PACIFIC JOURNAL OF MATHEMATICS
Vol 111, No. 2, 1984

REPRESENTATIONS AND AUTOMORPHISMS OF THE IRRATIONAL ROTATION ALGEBRA

BERNDT A. BRENKEN

Given an irrational number α, A_α is the unique C^*-algebra generated by two unitary operators, U and V, satisfying the twisted commutation relation $UV = \exp(2\pi i\alpha)VU$. We investigate separable representations of A_α which, when restricted to the abelian C^* algebra generated by V, are of uniform multiplicity m. These representations are classified by their multiplicity, a quasi-invariant Borel measure on the circle (w.r.t. rotation by the angle $2\pi\alpha$) and a unitary one cocycle.

Separable factor representations lie in this class, the measure being ergodic in this case. A factor representation is of uniform multiplicity m' on the C^* algebra generated by U, and if m, m' are relatively prime, the representation is irreducible. By use of an action of $\mathrm{SL}(2, \mathbf{Z})$ as *-automorphisms of A_α, that we construct, we arrive at a separating family of pure states of A_α whose corresponding irreducible representations provide explicit examples with m and m' occurring as any given pair of nonzero relatively prime numbers.

Introduction. We study representations of the irrational rotation algebras, a special class of C^*-algebras that has received a great deal of attention in recent years [**13–16**]. Our focus is primarily, though not exclusively, on factor and, in particular, irreducible, representations of algebras in this family. This class of algebras is parametrized by the irrational numbers in $[0, 1]$. To each irrational number α in $[0, 1]$, we make correspond the C^*-algebra A_α generated by multiplications by continuous functions on $\mathbf{T}$, the unit circle in the plane of complex numbers, and the unitary transformation on $L_2(\mathbf{T}, \nu)$ arising from rotation of $\mathbf{T}$ through the angle $2\pi\alpha$, where ν is (normalized) Haar measure on $\mathbf{T}$. More specifically, let $M_f g$ be fg where $f \in C(\mathbf{T})$ and $g \in L_2(\mathbf{T}, \nu)$, and let $(Ug)(\exp(2\pi i\theta))$ be $g(\exp(2\pi i(\theta + \alpha)))$ for each θ in $[0, 1]$. Then A_α is the C^*-algebra generated by $\{M_f, U: f \in C(\mathbf{T})\}$.

Although we have described A_α in a particular representation, in the first instance, it can be characterized (uniquely, as it turns out) as a C^*-algebra generated by two unitary operators U and V satisfying a "twisted" commutation relation $UV = (\exp 2\pi i\alpha)VU$. In the representation of A_α we described, U is as noted and V is multiplication by z (the identity transform on $\mathbf{T}$). There are several other ways of viewing A_α that will be useful to us. The rotation of $\mathbf{T}$ through the angle $2\pi\alpha$ is a homeomorphism that engenders an automorphism of $C(\mathbf{T})$. We use 'α' to denote this automorphism as well as the parametrizing irrational number. The mapping that assigns α^n to the integer n is a homomorphism of $\mathbf{Z}$, the additive group of integers, into Aut $C(\mathbf{T})$, the group of automorphisms of (the abelian C^*-algebra) $C(\mathbf{T})$. Such a homomorphism is termed an "action" of $\mathbf{Z}$ on $C(\mathbf{T})$ and the triple $(C(\mathbf{T}), \alpha, \mathbf{Z})$ is a C^*-dynamical system. A general construction associates a C^*-algebra with such a dynamical system, the crossed product, $C(\mathbf{T}) \times_\alpha \mathbf{Z}$, of $C(\mathbf{T})$ by $\mathbf{Z}$ via the action. In the present case the crossed product is another description of A_α.

From [5] the representations of A_α are in one-to-one (canonical) correspondence with the "covariant representations" of the C^*-dynamical system $(C(\mathbf{T}), \alpha, \mathbf{Z})$. That is, a representation $\tilde{\pi}$ of A_α corresponds to a representation π of $C(\mathbf{T})$ and a unitary representation of $\mathbf{Z}$ such that the action of $\mathbf{Z}$ on $C(\mathbf{T})$ is implemented by the unitary operators representing $\mathbf{Z}$ on $\pi(C(\mathbf{T}))$. In our situation $C(\mathbf{T})$ "appears" as a subalgebra of A_α, and $\mathbf{Z}$ "appears" as a group of unitary elements in A_α (and together they generate A_α) — much as in our initial (representation) description of A_α. Given $\tilde{\pi}$, then, π is the restriction of $\tilde{\pi}$ to $C(\mathbf{T})$ and the unitary representation of $\mathbf{Z}$ is the restriction of $\tilde{\pi}$ to the unitary elements representing $\mathbf{Z}$ in A_α. We make use of the well-developed representation theory of abelian C^*-algebras to single out those representations $\tilde{\pi}$ of A_α whose associated π has "uniform multiplicity". That is, we study those $\tilde{\pi}$ for which $\pi(C(\mathbf{T}))$ acts as an m-fold copy of a representation of $C(\mathbf{T})$ in which the strong-operator closure of the image is maximal abelian — in another form, the commutant $\pi(C(\mathbf{T}))'$ of $\pi(C(\mathbf{T}))$ is a von Neumann algebra of type I_m. Here, m is a given cardinal number and the representation $\tilde{\pi}$ is said to be a "uniform multiplicity m representation" (of A_α). We will be concerned exclusively with the case of separable Hilbert space representations, so that m is either finite or $\aleph_0$.

To each (separable) uniform multiplicity representation $\tilde{\pi}$ of A_α, we associate a triple (m, ν, b), where m is the multiplicity of π, ν is a quasi-invariant regular Borel measure on $\mathbf{T}$, and b is a certain "unitary cocycle" (on $\mathbf{Z}$ with values in the unitary group of the algebra of $m \times m$ matrices with entries in $L^\infty(\mathbf{T}, \nu)$). We prove that each factor representation of A_α is a uniform multiplicity representation, and ν is ergodic under the action of α, in this case. Two uniform multiplicity representations of A_α are unitarily equivalent if and only if they have "equivalent" triples: the multiplicities are the same, the measures on $\mathbf{T}$ are equivalent (absolutely continuous with respect to one another), and the unitary cocycles are cohomologous. We establish corresponding criteria for quasi-equivalence of uniform multiplicity representations. In case m is finite we show that the representation $\tilde{\pi}$ is of type I (that is, $\tilde{\pi}(A_\alpha)''$ is a von Neumann algebra of type I).

Of course A_α has pure states and each of these gives rise to an irreducible representation (by means of the GNS construction). Each irreducible representation is a factor representation (of type I_∞) and is, accordingly, a uniform multiplicity representation. By exploiting the symmetry of the two generating unitary elements U and V (the abelian C^*-algebra generated by U is also isomorphic to $C(\mathbf{T})$), we may associate with each factor representation of A_α a second cardinal number m' (determined by the multiplicity of the restriction of the representation to the subalgebra generated by U). We prove that the representation is irreducible when m and m' are relatively prime.

It is clear that there are multiplicity one representations (that is, ones for which $\pi(C(T))''$ is maximal abelian). These abound — indeed, the representation in which we first described A_α is such. What is not clear *a priori* is that there are irreducible representations of multiplicity greater than one [**1**]. In §2.5 we construct such a representation for each finite multiplicity m, where ν is Haar measure. In effect we have found explicit ergodic actions of $\mathbf{Z}$ on the algebra of $m \times m$ matrices with entries in $L^\infty(\mathbf{T}, \nu)$. In fact we find irreducible representations of A_α such that the pair of multiplicities m and m' obtained by restricting the representation to the C^*-algebras generated by V and U respectively are any given pair of relatively prime numbers. This is accomplished with the aid of an action of $\mathrm{SL}(2, \mathbf{Z})$ on A_α that we construct. Loosely speaking, we let an element (matrix) of $\mathrm{SL}(2, \mathbf{Z})$ act on the "vector" (p, q) of integers appearing in V^pU^q and modify this by a (carefully chosen) phase factor to determine the automorphism of A_α corresponding to that element of $\mathrm{SL}(2, \mathbf{Z})$. With θ in $[0, 1]$, "evaluation" at $\exp 2\pi i\theta$ is a pure state of $C(\mathbf{T})$ and has a pure state extension φ_θ to A_α. For each g in $\mathrm{SL}(2, \mathbf{Z})$, $\varphi_\theta \circ \beta_g$ is a pure state of A_α, where β_g is the automorphism of A_α corresponding to g. With g suitably selected, the GNS representation of A_α corresponding to $\varphi_\theta \cdot \beta_g$ has the given multiplicities m and m'.

