb{S}$ have finite measure iff T is m-fold mixing $\forall m \geq 1$ iff T is mixing.*

We provide a (different) proof of this theorem in §1. In §2 some technical results are recalled ([7]) and brief proofs are provided. §§3 and 4 contain the main results of this paper. It is shown that if τ acts on $(X, \mathcal{S}, \mu)$ as a Markov chain with transition matrix $(p_{i,j})_{i,j \in \mathbf{N}}$, then the corresponding Poisson dynamical system is isomorphic to a Bernoulli shift iff $p_{i,j}^{(n)} \to 0\ \forall i, j \in \mathbf{N}$ as $n \to \infty$. If in addition to the last condition, either τ has a recurrent state or it is transient and $(\mathcal{S}, \mu)$ is not completely atomic, then the corresponding Poisson process has infinite entropy. The analogous results remain valid for ν-step Markov chains preserving a σ-finite measure ($\nu > 1$).

Poisson processes over Markov chains have been considered by several authors. S. Goldstein and V. L. Lebowitz [2] examined the case in which τ is the $(\frac{1}{2}, \frac{1}{2})$ random walk; they showed that the corresponding Poisson transformation is a K-automorphism. F. A. Marchat [7] obtained the same result for any Markov chain preserving a σ-finite measure. S. Kalikow [6] showed, for the case where τ is a recurrent random walk, that the process $\{N_{\{x(n)=a\}}: n \in \mathbf{Z}, a \in \mathbf{Z}\}$ forms a stationary Markov chain whose shift is Bernoulli. This process is a factor of the Poisson process over τ, so his result is a corollary of ours; we don't know, however, whether the factor is proper.

Kalikow's work was earlier than ours, but we learned of each others' results later, and the arguments are different.

This work is part of the author's Ph.D. thesis [3] done under the supervision of Professor Jacob Feldman, to whom I express my appreciation for his patience and much encouragement.

1. Ergodicity and mixing. We provide a different proof of Theorem 1, based on the computation of a dependence coefficient for certain σ-algebras contained in $\mathcal{C}$. We need some definitions and notation. Let $\mathcal{F} = \{F \in \mathcal{S}: \mu(F) < \infty\}$ denote the ring of sets of finite measure and let $\Sigma_F = \sigma\{N(H): H \in \mathcal{S} \cap F\}, F \in \mathcal{F}$.

Define

$$\rho(\Sigma_F, \Sigma_G) = \sup\{|p(M \cap M') - p(M)p(M')|: M \in \Sigma_F, M' \in \Sigma_G\}.$$

Clearly $\rho(\Sigma_F, \Sigma_G) = 0$ iff $\mu(F \cap G) = 0$.

LEMMA 1.1.

$$\lim_{\mu(H) \to 0} \frac{\rho(\Sigma_H, \Sigma_H)}{\mu(H)} = 1.$$

Proof. Since $0 \leq N(G) \leq N(H)$ whenever $G \subset H$, then $\{N(G) = n\}$ contains or is disjoint from $\{N(H) = 0\}$ according to whether $n = 0$ or $n \neq 0$, hence any set $M \in \Sigma_H$ which is not disjoint from $\{N(H) = 0\}$ must contain it, i.e. $\{N(H) = 0\}$ is an atom of Σ_H. Therefore $p(M) \geq p(\{N(H) = 0\})$ or $p(M) \leq 1 - p(\{N(H) = 0\})$ and hence

$$\rho(\Sigma_H, \Sigma_H) \leq 1 - p(\{N(H) = 0\}) = 1 - \exp(-\mu(H)).$$

On the other hand setting $M = M' = \{N(H) = 0\}$ we obtain

$$\exp(-\mu(H))(1 - \exp(-\mu(H))) \leq \rho(\Sigma_H, \Sigma_H).$$

Dividing by $\mu(H)$ and taking limits the result follows. □

LEMMA 1.2. *Let F_1, F_2 and G be such that $\mu(F_1 \cap F_2) = 0$. Then*

$$\rho(\Sigma_{F_1}, \Sigma_{G \cup F_2}) = \rho(\Sigma_{F_1}, \Sigma_G) = \rho(\Sigma_{F_1}, \Sigma_{G - F_2}).$$

Proof. By the definition of ρ, it is enough to show that

$$\rho(\Sigma_{F_1}, \Sigma_{G \cup F_2}) \leq \rho(\Sigma_{F_1}, \Sigma_{G - F_2}). \tag{1}$$

Let

$$\mathcal{C} = \left\{ \bigcup_j C_j \cap D_j \colon C_j \in \Sigma_{F_1}, D_j \in \Sigma_{G - F_2}, C_i \cap C_j = \varnothing, i \neq j, J \text{ finite} \right\}.$$

Then $\mathcal{C}$ is an algebra of subsets and is such that $\Sigma_{G \cup F_2} = \sigma(\mathcal{C})$. Let $M \in \Sigma_{F_1}$ and $M' \in \mathcal{C}$ be arbitrary. Then by independence of C_j and $D_j \cap M$ one gets

$$|p(M \cap M') - p(M)p(M')| \leq \sum_J p(C_j)\rho(\Sigma_{F_1}, \Sigma_{G - F_2}) \leq \rho(\Sigma_{F_1}, \Sigma_{G - F_2})$$

and (1) follows by approximation. □

COROLLARY 1.3.

$$\rho(\Sigma_F, \Sigma_{\tau^{-n}G}) \to 0 \quad \textit{as } n \to \infty \quad \textit{iff} \quad \mu(F \cap \tau^{-n}G) \to 0 \quad \textit{as } n \to \infty.$$