We show that the pure states φ_θ and $\varphi_{\tilde{\theta}}$ of A_α are unitarily equivalent if and only if $\exp 2\pi i\theta$ lies in the orbit of $\exp 2\pi i\tilde{\theta}$ under the action of α (on $\mathbf{T}$). We determine when φ_θ and $\varphi_{\tilde{\theta}} \circ \beta_g$ are unitarily equivalent. If H is the subgroup of $\mathrm{SL}(2, \mathbf{Z})$ consisting of matrices of the form $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$ $(n \in \mathbf{Z})$ and $g \in H$, we show that $\varphi_\theta \circ \beta_g = \varphi_\theta$ and $\varphi_\theta \circ \beta_{-g} = \varphi_{1-\theta}$. We prove that if $\varphi_{\tilde{\theta}} \circ \beta_g$ and φ_θ are unitarily equivalent, then g or $-g$ is in H, whence

$\varphi_{\tilde{\theta}} \circ \beta_g$ and φ_θ are unitarily equivalent if and only if either $g \in H$ and $\exp 2\pi i\tilde{\theta}$ is in the orbit of $\exp 2\pi i\theta$ or $-g \in H$ and $\exp - 2\pi i\tilde{\theta}$ is in the orbit of $\exp 2\pi i\theta$. It is also proved that the pure states of the form $\varphi_\theta \circ \beta_g$ "separate" A_α.

Explicit computation of the unitary cocycle is made in several cases, in particular, for the "multiplicity m" irreducible representations described above. We define certain cocycles to be "diagonal" and compute for uniform multiplicity m representations ($m \in \{1, 2, \ldots, \aleph_0\}$) with diagonal cocycles the structure of the commutant of the image of the representation. The particular cocycle (its diagonal entries) imposes an equivalence relation on the set of pairs of integers $1, \ldots, m$. This relation consists of all pairs if and only if the representation is factorial. If the relation coincides with "equality", then the commutant is abelian. When a uniform multiplicity representation has multiplicity $\aleph_0$ and the cocycle arises from the "bilateral shift" the representation is the familiar "group-measure space construction" of Murray and von Neumann [**17**]. It is of type II_1 and arises from the trace on A_α. In this connection, we make use of several basic algebraic facts about A_α (see, for example, [**12**, **20**]). It possesses a unique tracial state, whence all finite representations of A_α are type II_1 factor representations and each is unitarily equivalent to the trace representation if it is cyclic [**8**]. In addition, A_α is simple, and, as noted earlier, it is characterized as being generated by two unitary elements U and V satisfying the twisted commutation relation $UV = (\exp 2\pi i\alpha)VU$.

The recently developed K-theory of C^*-algebras (see, for example, [**13**]) supplies us with a natural map of Aut A_α to the group of automorphisms of $\mathbf{Z} \times \mathbf{Z}$ $(= K_1(A_\alpha))$. This map is the identity on $\mathrm{SL}(2, \mathbf{Z})$ and we show that the kernel of this map (a closed subgroup of Aut A_α in an appropriate topology) contains the centrally trivial automorphisms (see [**3**, **7**]) and, thus, also the approximately inner automorphisms of A_α.

There are many questions related to the representation theory of A_α on which we have not touched (the conjugacy of the maximal abelian subalgebras of A_α and criteria for type III representations, for example) and some for which our information is not complete (a computable necessary and sufficient condition for factoriality of a representation, for example). We expect to return to these questions in later publications.

During the course of these investigations, we have benefited from conversations with several of our colleagues at the University of Pennsylvania. It is a pleasure to express our gratitude to Joachim Cuntz, Vaughn Jones, Robert Powers, Jonathan Rosenberg, and Antony Wassermann. A special debt of gratitude is due to R. Kadison who suggested the general

topic and whose patient and careful supervision of my thesis, from which this paper is adapted, has led to several simplifications and improvements.

1. The irrational rotation algebra. A *C^*-dynamical system* is a triple (A, α, G) where A is a C^*-algebra, G a locally compact group and α a homomorphism of G into the group of $*$-automorphisms of A such that $x \to \alpha_x(b)$ is continuous from G to A for each $b \in A$. Given a C^*-dynamical system we may form a Banach-$*$-algebra $L^1(G, A)$. If $C_c(G, A)$ is the vector space of continuous functions with compact supports from G to A we endow it with a norm, involution and "twisted" convolution product as follows [**5**, **12**, **20**]

$$\|f\|_1 = \int |f(x)|\,dx,$$

$$f^*(x) = \Delta(x)^{-1}\alpha_x(f(x^{-1}))^*,$$

$$(f * h)(x) = \int_G f(s)\alpha_s(h(s^{-1}x))\,ds \qquad (f, h \in C_c(G, A), x \in G).$$

Here dg and $\Delta(g)$ denote the left Haar measure and the modular function on G respectively. With this structure $C_c(G, A)$ is a normed $*$-algebra with isometric involution and $L^1(G, A)$ denotes its completion. A covariant representation of a C^*-dynamical system (A, α, G) is a triple (π, U, H) where π is a representation of A as bounded operators on the Hilbert space H, U is a strongly continuous unitary representation of G on H, and $U_x\pi(b)U_x^* = \pi(\alpha_x(b))$ for x in G, b in A.

PROPOSITION 1.1 [**5**]. *If (π, U, H) is a covariant representation of the C^*-dynamical system (A, α, G), there is a nondegenerate representation $(\pi \times U, H)$ of $L^1(G, A)$ such that*

$$(\pi \times U)(f) = \int_G \pi(f(x))U_x\,dx \quad \textit{for } f \textit{ in } C_c(G, A).$$

The correspondence $(\pi, U, H) \to (\pi \times U, H)$ is a bijection onto the set of nondegenerate representations of $L^1(G, A)$.

The crossed product associated with the system (A, α, G), denoted by '$A \times_\alpha G$', is the enveloping C^*-algebra of $L^1(G, A)$. The following "universal" property for $A \times_\alpha G$ is a consequence of the preceding proposition and the universal property of enveloping C^*-algebras [**4**].

PROPOSITION 1.2. *Let τ be the canonical map of $L^1(G, A)$ into $A \times_\alpha G$. If (π, U, H) is a covariant representation of (A, α, G), there is a unique representation $(\tilde{\pi}, H)$ of $A \times G$ such that the representation $\tilde{\pi} \circ \tau$ of $L^1(G, A)$ is $\pi \times U$.*

Notice that if the group G is discrete and A is a C^*-algebra with unit, then G and A have canonical isomorphic copies in $A \times_\alpha G$. For if $a \in A$, the function from G to A defined at e (the unit of G) by a, and taking the value 0 elsewhere, is an element of $C_c(G, A)$ $(\subseteq L^1(G, A))$ representing a. We represent x in G as the element of $C_c(G, A)$ that maps x to Id_A and has the value 0 elsewhere on G. With these identifications the unit of A is a unit for $A \times_\alpha G$.