Proof. It follows from Lemma 1.2 that

$$\rho(\Sigma_F, \Sigma_{\tau^{-n}G}) = \rho(\Sigma_{F \cap \tau^{-n}G}, \Sigma_{F \cap \tau^{-n}G}).$$

Then apply Lemma 1.1. □

We are now ready to establish the following

THEOREM 1.4. (1) *τ has no invariant sets of positive finite measure iff T is ergodic iff T is weakly mixing.*

(2) *τ satisfies the mixing condition iff T is m-fold mixing for all $m \geq 1$ iff T is mixing.*

Proof. (1) If τ has no invariant sets of positive finite measure, it follows from the mean ergodic theorem that

$$\frac{1}{n}\sum_{j=0}^{n-1} \mu(F \cap \tau^{-j}F) \to 0 \quad \text{as } n \to \infty \text{ for all } F \in \mathfrak{F}.$$

Let $A, B \in \Sigma_F$ be arbitrary. Then by Corollary 1.3

$$\frac{1}{n}\sum_{j=0}^{n-1} |p(A \cap \tau^{-j}B) - p(A)p(B)| \leq \frac{1}{n}\sum_{j=0}^{n-1} \rho(\Sigma_F, \Sigma_{\tau^{-j}F}) \to 0 \quad \text{as } n \to \infty.$$

For general $A, B \in \mathcal{C}$, the same result holds by approximating by sets in Σ_F for large $F \in \mathfrak{F}$. Hence T is weakly mixing and, in particular, ergodic. Conversely if $F \in \mathfrak{F}$ is τ-invariant and has positive measure, then $N(F)$ is T-invariant and non-constant so T is not ergodic.

(2) Assume τ satisfies the mixing condition. Fix $m \geq 1$ and $F \in \mathfrak{F}$. Let $0 = n_0 \leq n_1 < \cdots \leq n_m$ be non-negative integers and put $F_j = \bigcup_{i=j}^m \tau^{-n_i}F$, $j = 1,\ldots,m$. Then $\mu(F \cap F_j) \to 0$ as $\min\{n_s - n_{s-1}\colon s = 1,\ldots,m\} \to \infty$. Let $A_0, A_1,\ldots,A_m \in \Sigma_F$ and $C_j = \bigcap_{i=j}^m T^{-n_i}A_i \in \Sigma_{F_j}$. By adding and subtracting $p(A_0)p(C_1)$ and using the triangle inequality one gets

$$\left|p\left(\bigcap_{i=0}^m T^{-n_i}A_i\right) - \prod_{i=0}^m p(A_i)\right| \leq \rho(\Sigma_F, \Sigma_{F_1}) + \left|p(C_1) - \prod_{i=1}^m p(A_i)\right|.$$

Repeating the same argument with $T^{-n_1}A_1$ in the role of A_0, and C_2 in the role of C_1, one gets

$$\left|p(T^{-n_1}A_1 \cap C_2) - \prod_{i=1}^m p(A_i)\right| \leq \rho(\Sigma_F, \Sigma_{F_2}) + \left|p(C_2) - \prod_{i=2}^m p(A_i)\right|.$$

Continuing in this fashion it follows that

$$\left| p\left(\bigcap_{i=0}^{m} T^{-n_i} A_i \right) - \prod_{i=0}^{m} p(A_i) \right| \le \sum_{j=1}^{m} \rho(\Sigma_F, \Sigma_{F_j}).$$

Therefore

$$p\left(\bigcap_{i=0}^{m} T^{-n_i} A_i \right) \to \prod_{i=0}^{m} p(A_i) \quad \text{as } \min\{n_s - n_{s-1}: s = 1,\dots,m\} \to \infty.$$

For general $A_0, A_1, \dots, A_m \in \mathcal{A}$ the same result follows by approximating by sets in Σ_F for large $F \in \mathfrak{F}$. Hence T is m-fold mixing $\forall m \ge 1$ and, in particular, it is mixing. Conversely, if T is mixing then, as $n \to \infty$, we have

$$(N(G) \circ T^n, N(F)) \to (N(G), 1)(1, N(F)) = \mu(F)\mu(G) \quad \forall F, G \in \mathfrak{F},$$

but one easily computes that

$$(N(G) \circ T^n, N(F)) = \mu(F)\mu(G) + \mu(F \cap \tau^{-n} G).$$

Therefore τ satisfies the mixing condition. □

REMARK 1. There are τ's that satisfy the mixing condition but the associated T's are not K-automorphisms. In [3] an ergodic τ satisfying the mixing condition was constructed so that T has entropy zero.

2. Conditional expectations. In this section a formula for conditional expectations over certain σ-algebras of $\mathcal{A}$ is recalled. The results and proofs are essentially those of [7] and are included here for the sake of completeness.

For each σ-algebra $\mathcal{G} \subset \mathcal{S}$ such that $\mu | \mathcal{G}$ is σ-finite, let $\mathfrak{B}(\mathcal{G}) = \sigma\{N(F): F \in \mathfrak{F} \cap \mathcal{G}\} \subset \mathcal{A}$, and let $\mathcal{L}(\mathcal{G})$ be the linear space of simple functions $f = \sum_{i \in I} c_i X_{F_i}$, I finite, with finite $\mathcal{G}$-measurable support. Using linearity and setting $N(F) = N(X_F)$, we define $N(f)$ for $f \in \mathcal{L}(\mathcal{S})$; notice that $N(f)$ is also a Poisson random variable with mean $\int f \, d\mu$. Define

$$\phi(f) = \frac{\exp N(f)}{E \exp N(f)} \quad \text{for } f \in \mathcal{L}(\mathcal{S});$$

one easily verifies that $E \exp N(f) = \exp E\psi(f)$ where $\psi(x) = e^x - 1$. On the other hand, since $\psi(x + y) = \psi(x)\psi(y) + \psi(x) + \psi(y)$, it follows that

$$(\phi(f), \phi(g)) = E\left(\frac{\exp N(f+g)}{\exp E(\psi(f) + \psi(g))} \right) = \frac{\exp E\psi(f+g)}{\exp E(\psi(f) + \psi(g))}$$
$$= \exp(\psi(f), \psi(g)) \quad \forall f, g \in \mathcal{L}(\mathcal{S}).$$

Hence $\phi(f) \in L_2(\Omega, \mathcal{A}, p)$. Notice that we have used inner products in two different L_2-spaces.

PROPOSITION 2.1. *The class* $\{\phi(f)\colon f \in \mathcal{L}(\mathcal{G})\}$ *generates* $L_2(\Omega, \mathcal{B}(\mathcal{G}), p)$ *for every sub-σ-algebra* $\mathcal{G}$ *of* $\mathcal{S}$.

Proof. Assume $\phi \in L_2(\Omega, \mathcal{B}(\mathcal{G}), p)$ is such that $(\phi(f), \phi) = 0\ \forall f \in \mathcal{L}(\mathcal{G})$. We must show $\phi = 0$. Given $F_1, \ldots, F_n \in \mathcal{F} \cap \mathcal{G}$ define a signed measure on $(\mathbf{N} \cup \{0\})^n$ by placing

$$\nu_{F_1,\ldots,F_n}(B) = \int_{(N(F_1),\ldots,N(F_n)) \in B} \phi\, dp.$$

Let $\bar{u} \in \mathbf{R}^n$ arbitrary. Then the Laplace transform of $\nu_{F_1,\ldots,F_n}$ is equal to

$$\int \left(\exp \sum_{i=1}^{n} u_i N(F_i)\right) \phi\, dp = \exp E\psi(f)(\phi(f), \phi) = 0$$

with

$$f = \sum_{i=1}^{n} u_i X_{F_i}.$$

Consequently $\phi\, dp$ is 0 on $\sigma\{N(F_1), \ldots, N(F_n)\}$ and, hence, is 0 on $\mathcal{B}(\mathcal{G})$. Thus $\phi = 0$. □

In order to obtain a formula for the conditional expectation $E(\phi(f) \mid \mathcal{B}(\mathcal{G}))$ we need to extend the definition of $\phi(f)$ for all $f \in L_2(X, \mathcal{S}, \mu)$. Let $f, g \in L_2(\mathcal{S})$ and find sequences (f_n) and (g_n) in $\mathcal{L}(\mathcal{S})$ such that $f_n \to f$ and $g_n \to g$ in mean. Then

$$\begin{aligned} \|\phi(f_n) - \phi(f_m)\|_\Omega^2 &= \exp\|\psi(f_n)\|_X^2 + \exp\|\psi(f_m)\|_X^2 \\ &\quad - 2\exp(\psi(f_n), \psi(f_m)). \end{aligned}$$

Therefore $(\phi(f_n))$ is fundamental in mean and hence converges in mean to some limit in $L_2(\Omega, \mathcal{A}, p)$, which we define as $\phi(f)$. Similar arguments show that the identity $(\phi(f), \phi(g)) = \exp(\psi(f), \psi(g))$ remains true.

THEOREM 2.2. *Let* $\mathcal{G} \subset \mathcal{S}$ *be a σ-finite, sub- σ-algebra. Then*

$$E(\phi(f) \mid \mathcal{B}(\mathcal{G})) = \phi\left(\psi^{-1} \operatorname{pr}(\psi(f) \mid L_2(\mathcal{G}))\right) \forall f \in \mathcal{L}(\mathcal{S}),$$

where $\operatorname{pr}(\circ \mid L_2(\mathcal{G}))$ *denotes orthogonal projection onto the indicated subspace.*

Proof. For simplicity write $\psi_{\mathcal{G}}(f) = \mathrm{pr}(\psi(f) \mid L_2(\mathcal{G}))$. We first show that $\psi^{-1}\psi_{\mathcal{G}}(f) \in L_2(\mathcal{G})$. Since $\psi(f) \in \mathcal{L}(\mathcal{S})$, there exists $c > -1$ such that $\psi(f) \geq c$. Hence $\psi_{\mathcal{G}}(f) \geq c$, also. On the other hand there exists $d \geq 1$ such that $|\psi^{-1}(x)| = |\log(1 + x)| \leq d|x|$ whenever $x \geq c > -1$. Therefore $\psi^{-1}\psi_{\mathcal{G}}(f) \in L_2(\mathcal{G})$.

Let $g \in \mathcal{L}(\mathcal{G})$ be arbitrary. We have

$$\begin{aligned}(\phi(f), \phi(g)) &= \exp(\psi(f), \psi(g)) = \exp(\psi_{\mathcal{G}}(f), \psi(g)) \\ &= \exp\big(\psi(\psi^{-1}\psi_{\mathcal{G}}(f)), \psi(g)\big) = \big(\phi(\psi^{-1}\psi_{\mathcal{G}}(f)), \phi(g)\big).\end{aligned}$$

Since $\{\phi(g) \colon g \in \mathcal{L}(\mathcal{G})\}$ spans $L_2(\mathcal{B}(\mathcal{G}))$ one has

$$\phi\big(\psi^{-1}\psi_{\mathcal{G}}(f)\big) = \mathrm{pr}(\phi(f) \mid L_2(\mathcal{B}(\mathcal{G}))) = E\big(\phi(f) \mid \mathcal{B}(\mathcal{G})\big). \qquad \square$$

REMARK. Analogous results for the case $\mu(X) < \infty$ are worked out in Neveu [**9**, pp. 162–168].