We now define the irrational rotation algebra A_α. If α is an irrational number in $[0, 1]$ define an action of the discrete abelian group $\mathbf{Z}$ via homeomorphisms of the circle group $\mathbf{T}$ by n: $e^{2\pi i t} \to e^{2\pi i(t+n\alpha)}$. This results in an action α of $\mathbf{Z}$ on Borel functions f on $\mathbf{T}$ given by $(\alpha_n f)(e^{2\pi i t}) = f(e^{2\pi i(t+n\alpha)})$. This action is by $*$-automorphisms on the abelian C^*-algebra $C(\mathbf{T})$. Define A_α to be the crossed product $C(T) \times_\alpha \mathbf{Z}$ of the C^*-dynamical system $(C(\mathbf{T}), \alpha, \mathbf{Z})$. As $\mathbf{Z}$ is discrete and $C(\mathbf{T})$ has a unit, $\mathbf{Z}$ and $C(\mathbf{T})$ have canonical 'copies' in $C(\mathbf{T}) \times_\alpha \mathbf{Z}$. Denote by U_n the unitary operator in $C(\mathbf{T}) \times_\alpha \mathbf{Z}$ corresponding to the element n of $\mathbf{Z}$. For $f \in C_c(\mathbf{Z}, C(\mathbf{T}))$, $f = \Sigma f(n) U_n$ (where the product $f(n)U_n$ is the "twisted" convolution product).

The one-to-one correspondence between representations π of A_α and covariant representations (ρ, T) of $(C(\mathbf{T}), \alpha, Z)$ is given by restriction, i.e. $\pi|_{C(\mathbf{T})} = \rho$ and $\pi(U_n) = T(n)$ $(n \in \mathbf{Z})$. There is an action $t \to \hat{\alpha}_t$ of $\mathbf{T}$ $(= \hat{\mathbf{Z}})$ on A_α, the dual action, given by

$$\hat{\alpha}_t(f)(n) = t^n f(n) \qquad (f \in C_c(\mathbf{Z}, C(\mathbf{T})), t \in \mathbf{T}, n \in \mathbf{Z}) \qquad [\mathbf{12}].$$

We may define a conditional expectation E: $A_\alpha \to C(T)$ by

$$E(x) = \int_{\mathbf{T}} \hat{\alpha}_t(x)\, dt \qquad (x \in A_\alpha),$$

where dt denotes normalized Haar measure on $\mathbf{T}$. We have $E(f) = f(0)$ for f in $C_c(\mathbf{Z}, C(\mathbf{T}))$. The algebra A_α is known to be simple [**12**, **20**] and to have a unique normalized trace τ. For f in $C_c(\mathbf{Z}, C(\mathbf{T}))$ the trace is given by $f \to \int_{\mathbf{T}} f(0)\, dt$.

For W a unitary on a Hilbert space $\mathcal{H}$, B a C^*-algebra on $\mathcal{H}$, we let ad W denote the $*$-automorphism of B defined by $x \to WxW^*$ $(x \in B)$.

If V_1 is the unitary in $C(\mathbf{T})$ defined by $t \to t$ $(t \in \mathbf{T})$, then $U_1 V_1 = \exp(2\pi i\alpha) V_1 U_1$. Suppose U, V are two unitary operators satisfying this 'twisted commutation relation'. As $\operatorname{ad} U$ is a *-automorphism of $C^*(V)$, the C^*-algebra generated by V, we have $\operatorname{sp}(V)$, the spectrum of V, is equal to $\operatorname{sp}(\operatorname{ad} U(V)) = \exp(2\pi i\alpha)\operatorname{sp}(V)$. Thus $\operatorname{sp}(V) = \mathbf{T}$ and $C(\mathbf{T}) \cong C^*(V)$, the *-isomorphism mapping V_1 to V. Using Proposition 1.2. and the fact that A_α is simple, we conclude that A_α is *-isomorphic to the C^*-algebra generated by U and V, the isomorphism mapping V_1 to V and U_1 to U. Dropping the subscripts from here on, we write U and V for U_1, V_1.

Let X be a locally compact (2nd countable) Hausdorff space, μ a σ-finite positive Borel measure and G a discrete countable group of homeomorphisms of X (acting on the right by $x \to xs$ ($x \in X$, $s \in G$)).

The measure μ is *quasi-invariant* under G if $\mu(E) = 0$ implies $\mu(Es) = 0$ ($s \in G$, E a measurable set). In this case the measure μ_s defined by $\mu_s(E) = \mu(Es)$ (E measurable) is absolutely continuous with respect to μ and the Radon-Nikodym derivative $d\mu_s/d\mu$ exists. The measure is *invariant* if $d\mu_s/d\mu \equiv 1$. The group action is *free* if $\mu(\{x \in X \mid xs = x\}) = 0$ for each s in $G \backslash \{e\}$, e the unit of G. The group action is *ergodic* if $f_s = f$ for all s in G implies f is a constant a.e. ($f \in L^\infty(X, \mu)$). Here $f_s(x) = f(xs)$ ($x \in X$, $s \in G$) for any function f on G.

2. A class of representations of A_α. A separable representation ρ of $C(\mathbf{T})$ is of multiplicity m, $m \in N \cup \{\infty\}$, if $\rho(C(\mathbf{T}))'$ is a type I_m von Neumann algebra. Given ν a regular Borel measure on $\mathbf{T}$, let M_f denote the operator on the Hilbert space $L^2(\mathbf{T}, \nu)$ assigning fg to g; $g \in L^2(\mathbf{T}, \nu)$, $f \in L^\infty(\mathbf{T}, \nu)$. For H_m a Hilbert space of dimension m, the representation ρ_m of $C(\mathbf{T})$ defined by mapping f to $M_f \otimes \mathrm{Id}_{H_m}$ is a uniform multiplicity m representation on the Hilbert space $L^2(\mathbf{T}, \nu) \otimes H_m$. Conversely **[10]** if ρ is a representation of $C(\mathbf{T})$ of uniform multiplicity m, there is a regular Borel measure ν on $\mathbf{T}$, uniquely determined up to equivalence of measures, such that ρ is unitarily equivalent to the representation ρ_m above.

We consider those separable representations π of A_α whose restriction to $C(\mathbf{T})$ is of uniform multiplicity m, so we may suppose there is a regular Borel measure ν on $\mathbf{T}$ with $\pi|_{C(\mathbf{T})} = \rho_m$. The measure ν is necessarily quasi-invariant under $\mathbf{Z}$, for if $f \in C(\mathbf{T})$ we have $\operatorname{ad} \pi(U)(\rho_m(f)) = \rho_m(\alpha(f))$, so $\rho_m(f) = 0$ iff $\rho_m(\alpha(f)) = 0$. The regularity of ν ensures that $\nu_\alpha \sim \nu$.

Our interest in singling out those representations π of A_α whose restriction to $C(\mathbf{T})$ is of uniform multiplicity lies in the fact that any factor representation is such. The argument ([**11**] §3.8) shows that the projections occurring in the central decomposition of the type I von Neumann algebra $\pi(C(\mathbf{T}))'$ are invariant under $\operatorname{ad} \pi(U)$ and so lie in the center of $\pi(A_\alpha)''$. A short argument shows that the measure ν in this case must be ergodic, for if $f \in L^\infty(\mathbf{T}, \nu)$ with $\alpha(f) = f$, then

$$\operatorname{ad} \pi(U)\big(M_f \otimes \mathrm{Id}_{H_m}\big) = M_{\alpha(f)} \otimes \mathrm{Id}_{H_m} = M_f \otimes \mathrm{Id}_{H_m}.$$

Thus $M_f \otimes \mathrm{Id}_{H_m}$ lies in the center of $\pi(A_\alpha)'$ and f is constant (a.e. ν).

Given m in $\mathbf{N} \cup \{\infty\}$ and ν a quasi invariant regular Borel measure on $\mathbf{T}$, we may form a covariant representation $(\rho_m, \tilde{U} \otimes \mathrm{Id}_{H_m})$ of $(C(\mathbf{T}), \alpha, \mathbf{Z})$ by

$$\tilde{U}k = \sqrt{d\nu_\alpha/d\nu}\,\alpha(k) \qquad (k \in L^2(\mathbf{T}, \nu)) \qquad ([\mathbf{17}, \mathbf{19}]).$$

When there is no possibility of confusion, we write $\tilde{U}$ instead of $\tilde{U} \otimes \mathrm{Id}_{H_m}$. Note that $n \to \operatorname{ad} \tilde{U}_n$ is then an action of $\mathbf{Z}$ by *-automorphisms on the $m \times m$ matrix algebra over $L^\infty(\mathbf{T}, \nu)$, $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$.