3. Poisson process over Markov chains. We introduce some notation that will be used throughout the sequel. Let $P = (p_{i,j})$, $i, j \in \mathbf{N}$, be a stochastic matrix and let $\bar{\mu} = (\mu_i(P))$ denote a stationary measure for P, i.e. $\sum_i \mu_i p_{ij} = \mu_j$ $\forall j \in \mathbf{N}$. By a well-known result of T. E. Harris and H. Robbins [**4**], every irreducible recurrent stochastic matrix has a stationary measure unique up to multiplication by a constant (see [**1**] for terminology). The pair (P, μ) is called positive or null according to whether $(\sum_i \mu_i(P))^{-1}$ is positive or zero, respectively. Let (P, μ) denote a null pair. We define the (two-sided) Markov shift $\tau = \tau_{(P,\mu)}$ as follows: Let $X = \mathbf{N}^{\mathbf{Z}}$, $\mathcal{S}$ = the σ-algebra generated by cylinder sets, and let μ be the unique σ-finite measure satisfying

$$\begin{aligned}&\mu\{x \in X \colon x(n) = i_n, \ldots, x(n+k) = i_{n+k}\} \\ &\quad = \mu_{i_n} p_{i_n, i_{n+1}} \cdots p_{i_{n+k-1}, i_{n+k}} \quad \forall n \in \mathbf{Z}, k \geq 1, i_n, \ldots, i_{n+k} \in \mathbf{N}\end{aligned}$$

and $\tau \colon X \to X$ given by $(\tau(x))(i) = x(i-1)$, $i \in \mathbf{Z}$. It is well known that τ is an automorphism which is ergodic iff P is irreducible and recurrent.

Let $G_i = \{x \in X \colon x(0) = i\}$, $i \in \mathbf{N}$, and let $\mathcal{P} = \{G_1, G_2, \ldots\}$ denote the 0-time partition; so $\mu_i = \mu(G_i)$ and

$$p_{i,j} = \mu\{x(1) = j \mid x(0) = i\} = \mu\big(G_j \mid \tau^{-1}G_i\big).$$

Define $\mathcal{H}_0 = \sigma(\mathcal{P})$, the 0-time σ-algebra, and

$$\mathcal{H}_a^b = \bigvee_{i=a}^{b} \tau^i \mathcal{H}_0, \quad a \leq b \in \mathbf{Z}, \quad \text{and} \quad \mathcal{P}_a^b = \bigvee_{i=a}^{b} \tau^i \mathcal{P};$$

as usual $\mathcal{P}_0^\infty = \sigma\{\mathcal{P}_0^a: a \in \mathbf{N}\}$ and $\mathcal{P}_{-\infty}^\infty = \sigma\{\mathcal{P}_a^b: a < b \in \mathbf{Z}\}$. Finally set $\gamma_s = \mathcal{B}(\mathcal{H}_0^s)$ and $\mathcal{Q}_s = \bigvee_{i=-\infty}^{\infty} T^{-i}\gamma_s$; Notice that $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_s$ and $\mathcal{Q}_s \uparrow \mathcal{Q}$ as $s \to \infty$. (*Proof*:

$$\begin{aligned}\mathcal{Q} = \mathcal{B}(\mathbb{S}) = \mathcal{B}(\mathcal{P}_{-\infty}^\infty) &= \bigvee_{i=-\infty}^{\infty} T^i\mathcal{B}(\mathcal{P}_0^\infty) = \bigvee_{i=-\infty}^{\infty} T^{-i} \bigvee_{s=0}^{\infty} \gamma_s \\ &= \bigvee_{s=0}^{\infty} \bigvee_{i=-\infty}^{\infty} T^{-i}\gamma_s = \bigvee_{s=0}^{\infty} \mathcal{Q}_s.)\end{aligned}$$

We need the following lemma.

LEMMA 3.1.

$$p(A \mid T^{-N}\gamma_s) = p(A \mid T^{-N}\gamma_0) \quad a.s. \qquad \forall A \in \bigvee_{i=-k}^{0} T^{-i}\gamma_r,$$

s, r, $k \geq 0$, whenever $N > r$.

Proof. Any atom of $\mathcal{P}_0^s$ is of the form $H_{\bar{a}}$, where $\bar{a} = (a(0), \ldots, a(s)) \in \mathbf{N}^{s+1}$ and $H_{\bar{a}} = \bigcap_{i=0}^{s} \tau^{-i} G_{a_i}$. Hence the family

$$\left\{ \frac{\chi_{H_{\bar{a}}} \circ \tau^n}{\sqrt{\mu(H_{\bar{a}})}} : \bar{a} \in \mathbf{N}^{s+1} \text{ such that } \mu(H_{\bar{a}}) > 0 \right\}$$

is an orthonormal basis for $L_2(\tau^{-N}\mathcal{H}_0^s)$. Consequently, $\forall F \in \mathcal{F} \cap \mathcal{H}_{-k}^r$ one has

$$\mathrm{pr}(\chi_F \mid L_2(\tau^{-N}\mathcal{H}_0^s)) = \sum_{\bar{a}} \mu(F \mid \tau^{-N}H_{\bar{a}})\chi_{H_{\bar{a}}} \circ \tau^N.$$

By the Markov property $\mu(F \mid \tau^{-N}H_{\bar{a}}) = \mu(F \mid \tau^{-N}G_{a(0)})$ whenever $N > r$; substituting we obtain

$$\mathrm{pr}(\chi_F \mid L_2(\tau^{-N}\mathcal{H}_0^s)) = \mathrm{pr}(\chi_F \mid L_2(\tau^{-N}\mathcal{H}_0)).$$

By linearity of the projections we obtain

$$\mathrm{pr}(f \mid L_2(\tau^{-N}\mathcal{H}_0^s)) = \mathrm{pr}(f \mid L_2(\tau^{-N}\mathcal{H}_0)) \quad \forall f \in \mathcal{L}(\mathcal{H}_{-k}^r).$$

Consequently, by Theorem 2.2,

$$E(\phi(f) \mid T^{-N}\gamma_s) = E(\phi(f) \mid T^{-N}\gamma_0) \quad \text{all } f \in \mathcal{L}(\mathcal{H}_{-k}^r)$$

whenever $N > r$. Since $\{\phi(f): f \in \mathcal{L}(\mathcal{H}_{-k}^r)\}$ is a basis for $L_2(\mathcal{B}(\mathcal{H}_k^r))$, it follows, by approximating any χ_A with $A \in \bigvee_{i=-k}^{0} T^{-i}\gamma_r \subset \mathcal{B}(\mathcal{H}_k^r)$ by basic functions and convergence properties of conditional expectation, that

$$p(A \mid T^{-N}\gamma_s) = p(A \mid T^{-N}\gamma_0) \quad \text{a.s.} \qquad \forall A \in \bigvee_{i=-k}^{0} T^{-i}\gamma_r$$

whenever $N > r$. $\square$

We now come to one of the main results.

THEOREM 3.2. *Let τ be a Markov chain with stochastic matrix P and measure μ. Then T is Bernoulli iff $p_{i,j}^{(n)} \to 0$, $\forall i, j \in \mathbf{N}$, as $n \to \infty$.*

Proof. If T is Bernoulli, then it is mixing and, hence, τ satisfies the mixing condition. In particular,

$$p_{i,j}^{(n)} = \mu(G_i \cap \tau^{-n}G_j)\mu(G_i)^{-1} \to 0 \quad \forall i, j \in \mathbf{N} \text{ as } n \to \infty.$$

Conversely, we first show that the system $((\Omega, \mathcal{C}_r, p), T)$ is Bernoulli, by showing that *every* finite partition Q which is γ_r-measurable is weak Bernoulli, i.e. we must show that given $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that $\forall k \geq 1$, $Q_{-k}^{0} \perp^{\varepsilon} Q_N^{N+k}$ (see [**10**] for definitions).

Let $k \geq 1$. Since $\gamma_0 \subset \bigvee_{i=0}^{k} T^{-i}\gamma_r \subset \mathfrak{B}(\mathcal{K}_0^{r+k})$, it follows from the last lemma, by taking conditional expectations with respect to $\bigvee_{i=0}^{k} T^{-i}\gamma_r$ and replacing s by $r + k$, that

$$p\left(A \mid T^{-N} \bigvee_{i=0}^{k} T^{-i}\gamma_r\right) = p(A \mid T^{-N}\gamma_0) \text{ whenever } N > r;$$

for $s = r$,

$$p(A \mid T^{-N}\gamma_r) = p(A \mid T^{-N}\gamma_0).$$

Consequently,

$$p\left(A \mid T^{-N} \bigvee_{i=0}^{k} T^{-i}\gamma_r\right) = p(A \mid T^{-N}\gamma_r), \qquad k \geq 0, \forall A \in \bigvee_{i=-k}^{0} T^{-i}\gamma_r$$

whenever $N > r$. Therefore for any atoms $B \in T^{-N}Q$ and $B_k \in T^{-N}Q_0^k$ with $\varnothing \neq B_k \subset B$ and $N > r$, one has

$$\operatorname{dist}\left(\bigvee_{i=-k}^{0} T^{-i}Q \mid B\right) = \operatorname{dist}\left(\bigvee_{i=-k}^{0} T^{-i}Q \mid B_k\right).$$

Hence it is enough to verify that $Q_{-k}^{0} \perp^{\varepsilon} T^{-N}Q$, $\forall k \geq 1$, for some $N = N(\varepsilon) > r$. Equivalently $Q \perp^{\varepsilon} Q_{-N-k}^{-N}$, $\forall k \geq 1$, for some $N > r$. On the other hand, by applying the same arguments to the reversed Markov chain one has that $\forall C \in T^N Q$ and $C_k \in T^N Q^0_{-k}$ with $\varnothing \neq C_k \subset C$, $\operatorname{dist}(Q \mid C) = \operatorname{dist}(Q \mid C_k)$ whenever $N > r$. Consequently we need only show $Q \perp^{\varepsilon^2} T^N Q$ for some $N > r$. By assumption $\mu(G_i \cap \tau^{-n} G_j) \to 0$, $\forall i, j \in \mathbf{N}$, as $n \to \infty$, and since $\mathcal{G}$ generates $\mathcal{S}$ under τ, it follows that τ satisfies the mixing condition; therefore T is mixing and $Q \perp^{\varepsilon^2} T^N Q$ if N is chosen large enough.