We recall ([**19**]) that a map b of $\mathbf{Z}$ into the (non-abelian) group of unitaries in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ is a unitary 1-cocycle if $b_{n+r} = b_n \operatorname{ad} \tilde{U}_n(b_r)$ for n, r in $\mathbf{Z}$. Two such cocycles b, c are equivalent if there is a unitary Y in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ with $b_n = Y^{-1}c_n \operatorname{ad} \tilde{U}_n(Y)$. With m and ν fixed, there is a one-to-one correspondence between unitary equivalence classes of covariant representations (ρ_m, T) of $(C(\mathbf{T}), \alpha, \mathbf{Z})$ and equivalence classes of unitary 1-cocycles b of $\mathbf{Z}$ in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ given by $T_n = b_n(\tilde{U}_n \otimes \mathrm{Id}_{H_m})$ $(n \in \mathbf{Z})$ ([**19**]).

Thus (unitary equivalence classes of) representations π of A_α with $\pi|_{C(\mathbf{T})}$ of uniform multiplicity are completely described by triples (m, ν, b) with m in $N \cup \{\infty\}$, ν a quasi-invariant Borel measure (class) and b a cocycle (equivalence class). In particular any factor representation of A_α may be described by such a triple.

Suited to the study of factor representations is the notion of quasi-equivalence. Let π and $\tilde{\pi}$ be two representations of a C^*-algebra $\mathfrak{A}$ and $\mathfrak{R}$, $\tilde{\mathfrak{R}}$ the von Neumann algebras generated by $\pi(\mathfrak{A})$, $\tilde{\pi}(\mathfrak{A})$, respectively. Recall π is quasi-equivalent to $\tilde{\pi}$ (write $\pi \approx \tilde{\pi}$) if there is a *-isomorphism $\varphi\colon \mathfrak{R} \to \tilde{\mathfrak{R}}$ with $\tilde{\pi}(x) = \varphi(\pi(x))$ for all x in $\mathfrak{A}$, or, equivalently, if $\bigoplus^{c_1} \pi$ is unitarily equivalent to $\bigoplus^{c_2} \tilde{\pi}$ for some c_1, $c_2 \in \mathbf{N} \cup \{\infty\}$ (for π, $\tilde{\pi}$ separable) [**4**].

If π and $\tilde{\pi}$ are two separable factor representations of A_α that are quasi-equivalent, then we see (as a *-isomorphism of von Neumann algebras is weakly bicontinuous on the unit balls of the algebras) that $\pi|_{C(\mathbf{T})}$ is quasi-equivalent to $\tilde{\pi}|_{C(\mathbf{T})}$. Now these representations of $C(\mathbf{T})$ are of uniform multiplicity (as π, $\tilde{\pi}$ are factor representations). Thus the measures ν and $\tilde{\nu}$ on $\mathbf{T}$ arising from these representations of $C(\mathbf{T})$ must be equivalent [**10**]. We can therefore assume, as far as quasi-equivalence of factor representations of A_α are concerned, that $\nu = \tilde{\nu}$.

PROPOSITION 2.1. *Let π, $\tilde{\pi}$ be two separable factor representations of A_α. Let (ρ_m, T) and $(\rho_{\tilde{m}}, \tilde{T})$ be the associated covariant representations on $L^2(\mathbf{T}, \nu) \otimes H_m$ and $L^2(\mathbf{T}, \tilde{\nu}) \otimes H_{\tilde{m}}$, respectively, where ν, $\tilde{\nu}$ are ergodic quasi-invariant regular Borel measures on $\mathbf{T}$ and $T_n = b_n(\tilde{U} \otimes \mathrm{Id}_{H_m})_n$, $\tilde{T}_n = \tilde{b}_n(\tilde{U} \otimes \mathrm{Id}_{H_{\tilde{m}}})_n$ with b_n, $\tilde{b}_n$ unitary cocycles. Then $\pi \approx \tilde{\pi}$ iff $\nu \sim \tilde{\nu}$ (so by the preceding comments we can assume $\nu = \tilde{\nu}$), and there exist c_1, c_2 in $\mathbf{N} \cup \{\infty\}$ with $mc_1 = \tilde{m}c_2 = r$ and a unitary W in $\mathfrak{M}_r(L^\infty(\mathbf{T}, \nu))$ on the Hilbert space $L^2(\mathbf{T}, \nu) \otimes H_r$ such that*

$$W\left(\bigoplus^{c_1} b_1\right) = \left(\bigoplus^{c_2} \tilde{b}_1\right)\mathrm{ad}\left(\tilde{U} \otimes \mathrm{Id}_{H_r}\right)(W).$$

Proof. We have $\pi \approx \tilde{\pi}$ iff there are c_1, c_2 with $mc_1 = \tilde{m}c_2 = r$ and a unitary W on $L^2(\mathbf{T}, \nu) \otimes H_r$ with $W(\oplus^{c_1}\pi) = (\oplus^{c_2}\tilde{\pi})W$. This is equivalent to

$$W\left(\bigoplus^{c_1} \rho_m\right) = \left(\bigoplus^{c_1} \rho_{\tilde{m}}\right)W$$

and

$$W\left(\bigoplus^{c_1} b_n\left(\tilde{U} \otimes \mathrm{Id}_{H_m}\right)_n\right) = \left(\bigoplus^{c_2} \tilde{b}_n\left(\tilde{U} \otimes \mathrm{Id}_{H_{\tilde{m}}}\right)_n\right)W.$$

The first equation means

$$W \in \rho_r(C(\mathbf{T}))' = \mathfrak{M}_r(L^\infty(\mathbf{T}, \nu)),$$

and the second is equivalent to

$$W\left(\bigoplus^{c_1} b_1\right) = \left(\bigoplus^{c_2} \tilde{b}_1\right)\mathrm{ad}\left(\tilde{U} \otimes \mathrm{Id}_{H_r}\right)W. \qquad \square$$

We return to examining covariant representations (ρ_m, T) of $(C(\mathbf{T}), \alpha, \mathbf{Z})$, where $m \in \mathbf{N} \cup \{\infty\}$ and ν is a regular Borel measure of $\mathbf{T}$. We suppose from now on, without loss of generality, that $\nu(\mathbf{T}) = 1$. The next proposition shows that for m finite we always have a type I representation.

PROPOSITION 2.2. *Let π denote the representation of A_α corresponding to a covariant representation (ρ_m, T) of $(C(\mathbf{T}), \alpha, \mathbf{Z})$ where $m \in \mathbf{N}$ and ν is a regular Borel measure of $\mathbf{T}$. If $\mathfrak{R}$ is the von Neumann algebra $\{\pi(x) \mid x \in A_\alpha\}''$, then $\mathfrak{R}$ is type I. Furthermore there are central projections P_s, $s \in \{1, \ldots, m\}$, of $\mathfrak{R}$ with $\Sigma P_s = 1$ and $\mathfrak{R}'P_s$ of type I_s or $P_s = 0$.*

Proof. To show $\mathfrak{R}$ is of type I it is enough to show that $\mathfrak{R}'$ is type I. If a bounded operator C on $L^2(\mathbf{T}, \nu) \otimes H_m$ lies in $\mathfrak{R}'$, then $C \in \rho_m(C(\mathbf{T}))' = \mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$. Thus $\mathfrak{R}' \subseteq \mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$, which is a type I_m von Neumann algebra. It is well known that $\mathfrak{R}'$ must then be a direct sum of type I_s $(s \leq m)$ von Neumann algebras ($\mathfrak{R}'$ inherits a faithful scalar trace from $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$). □

Let π be a representation of A_α with $\pi|_{C(\mathbf{T})}$ a uniform multiplicity representation. Conditions on the triple (m, ν, b) associated with π are reflected in the structure of the von Neumann algebra $\mathfrak{R} = \pi(A_\alpha)''$. We have seen, for example, that $\mathfrak{R}$ is type I if $m < \infty$. Another case considered below is when ν is ergodic and b is a "diagonal cocycle". In this case $\mathfrak{R}$ will be type I and we can say exactly when $\mathfrak{R}$ is a factor by examining an equivalence relation on $\{1, \ldots, m\}$. Note $m = \infty$ is not excluded in these considerations.