Hence $((\Omega, \mathcal{C}_r, p)T)$ is Bernoulli, and since $\mathcal{C}_r \uparrow \mathcal{C}$ as $r \to \infty$, it follows by a theorem of Ornstein ([**10**] page 53), that $((\Omega, \mathcal{C}, p), T)$ is Bernoulli. □

REMARKS. An irreducible null-recurrent Markov chain or a transient Markov chain preserving a σ-finite measure satisfies the condition $p_{i,j}^{(n)} \to 0$ as well as Markov chains with periodic states that are recurrent and null (see [**1**] page 33, Theorem 4 for a complete study of the limiting behavior of $p_{i,j}^{(n)}$).

If τ is a ν-step Markov chain ($\nu > 1$), then after minor modifications in the conclusions of Lemma 3.1, we obtain the following

THEOREM 3.3. *Let τ be a ν-step Markov chain ($\nu > 1$) preserving a σ-finite measure μ. Then T is Bernoulli iff $p_{a,b}^{(n)} \to 0$, $\forall a, b \in \mathbf{N}^\nu$, as $n \to \infty$, where*

$$p_{a,b}^{(n)} = \mu\{x(n+j) = b(j) \mid x(j) = a(j) \colon 0 \le j \le \nu - 1\}.$$

4. Entropy of the Poisson process over Markov chains. We start by establishing a formula for the entropy of T, if τ is a Markov chain.

PROPOSITION 4.1. *Let τ be a Markov chain and T its Poisson transformation. Then*

$$h_p(T) = \lim_{r \to \infty} \frac{1}{r+1} \sup\left\{H_p\left(Q \mid T^{-r-1} Q\right) \colon \sigma(Q) \subset \gamma_r, H_p(Q) < \infty\right\}.$$

Proof. Since $\mathcal{C}_r \uparrow \mathcal{C}$ it follows by well-known properties of entropy that $h_p(T) = \lim_{r \to \infty} h_p(T, \mathcal{C}_r)$. Let Q be a γ_r-measurable partition with finite entropy. We have by Lemma 3.1 that

$$\operatorname{dist}\left(Q \mid T^{-N} Q\right) = \operatorname{dist}\left(Q \mid T^{-N} Q_0^k\right) \quad \forall k \ge 1 \text{ whenever } N > r;$$

so for $N = r + 1$ it follows that

$$\operatorname{dist}\left(Q \mid T^{-r-1} Q\right) = \operatorname{dist}\left(Q \mid \bigvee_{i=1}^{n} T^{-(r+1)i} Q\right) \quad \forall n \ge 1.$$

Hence $h_p(T^{r+1}, Q) = H_p(Q \mid T^{-r-1}Q)$ so

$$h_p(T, \mathcal{C}_r) = \frac{1}{r+1} \sup\{H_p(Q \mid T^{-r-1}Q) : \sigma(Q) \subset \gamma_r, H_p(Q) < \infty\},$$

from which the result follows □

We first evaluate $h_p(T)$ for some special kinds of Markov chains which we describe below.

Definition 4.2. Let (f_n) be a sequence of non-negative real numbers such that $\Sigma_{n=1}^{\infty} f_n = 1$; put

$$f(\lambda) = \sum_{n=1}^{\infty} f_n \lambda^n \quad \text{and} \quad u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n = \frac{1}{1 - f(\lambda)}.$$

The sequence $(u_n)_{n=0}^{\infty}$ is called a *recurrent renewal sequence*, as is any sequence obtained in this fashion from an (f_n) satisfying the above requirements. Observe that every probability distribution (f_n) determines a unique recurrent renewal sequence, and conversely every recurrent renewal sequence comes from a unique probability distribution. We will write $\bar{u} = (u_n)$. Given $\bar{u}$ and the probability distribution (f_n) from which it comes, define a doubly infinite matrix $P_{\bar{u}} = (p_{i,j}(\bar{u}))_{i,j \in \mathbf{N}}$ as follows:

$$p_{i,j}(\bar{u}) = \begin{cases} f_j & \text{if } i = 1, \\ 1 & \text{if } i \geq 2, j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $P_{\bar{u}}$ is stochastic, irreducible and recurrent and $p_{11}^{(n)} = u_n$. A stationary measure $m = (m_i(\bar{u}))$ for $P_{\bar{u}}$ is given by $m_i(\bar{u}) = \Sigma_{l=i}^{\infty} f_l$. Consequently, $P_{\bar{u}}$ is positive or null according to whether $(\Sigma_{n=1}^{\infty} n f_n)^{-1}$ is positive or zero, respectively.

Denote by $\tau_{\bar{u}}$ the Markov shift with stochastic matrix $P_{\bar{u}}$ and stationary measure $m = m(\bar{u})$ for a recurrent renewal sequence $\bar{u}$ and let $T_{\bar{u}}$ be its associated Poisson transformation.

Proposition 4.3. *If $P_{\bar{u}}$ is null, then $h_p(T_{\bar{u}}) = \infty$.*

Proof. Define $H: [0, \infty) \to [0, \infty)$ by

$$H(x) = -\sum_{n=0}^{\infty} e^{-x} \frac{x^n}{n!} \log_2 e^{-x} \frac{x^n}{n!}$$

where, as usual, $0 \log 0$ is defined to be 0. $H(x)$ is just the entropy of the Poisson distribution with parameter x. It is easy to show that H is continuous, $H(x+y) \leq H(x) + H(y)$, $H(x) \to \infty$ as $x \to \infty$ and $H(x)/x \to \infty$ as $x \to 0$.