We say an operator in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ is diagonal if it lies in the (von Neumann) subalgebra $\oplus^m L^\infty(\mathbf{T}, \nu)$. A unitary cocycle b of $\mathbf{Z}$ in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ will be called diagonal if b_1 is diagonal.

LEMMA 2.3. *Let π be a representation of A_α corresponding to a triple (m, ν, b) where, in addition, ν is ergodic and b is a diagonal cocycle. Let $b_1 = \oplus^m \lambda_j$ with $\lambda_j \in L^\infty(\mathbf{T}, \mu)$ and let $\mathfrak{R} = \pi(A_\alpha)''$. We have a relation $\sim$ on $\{r \in \mathbf{N} \mid 1 \leq r \leq m\}$ if $m < \infty$ and on $\mathbf{N}\backslash\{0\}$ if $m = \infty$ by defining $r \sim s$ iff there is a non-zero element d of $L^\infty(\mathbf{T}, \nu)$ with $\alpha(d) = \bar{\lambda}_r \lambda_s d$. The following hold.*

(1) *The relation $\sim$ is an equivalence relation.*

(2) *If $C \in \mathfrak{R}'$ and $j \nsim k$ then the jk entry of C (viewed as a matrix in $\mathfrak{B}(L^2(\mathbf{T}, \nu) \otimes H_m)$), c_{jk}, is zero.*

(3) *If $A \in \mathfrak{R} \cap \mathfrak{R}'$, the center of $\mathfrak{R}$, and $j \sim k$ with $j \neq k$, then $a_{jj} = a_{kk}$ and $a_{jk} = 0$.*

Proof. (1) Call d a "(j, k) solution" if d is a non-zero element of $L^\infty(\mathbf{T}, \nu)$ with $\alpha(d) = \bar{\lambda}_j \lambda_k d$. As b is unitary, we have $|\lambda_j| = 1$, so $j \sim j$, as constants are (j, j) solutions. If d is a (j, k) solution, $\bar{d}$ is a (k, j) solution. Note that $|d|^2 = d\bar{d}$ is a (k, k) solution and thus a (non-zero) constant as ν is ergodic. Thus if d and h are (j, k), (k, r) solutions, respectively, dh is non-zero and so a (j, r) solution.

(2) A bounded operator C on $L^2(\mathbf{T}, \nu) \otimes H_m$ lies in $\mathfrak{R}'$ if $C \in \rho_m(C(\mathbf{T}))' = \mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ and $\operatorname{ad}(b_1\tilde{U})(C) = C$. The latter equation holds iff $\operatorname{ad}\tilde{U}(C) = b_1^* C b_1$ iff $\alpha(c_{jk}) = \bar{\lambda}_j \lambda_k c_{jk}$ for each j, k. By definition $c_{jk} = 0$ if $j \nsim k$.

(3) Let d be a (j, k) solution. We define an element C of $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ as follows: the j, k entry of C is d, c_{jj}, c_{kk} are two arbitrary constants and $c_{rs} = 0$ for all other r, s. As $\alpha(c_{rs}) = \bar{\lambda}_r \lambda_s c_{rs}$ for all r, s, $C \in \mathfrak{R}'$. As A commutes with the elements C, we have, upon examining j, k entries from the matrix equation $AC = CA$, $(a_{jj} - a_{kk})c_{jk} = a_{jk}(c_{jj} - c_{kk})$. We may vary the constant $c_{jj} - c_{kk}$ at will, while the left side of the equation remains unchanged. Thus $a_{jk} = 0$ (a.e. ν). Now $c_{jk} = d$ is non-zero and a_{jj}, a_{kk} are constants (as $A \in \mathfrak{R}'$). Thus $a_{jj} - a_{kk} = 0$. $\square$

PROPOSITION 2.4. *With the hypothesis of the previous lemma, the following are true.*

(1) *If $k \nsim j$ for all $k \neq j$ then $\mathfrak{R}' = \oplus^m \mathbf{C}$ so $\mathfrak{R} = \oplus^m \mathfrak{B}(L^2(\mathbf{T}, \nu))$.*

(2) *If $k \sim j$ for all k, j then $\mathfrak{R}'$ is a type I_m factor.*

(3) *If $r \nsim s$, some $r \neq s$, then $\mathfrak{R}$ is not a factor.*

(4) *$\mathfrak{R}$ is type* I.

Proof. (1) For C in $\mathfrak{R}'$ we have c_{kk} is a constant for each k as ν is ergodic. By the lemma, $c_{rs} = 0$ for all r, s with $r \neq s$. Thus $\mathfrak{R}' = \oplus^m \mathbf{C}$ and $\mathfrak{R} = \oplus^m \mathfrak{B}(L^2(\mathbf{T}, \nu))$.

(2) The lemma shows for A in $\mathfrak{R} \cap \mathfrak{R}'$ that $a_{rs} = 0$ and $a_{rr} = a_{ss}$ (is a constant) for all r, s with $r \neq s$. Thus $\mathfrak{R}$ is a factor. Let E be the element of $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ described by defining all entries e_{rs} to be zero except the (j, k) entry which is a (j, k) solution d with $|d| = 1$ (a.e. ν). Again, as $\alpha(e_{rs}) = \bar{\lambda}_r \lambda_s e_{rs}$, E is in $\mathfrak{R}'$. The (k, k) entry of E^*E, $d\bar{d}$, is equal to 1 (a.e. ν), while all other entries are zero. Thus E is a partial isometry with initial projection P_k and final projection P_j. Here P_r denotes the projection in $\mathfrak{R}'$ that has all zero entries except the (r, r) diagonal entry, which is 1. The projection P_r is abelian in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$, thus abelian in $\mathfrak{R}'$. As $\sum_1^m P_r = \mathrm{Id}$, $\mathfrak{R}'$ is type I_m.

(3) We display non-scalar elements A in the center of $\mathfrak{R}$. If h_1, h_2 are two distinct constants, define $a_{jj} = h_1$ if $r \sim j$, $a_{jj} = h_2$ if $j \sim s$ and $a_{pq} = 0$, all other p, q. As $r \nsim s$, a_{jj} is well defined. Note that A is not a

scalar and, since $\oplus^m \mathbf{C} \subseteq \mathfrak{R}'$, $A \in \mathfrak{R}'$. To conclude that $A \in \mathfrak{R}$, we prove $AC = CA$ for all C in $\mathfrak{R}'$. We show that the (p, q) entry of AC,

$$a_{pp}c_{pq} = \begin{cases} h_1 C_{pq} & \text{iff } r \sim p, \\ h_2 C_{pq} & \text{iff } p \sim s, \\ 0 & \text{otherwise,} \end{cases}$$

is equal to the (p, q) entry of CA,

$$c_{pq}a_{qq} = \begin{cases} h_1 C_{pq} & \text{iff } r \sim q, \\ h_2 C_{pq} & \text{iff } q \sim s, \\ 0 & \text{otherwise.} \end{cases}$$

If $r \sim p$ and $r \nsim q$, then $p \nsim q$ so $c_{pq} = 0$. Similarily, if $p \sim s$ and $q \nsim s$, then $p \nsim q$ and $c_{pq} = 0$, and we have $a_{pp}c_{pq} = c_{pq}a_{qq}$. If both $r \nsim p$ and $p \nsim s$, then we must show $c_{pq}a_{qq} = 0$. This can be non-zero only if $r \sim q$ or $q \sim s$. In both cases, $p \nsim q$ (otherwise $r \sim p$ or $p \sim s$, contradicting our present assumption). Thus $c_{pq} = 0$ and $c_{pq}a_{qq} = 0$.

(4) Changing to an equivalent cocycle by a permutation unitary, we may assume that if $j \sim k$, then $j \sim s$ for all s with $j \leq s \leq k$. Combining the arguments in (2) and (3) shows $\mathfrak{R}'$ is a direct sum of type I factors. □

An example where $\mathfrak{R}$ is a factor is $b_1 = \mathrm{Id}$. For an example with $\mathfrak{R}$ not a factor let $\nu =$ Haar measure on $\mathbf{T}$ and $b_1 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then $1 \nsim 2$, for if $d \in L^\infty(\mathbf{T}, \nu)$ with $\alpha(d) = -d$, then $\alpha_2(d) = d$. As $2\alpha \bmod 1$ is still irrational, d is a constant. As $d = -d$, $d = 0$.

Suppose the 1-cocycle b is cohomologous to a diagonal cocycle, i.e., there is a unitary Y in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ such that $Y^{-1}b_n \operatorname{ad} \tilde{U}_n(Y)$ is a diagonal cocycle. Then the covariant representation (ρ_m, T) is unitarily equivalent to a covariant representation with a diagonal cocycle. A result of Kadison **[9]** shows that if m is finite, there is a unitary W in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ such that $W^* b_1 W$ is diagonal. Note that, in general, this is not enough to ensure that our cocycle is cohomologous to a diagonal cocycle. However, if the unitary W had constant entries, i.e., $W \in \mathfrak{M}_m(\mathbf{C}) \otimes \mathrm{Id}_{L^2(\mathbf{T},\nu)}$, then $\operatorname{ad} \tilde{U}_n(W) = W$ and the cocycle b is cohomologous to a diagonal cocycle. If m is finite and b_1 is a unitary in $\mathfrak{M}_m(\mathbf{C}) \otimes \mathrm{Id}_{L^2(\mathbf{T},\nu)}$, such a unitary W with constant entries always exists.

If the measure ν is ergodic, the unitary b_1 has constant entries iff T_1 commutes with $\tilde{U}_1$. This follows from the fact that T_1 commutes with $\tilde{U}_1$ iff $\operatorname{ad} \tilde{U}_1(b_1) = b_1$ iff $\alpha((b_1)_{jk}) = (b_1)_{jk}$ for each j, k.

3. Automorphisms of A_α. A representation of the discrete group $SL(2, \mathbf{Z})$ as automorphisms of A_α arises naturally when viewing A_α as a twisted covariance algebra [6]. We also investigate a few simple properties of the group homomorphism of $\mathrm{Aut}(A_\alpha)$ to $\mathrm{Aut}(K_1(A_\alpha))$ defined by mapping an automorphism γ of A_α to γ_*, the induced isomorphism of the abelian group $K_1(A_\alpha)$. We show that the kernel of this map, $\gamma \to \gamma_*$, is closed and contains the centrally trivial automorphisms.

If G is a separable locally compact group and $\mathbf{T}$ the circle group, a Borel function ω from $G \times G$ to $\mathbf{T}$ is a *multiplier* (a 2-cocycle on G with coefficients in $\mathbf{T}$) [11] if

(a) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z), (x, y, z) \in G$;

(b) $\omega(x, e) = \omega(e, x) = 1$ with e the unit of G, $x \in G$.

Given a multiplier ω on G, an *ω-representation of G* is a strongly continuous map L of G to the unitary group on a separable Hilbert space with $L_e = \mathrm{Id}$ and $L_x L_y = \omega(x, y)L_{xy}$ for x, y in G.

Define a multiplier on the discrete group $G = \mathbf{Z} \times \mathbf{Z}$ by

$$\omega((m, n), (m', n')) = \exp[\pi i\alpha(m'n - mn')] = \exp\left[\pi i\alpha\begin{vmatrix} m' & m \\ n' & n \end{vmatrix}\right],$$

where $\left|\begin{smallmatrix} m' & m \\ n' & n \end{smallmatrix}\right|$ denotes the determinant of the matrix $\left[\begin{smallmatrix} m' & m \\ n' & n \end{smallmatrix}\right]$, m, n, m', n' in $\mathbf{Z}$. For g in $SL(2, \mathbf{Z})$, we note $\omega(g(m, n), g(m', n')) = \omega((m, n)(m', n'))$, where $SL(2, \mathbf{Z})$ acts on $\mathbf{Z} \times \mathbf{Z}$ in the usual manner. We form the Mackey obstruction group, a locally compact separable group $G(\omega)$ with underlying set $\{(x, t) \mid x \in G, t \in \mathbf{T}\}$ and multiplication given by $(x, t)(y, t') = (xy, \omega(x, y)tt')$ for x, y in G and t, t' in $\mathbf{T}$. The usefulness of $G(\omega)$ lies in the fact that the ω-representations of G, L, are in one-to-one correspondence with the unitary representations W of $G(\omega)$ restricting on T to a multiple of the one-dimensional representation $t \to t$. The correspondence is given by $W(x, t) = tL_x$ [11].

Let $C_c(G(\omega), \mathbf{T})$ be the set of functions from $G(\omega)$ to $\mathbf{C}$ with $f(tx) = t^{-1}f(x)$ $(t \in \mathbf{T}, x \in G(\omega))$ such that $f(m, n, t)$ is non-zero for only finitely many m, n in $\mathbf{Z}$. Define a normed *-algebra structure on $C_c(G(\omega), \mathbf{T})$ as follows: for f, h in $C_c(G(\omega), \mathbf{T})$,

$$\|f\| = \sum_{m, n \in \mathbf{Z}} |f(m, n, 1)|,$$

$$f^*(m, n, t) = \overline{f(-m, -n, t^{-1})},$$

$$f * h(m, n, t) = \sum_{a, b \in \mathbf{Z}} f(a, b, 1)h[(-a, -b, 1)(m, n, t)].$$

The enveloping C^*-algebra of the completion of this normed *-algebra, denoted $C^*(G(\omega),\mathbf{T})$, is a twisted covariance algebra. There is a one-to-one correspondence between *-representations of $C^*(G(\omega),\mathbf{T})$ and representations of $G(\omega)$ restricting to a multiple of $t \to t$ on $\mathbf{T}$ [**6**].

The C^*-algebra $C^*(G(\omega),\mathbf{T})$ is *-isomorphic to A_α. If $f_{(m,n)}$ is the element of $C_c(G(\omega),\mathbf{T})$ defined by mapping all elements to zero except (m, n, t), which is mapped to $t^{-1}\xi(m, n)$, where $\xi(m, n) = \exp(\pi i\alpha mn)$, we have

$$f_{(m,n)} * f_{(\tilde{m},\tilde{n})} = \exp(2\pi i\alpha\tilde{m}n)f_{(m+\tilde{m},n+\tilde{n})}.$$

Thus $f_{(1,0)}$ and $f_{(0,1)}$ are two unitaries V, U, respectively, with $UV = e^{2\pi i\alpha}VU$. The C^*-algebra generated by $f_{(1,0)}$ and $f_{(0,1)}$ is $C^*(G(\omega),\mathbf{T})$. Thus $C^*(G(\omega),\mathbf{T}) \cong A_\alpha$.

For g in $\mathrm{SL}(2,\mathbf{Z})$ let $g(m,n) = (\tilde{m},\tilde{n})$ denote the usual action of $\mathrm{SL}(2,\mathbf{Z})$ on $\mathbf{Z}\times\mathbf{Z}$. Let γ_g be a map of $G(\omega)$ to $G(\omega)$ defined by $\gamma_g(m, n, t) = (g(m, n), t)$ $(g \in \mathrm{SL}(2,\mathbf{Z}))$. As $\omega(g(m, n), g(p, q)) = \omega((m, n)(p, q))$ for m, n, p, q in $\mathbf{Z}$, γ_g is a group automorphism of $G(\omega)$. The relation $\gamma_{g_1g_2} = \gamma_{g_1}\gamma_{g_2}$ for g_1, g_2 in $\mathrm{SL}(2,\mathbf{Z})$ shows that $\gamma\colon \mathrm{SL}(2,\mathbf{Z}) \to \mathrm{Aut}_{\mathrm{group}}(G(\omega))$ is a group homomorphism. Define an action of $\mathrm{SL}(2,\mathbf{Z})$ by isometric *-automorphisms of the normed *-algebra $C_c(G(\omega),\mathbf{T})$ by ${}^gf(x) = f(\gamma_{g^{-1}}(x))$ for f in $C_c(G(\omega),\mathbf{T})$, g in $\mathrm{SL}(2,\mathbf{Z})$, x in $G(\omega)$. This action extends to an action $g \to \beta_g$ of $\mathrm{SL}(2,\mathbf{Z})$ by *-automorphisms of the enveloping C^*-algebra A_α. For later use we note that

$$\begin{aligned}\beta_g(V^mU^n) = \beta_g(f_{(m,n)}) &= \xi(m,n)^{-1}\xi(\tilde{m},\tilde{n})f_{(\tilde{m},\tilde{n})}\\ &= \xi(m,n)^{-1}\xi(\tilde{m},\tilde{n})V^{\tilde{m}}U^{\tilde{n}}.\end{aligned}$$