Let (f_n) be the probability distribution associated with $\bar{u}$. By assumption, $\sum_{n=1}^{\infty} nf_n = \infty$. Consequently $\sum_{n=1}^{\infty} 2nf_{2n} = \infty$ or $\sum_{n=1}^{\infty}(2n+1)f_{2n+1} = \infty$. We will consider just the first case, the other being entirely similar. By the last proposition

$$H_p(T_{\bar{u}}) \geq \sup\left\{H_p\left(Q \mid T_{\bar{u}}^{-1}Q\right): \sigma(Q) \subset \gamma_0, H_p(Q) < \infty\right\}.$$

Let $\pi_G = \{\{N(G) = n\}\}_{n=0}^{\infty}$ $\forall G \in \mathfrak{F}$. Then $H_p(\pi_G) = H(\mu(G))$; put $Q_l = \pi_{G_2} \vee \cdots \vee \pi_{G_{2l}}$, where $G_i = \{x\colon x(0) = i\}$. Then because of the form of $P_{\bar{u}}$ we have $G_{2i} \cap \tau_{\bar{u}}^{-1} G_{2j} = \varnothing$, $i \neq j$; therefore, by independence,

$$H_p\left(Q_l \mid T_u^{-1}Q_l\right) = \sum_{i=1}^{l} H_p(\pi_{G_{2i}}) = \sum_{i=1}^{l} H(m_{2i}) \geq H\left(\sum_{i=1}^{l} m_{2i}\right).$$

Letting $l \to \infty$, we get $h_p(T_{\bar{u}}) = \infty$. □

DEFINITION 4.4. Let τ be an automorphism of a σ-finite measure space $(X, \mathbb{S}, \mu)$ and let $E \in \mathfrak{F}$. We say that E is a *recurrent set* iff for every sequence

$$0 = n_0 \leq n_1 \leq \cdots \leq n_k,$$

$$\mu_E\left(\bigcap_{j=1}^{k} \tau^{-n_j}E\right) = \prod_{j=1}^{k} \mu_E\left(\tau^{-(n_j - n_{j-1})}E\right)$$

where $\mu_E(F) = \mu(E \cap F)/\mu(E)$. It is clear that every set of the 0-time partition of a Markov shift is recurrent.

Assume E is a recurrent set of some conservative automorphism τ. Since $E \subset \bigcup_{n=1}^{\infty} \tau^{-n}E \pmod{\mu}$, we can define the induced transformation τ_E a.e. on E by setting $\tau_E(x) = \tau^{r_E(x)}(x)$ for $x \in \bigcup_{n=1}^{\infty} E \cap \tau^{-n}E$, where r_E denotes the return-time function defined by

$$r_E(x) = \min\{k \in \mathbf{N}\colon \tau^k x \in E\}$$

(See S. Kakutani [5].) Let $E_n = r_E^{-1}(n)$. Then $\mathfrak{R}(E) = \{E_1, E_2, \ldots\}$ is a partition of E, called the return time partition of E relative to τ. Since E is recurrent, it is not hard to show that the sequence $(u_n(E) = \mu_E(\tau^{-n}E))$ is a recurrent renewal sequence, with associated probability distribution $(f_n = \mu_E(E_n))$. Let $P_{\bar{u}(E)}$ denote the stochastic matrix associated with the sequence $\bar{u}(E)$. Then by Kac's theorem

$$\sum_{i=1}^{\infty} m_i(E) = \mu(E)^{-1} \sum_{i=1}^{\infty} i\mu(E_i) = \mu(E)^{-1} \int r_E \, d\mu.$$

Consequently, $P_{\bar{u}(E)}$ is positive or null according to whether or not $\int r_E \, d\mu$ is finite. On the other hand, $\mu(E^*) = \mu(E)^{-1}\int r_E \, d\mu$ where $E^* = \bigcup_{n=0}^{\infty} \tau^{-n}E$. Denote by $(X', \mathfrak{S}', m)$ the space of the shift $\tau_{\bar{u}(E)}$. Then one can define a.e. an onto map $\phi\colon E^* \to X'$ such that:

(i) $\phi^{-1}\mathfrak{S}' \subset \mathfrak{S} \cap E$;

(ii) $\phi \circ \tau|_{E^*} = \tau_{\bar{u}(E)} \circ \phi$; and

(iii) $\mu \circ \phi^{-1} = \mu(E)m$.

We multiply the stationary measure by $\mu(E)$ and, by abuse of notation, we still can call this new shift $\tau_{\bar{u}(E)}$. It is clear that also $h_p(\tau_{u(E)}) = \infty$ if $\mu(E^*) = \infty$. On the other hand, since τ is conservative, E^* is actually invariant and, hence, τ can be written as the union of the transformations restricted to E^* and $X - E^*$ and, therefore, T can be written as the direct product of the Poisson transformations associated with the restrictions to E^* and $X - E^*$. Collecting the above remarks we have the following:

PROPOSITION 4.5. *Let τ be a conservative automorphism and suppose it admits a recurrent set E with $\mu(E^*) = \infty$. Then $\tau|_{E^*}$ has a Markov shift as a factor for which the associated Poisson transformation has infinite entropy.*

The next proposition shows that "factors correspond to factors".