If $\mathfrak{A}$ is a C^*-algebra with unit, let $U_n(\mathfrak{A})$ be the unitary group of $\mathfrak{M}_n(\mathfrak{A})$ and $U_n^0(\mathfrak{A})$ the connected component of the Id in $U_n(\mathfrak{A})$. We define $K_1(\mathfrak{A}) = \varinjlim U_n(\mathfrak{A})/U_n^0(\mathfrak{A})$, the direct limit of the direct system of groups $(U_n(\mathfrak{A})/U_n^0(\mathfrak{A}), \varphi_n)$, where $\varphi_n\colon U_n(\mathfrak{A})/U_n^0(\mathfrak{A}) \to U_{n+1}(\mathfrak{A})/U_{n+1}^0(\mathfrak{A})$ is given by $x \to \left(\begin{smallmatrix} x & 0\\ 0 & 1\end{smallmatrix}\right)$. For d in $U_n(\mathfrak{A})$ we denote by $[d]$ its image in $K_1(\mathfrak{A})$. K_1 is a functor from the category of C^*-algebras (with unit, although this is not necessary) to the category of abelian groups [**18**].

Let U, V be two unitary operators with $UV = e^{2\pi i\alpha}VU$ that generate the C^*-algebra A_α. It is known ([**13**]) that $K_1(A_\alpha) = \mathbf{Z}\oplus\mathbf{Z}$, where $[U]$, $[V]$ correspond to $(0,1)$ and $(1,0)$, respectively. Examining the automorphisms β_g $(g \in \mathrm{SL}(2,\mathbf{Z}))$ of A_α, it is clear that $(\beta_g)_*$ is just the map g on $\mathbf{Z}\oplus\mathbf{Z}$.

Putting the topology of pointwise norm convergence on $\mathrm{Aut}(A_\alpha)$ (a net $\gamma_j \to \gamma$ iff $\|\gamma_j(x) - \gamma(x)\| \to 0$ for each x in A_α), we have that the subgroup $\{\gamma \in \mathrm{Aut}(A_\alpha) \mid \gamma_* = \mathrm{Id}\}$ is closed. To see this, suppose $\gamma_j \to \gamma$ in $\mathrm{Aut}(A_\alpha)$ with $(\gamma_j)_* = \mathrm{Id}$ for all j. Then there is a j_0 with $\|\gamma_{j_0}(U) - \gamma(U)\| < \varepsilon$ and $\|\gamma_{j_0}(V) - \gamma(V)\| < \varepsilon$, where ε is chosen small enough to ensure that $\gamma_{j_0}(U)$ (resp. $\gamma_{j_0}(V)$) lie in the path component of $\gamma(U)$ (resp. $\gamma(V)$) in $U_1(A_\alpha)$. Thus $[\gamma_{j_0}(U)] = [\gamma(U)]$ and $[U] = (\gamma_{j_0})_*\,[U] = [\gamma(U)]$. It is also seen that $[V] = [\gamma(V)]$ and thus $\gamma_* = \mathrm{Id}$.

DEFINITION. A norm bounded sequence $\{x_n\}$ in a C^*-algebra $\mathfrak{A}$ is *(norm) central sequence* if $\|x_n a - a x_n\| \to 0$ as $n \to \infty$ for each a in $\mathfrak{A}$. An automorphism γ of $\mathfrak{A}$ is *centrally trivial* (i.e., $\gamma \in \mathrm{Ct}(\mathfrak{A})$) iff $\|\gamma(x_n) - x_n\| \to 0$ as $n \to \infty$ for each central sequence x_n in $\mathfrak{A}$ [3], [7].

Note that if $\gamma \in \mathrm{Aut}(\mathfrak{A})$ and $\{x_n\}$ is a central sequence, then $\{\gamma(x_n)\}$ is again a central sequence. It follows that $\mathrm{Ct}(\mathfrak{A})$ is a subgroup of $\mathrm{Aut}(\mathfrak{A})$. Also $\mathrm{Ct}(\mathfrak{A})$ contains $\mathrm{Int}(\mathfrak{A})$, the group of inner automorphisms of $\mathfrak{A}$, for if W is a unitary in $\mathfrak{A}$ with $\gamma(a) = WaW^*$ $(a \in \mathfrak{A})$, then

$$\|\gamma(x_n) - x_n\| = \|W x_n W^* - x_n\| = \|W x_n - x_n W\| \to 0 \qquad (n \to \infty)$$

for any central sequence x_n.

In the C^*-algebra A_α there are ready examples of central sequences. If $p_j \in \mathbf{N}\backslash\{0\}$ is a sequence with $e^{2\pi i p_j \alpha} \to 1$ as $j \to \infty$, then both U^{p_j}, V^{p_j} are central sequences. For example

$$\|U^{p_j}V - VU^{p_j}\| = \|U^{p_j}VU^{-p_j} - V\| = \left\|(e^{2\pi i p_j \alpha} - 1)V\right\| \to 0 \quad \text{as } j \to \infty.$$

Thus $\|U^{p_j}x - xU^{p_j}\| \to 0$ for x in the $*$-algebra generated by U and V, and as this algebra is dense in A_α, this holds for all x in A_α.

PROPOSITION 3.1. *If* $\gamma \in \mathrm{Ct}(A_\alpha)$ *then* $\gamma_* = \mathrm{Id}$.

Proof. Fix a sequence $p_j \in \mathbf{N}\backslash\{0\}$ with $e^{2\pi i p_j \alpha} \to 1$ as $j \to \infty$. As γ is in $\mathrm{Ct}(A_\alpha)$, we must have $\|\gamma(U^{p_j}) - U^{p_j}\| \to 0$ and $\|\gamma(V^{p_j}) - V^{p_j}\| \to 0$ as $j \to \infty$. Thus for j large enough, $\gamma(U^{p_j})U^{-p_j} \in U_1^0(A_\alpha)$, so

$$[1] = [\gamma(U^{p_j})U^{-p_j}] = p_j\left(\left[\gamma_*(U)\right] - [U]\right).$$

As $K_1(A_\alpha)$ is torsion free, $\gamma_*[U] = [U]$. Similarly, $\gamma_*[V] = [V]$. □

We have seen that $(\beta_g)_*$ is the map g on $\mathbf{Z} \oplus \mathbf{Z}$. Thus β_g is not in the closure of $\mathrm{Ct}(A_\alpha)$ for $g \neq \mathrm{Id}$.

An interesting question is whether the image of the map $\varphi \to \varphi_*$ of $\mathrm{Aut}(A_\alpha)$ to $\mathrm{Aut}(\mathbf{Z} \times \mathbf{Z})$ is all of $GL(2, \mathbf{Z})$ or exactly $SL(2, \mathbf{Z})$. In other words, is there an automorphism φ of A_α with φ_* of determinant -1, in particular with $\varphi_* = [\begin{smallmatrix}0&1\\1&0\end{smallmatrix}]$? Rewording this amounts to asking if there are two unitaries, $\varphi(U)$ and $\varphi(V)$ say, of A_α with commutator $[\varphi(U), \varphi(V)]$ $(= \varphi(U)\varphi(V)\varphi(U)^{-1}\varphi(V)^{-1}) = \exp(2\pi i\alpha)$, and such that the equivalence class $[\varphi(U)] = [V]$ and $[\varphi(V)] = [U]$ (where U, V are generating unitaries of A_α with $[U, V] = \exp(2\pi i\alpha)$).