PROPOSITION 4.6. *Let τ and τ' be endomorphisms of σ-finite measure spaces $(X, \mathfrak{S}, \mu)$ and $(X', \mathfrak{S}', \mu')$, respectively. Let $((\Omega, \mathfrak{A}, p), T)$ and $((\Omega', \mathfrak{A}', p'), T')$ be the Poisson processes over the given bases, respectively. If τ' is a factor of τ, then T' is a factor of T. In particular $h_p(T, \mathfrak{A}) \geq h_{p'}(T', \mathfrak{A}')$.*

Proof. Let $\phi\colon (X, \mathfrak{S}, \mu) \to (X', \mathfrak{S}', \mu')$ be an onto map such that $\phi^{-1}\mathfrak{S}' \subset \mathfrak{S}$, $\phi \circ \tau = \tau' \circ \phi$ and $\mu \circ \phi^{-1} = \mu'$. Let $\mathfrak{A}'' = \mathfrak{B}(\phi^{-1}\mathfrak{S}')$. Then $\mathfrak{A}''$ is a sub-σ-algebra of $\mathfrak{A}$. Define a map $\tilde{\phi}\colon (\Omega, \mathfrak{A}'', p) \to (\Omega', \mathfrak{A}', p')$ by sending $\{N(\phi^{-1}G) = n\}$ onto $\{N(G) = n\}$, $\forall G \in \mathfrak{F} \cap \mathfrak{S}'$, $\forall n \geq 0$. Then $\tilde{\phi} \circ T = T' \circ \tilde{\phi}$ on $\{\{N(\phi^{-1}(G)) = n\}\colon G \in \mathfrak{F} \cap \mathfrak{S}', n \geq 0\}$ and $p \circ \tilde{\phi}^{-1} = p'$ on $\{\{N(G) = n\}\colon G \in \mathfrak{F} \cap \mathfrak{S}', n \geq 0\}$. But these classes generate $\mathfrak{A}''$ and $\mathfrak{A}'$, respectively. Consequently T' is a factor of T. $\square$

By the remarks and results of this section we obtain:

THEOREM 4.7. *Let τ denote a conservative automorphism that admits a recurrent set E with $\mu(E^*) = \infty$, or a Markov chain satisfying $p_{i,j}^{(n)} \to 0$, $\forall i, j \in \mathbf{N}$, as $n \to \infty$ such that it has a recurrent state or is transient and $(\mathfrak{S}, \mu)$ is not completely atomic. Then $h_p(T) = \infty$.*

Proof. Let T_{E^*} denote the Poisson transformation corresponding to $\tau|_{E^*}$. Then by the last proposition,

$$h_p(T) \geq h_p(T_{E^*}) = h_p(T_{\bar{u}(E)}) = \infty.$$

If τ is a Markov chain satisfying the hypotheses, then by earlier results it follows that τ does not have invariant sets of positive finite measure. Assume τ has a recurrent state $i_0 \in \mathbf{N}$. Let I denote the irreducible class containing i_0. Since I is closed (see [**1**]), i.e. $\Sigma_{k\in I} p_{j,k} = 1\ \forall j \in I$, and $I^{\mathbf{Z}}$ is τ-invariant, we have: $\mu(I^{\mathbf{Z}}) = \Sigma_{j\in I} \mu_j = \infty$ and $\tau|_{I^{\mathbf{Z}}}$ is an ergodic Markov shift. Clearly $E = \{x \in I^{\mathbf{Z}}\colon x(0) = i_0\}$ is a recurrent set with $\mu(E^*) = \mu(I^{\mathbf{Z}}) = \infty$. Therefore by the first part, $h_p(T) = \infty$.

Now assume τ is transient and $(\mathfrak{S}, \mu)$ is not completely atomic, i.e. $\exists$ a set $X_0 \in \mathfrak{S}$ with $\mu(X_0) > 0$, $\mathfrak{S} \cap X_0$ is non-atomic and $X - X_0$ is a countable union of atoms. Since X_0 is τ-invariant we have $\mu(X_0) = \infty$. Since τ is dissipative so is $\tau|_{X_0}$ acting on the σ-finite, non-atomic measure space $(X_0, \mathfrak{S} \cap X_0, \mu)$. Hence $\exists F \in \mathfrak{S} \cap X_0$ of positive finite measure such that $\{\tau^{-n}F\}_{n\in\mathbf{N}}$ is pairwise disjoint. For each $n > 1$ find disjoint subsets $G_1^{(n)}, \ldots, G_n^{(n)}$ whose union is F and such that $\mu(F) = n\mu(G_i^{(n)})$, $i = 1, \ldots, n$. Therefore, by independence,

$$h_p\left(T, \bigvee_{i=1}^{n} \pi_{G_i^{(n)}}\right) = \sum_{i=1}^{n} H_p(\pi_{G_i^{(n)}}) = nH\left(\frac{\mu(F)}{n}\right)$$

where H is the entropy function discussed above; since $\lim_{x\to 0}(H(x)/x) = \infty$, letting $n \to \infty$, we obtain $h_p(T) = \infty$.

Remark 1. The analogous results remain true for ν-step Markov chains ($\nu > 1$).

Remark 2. If $(X, \mathfrak{S}, \mu)$ is completely atomic and σ-finite, and if τ is a dissipative automorphism, it might happen that $h_p(T) < \infty$ or $h_p(T) = \infty$. For, X is (mod μ) the disjoint union of countably many atoms (E_i) of finite measure; since $\tau^n E_i$ is also an atom $\forall n \in \mathbf{Z}$, we can write $X = \bigcup_{n\in\mathbf{Z}} \tau^n W$ (disjoint) (mod μ) with W a union of atoms $\{E_j\}_{j\in J}$ such that $\tau^n E_j \cap E_{j'} = \varnothing$, $\forall n \in \mathbf{Z}$, $j \neq j'$ in J. Then $h_p(T) = h_p(T, \pi_W) = H(\mu(W))$ and so is finite or infinite according to whether $\mu(W) < \infty$ or $\mu(W) = \infty$.

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