4. Pure states of A_α. We define for each θ in $[0, 1]$ a pure state φ_θ of A_α. We show φ_θ is unitarily equivalent to $\varphi_{\theta'}$ (i.e., there is a unitary W in A_α with $\varphi_{\theta'} \circ \operatorname{ad} W = \varphi_\theta$) iff $e^{2\pi i\theta'} \in \{e^{2\pi i(\theta+n\alpha)} \mid n \in \mathbf{Z}\}$ (that is, iff θ' and θ lie in the same orbit of the α action of $\mathbf{Z}$ on $\mathbf{T}$), and also state similar conditions for $\varphi_{\theta'} \circ \beta_g$ to be unitarily equivalent to φ_θ (for g in $\mathrm{SL}(2, \mathbf{Z})$). We show, too, that the set of pure states $\{\varphi_\theta \circ \beta_g \mid \theta \in [0, 1],$ $g \in \mathrm{SL}(2, \mathbf{Z})\}$ separate the points of A_α.

Recalling the conditional expectation E: $A_\alpha \to C(\mathbf{T})$, we define a linear functional $\varphi_\theta = \rho_\theta \circ E$ for θ in $[0, 1]$ where ρ_θ is the (pure) state of $C(\mathbf{T})$ given by evaluation at the point $e^{2\pi i\theta}$ of $\mathbf{T}$. As $\varphi_\theta(1) = 1$ and

$$\varphi_\theta(x^*x) = \rho_\theta(E(x^*x)) = \int_{\mathbf{T}} \rho_\theta(\hat{\alpha}_t(x^*x))\, dt \geq 0$$

(since ρ_θ is positive), φ_θ is a state. The following proposition also follows from more general considerations [**2**].

PROPOSITION 4.1. *The state φ_θ is a pure state of A_α.*

Proof. We show that if η is a positive linear functional with $0 \leq \eta \leq \varphi_\theta$, then $\eta = \lambda\varphi_\theta$ for some λ in $\mathbf{C}$. For φ a positive linear functional on A_α, let $N(\varphi) = \{x \in A_\alpha \mid \varphi(x^*x) = 0\}$, the left kernel of φ. If K denotes the dense subalgebra $C_c(\mathbf{Z}, C(\mathbf{T}))$ of A_α, we show

$$(*) \qquad K \cap \varphi_\theta^{-1}(0) \subseteq K \cap N(\varphi_\theta) + K \cap N(\varphi_\theta)^*.$$

If $(*)$ is true then

$$\begin{aligned} K \cap \varphi_\theta^{-1}(0) &\subseteq K \cap N(\varphi_\theta) + K \cap N(\varphi_\theta)^* \\ &\subseteq K \cap N(\eta) + K \cap N(\eta)^* \subseteq K \cap \eta^{-1}(0), \end{aligned}$$

the last inclusion following from the Schwarz inequality. Thus $K \cap \varphi_\theta^{-1}(0)$ $\subseteq K \cap \eta^{-1}(0)$ and $\eta|_K = \lambda\varphi_\theta|_K$ for some $\lambda \in \mathbf{C}$. Continuity yields $\eta = \lambda\varphi_\theta$.

We prove the inclusion $(*)$.

$$\begin{aligned} K \cap N(\varphi_\theta) &= \{k \in K \mid \rho_\theta(k^*k(0)) = 0\} \\ &= \left\{k \in K \mid \varphi_\theta\left(\sum \alpha_n(|k(-n)|^2)\right) = 0\right\} \\ &= \left\{k \in K \mid k(-n)(e^{2\pi i(\theta+n\alpha)}) = 0, n \in \mathbf{Z}\right\}. \end{aligned}$$

Also

$$K \cap N(\varphi_\theta)^* = \{k \in K \mid \varphi_\theta(kk^*(0)) = 0\}$$
$$= \{k \in K \mid k(n)(e^{2\pi i\theta}) = 0, n \in \mathbf{Z}\}.$$

Now fix a k in $K \cap \varphi_\theta^{-1}(0)$. As $e^{2\pi i(\theta+n\alpha)} \neq e^{2\pi i\theta}$ for all n in $\mathbf{Z}\setminus\{0\}$, there are $f(n)$, $g(n)$ in $C(\mathbf{T})$ with $k(n) = f(n) + g(n)$ and $f(n)(e^{2\pi i\theta}) = g(n)(e^{2\pi i(\theta-n\alpha)}) = 0$ for all $n \in \mathbf{Z}\setminus\{0\}$. For example, if $n \in \mathbf{Z}\setminus\{0\}$ let $g(n) = k(n)h(n)$ and $f(n) = k(n)(1 - h(n))$, where $h(n)$ is an element of $C(\mathbf{T})$ with $h(n)(e^{2\pi i\theta}) = 1$ and $h(n)(e^{2\pi i(\theta-n\alpha)}) = 0$. Letting $f(0) = g(0) = \frac{1}{2}k(0)$ and defining f, g in $C_c(\mathbf{Z}, C(\mathbf{T}))$ in the obvious way, we have $f + g = k$, with $f \in K \cap N(\varphi_\theta)^*$ and $g \in K \cap N(\varphi_\theta)$. Thus the inclusion $(*)$ is true. □

Given θ, θ' in $[0, 1]$ with $e^{2\pi i\theta'} = e^{2\pi i(\theta+n\alpha)}$, for some n in $\mathbf{Z}$, then $\varphi_\theta \circ \operatorname{ad} U^n = \varphi_{\theta'}$ on A_α (as they agree on the dense subalgebra generated by U and V).

PROPOSITION 4.2. *For θ, θ' in* $[0, 1]$, $\|\varphi_\theta \circ \operatorname{ad} x - \varphi_{\theta'}\| \geq 1$ *for all x in A_α iff $e^{2\pi i\theta'} \notin \{e^{2\pi i(\theta+n\alpha)} \mid n \in \mathbf{Z}\}$.*

Proof. If $e^{2\pi i\theta'} = e^{2\pi i(\theta+n\alpha)}$ for some n in $\mathbf{Z}$, we have seen that there is an x, namely U^n, with $\varphi_\theta \circ \operatorname{ad} x = \varphi_{\theta'}$.

Now suppose $e^{2\pi i\theta'} \notin \{e^{2\pi i(\theta+n\alpha)} \mid n \in \mathbf{Z}\}$. For fixed x in $C_c(\mathbf{Z}, C(\mathbf{T}))$ we find b in $C(\mathbf{T})$ with $\varphi_\theta \circ \operatorname{ad} x(b) = 0$ and $\varphi_{\theta'}(b) = 1$. As x is in $C_c(\mathbf{Z}, C(\mathbf{T}))$, there is an m in $\mathbf{Z}$ with support $x \subseteq \{-m, \ldots, m\}$. The hypothesis on θ, θ' allows us to find b in $C(\mathbf{T})$ with $b \geq 0$, $\|b\| = 1$, $b(e^{2\pi i\theta'}) = 1$ and $b(e^{2\pi i(\theta+n\alpha)}) = 0$ for $n \in \{-m, \ldots, m\}$. Then

$$\varphi_\theta \circ \operatorname{ad} x(b) = \varphi_\theta(x * b * x^*(0))$$
$$= \sum_{n \in \{-m, \ldots, m\}} x(n)\alpha_n(b)\overline{x(n)}(e^{2\pi i\theta}) = 0$$

(as $\alpha_n(b)(e^{2\pi i\theta}) = 0$ for $n \in \{-m, \ldots, m\}$), while $\varphi_{\theta'}(b) = b(e^{2\pi i\theta'}) = 1$. Thus $\|\varphi_\theta \circ \operatorname{ad} x - \varphi_{\theta'}\| \geq 1$ for x in $C_c(\mathbf{Z}, C(\mathbf{T}))$. With x in A_α we choose a sequence x_j in $C_c(\mathbf{Z}, C(\mathbf{T}))$ converging to x in norm. Thus $\lim_{j\to\infty} \|\varphi_\theta \circ \operatorname{ad} x_j - \varphi_\theta \circ \operatorname{ad} x\| = 0$ and the proposition follows. □

If a state φ is unitarily equivalent to a state ψ, that is, if $\varphi \circ \operatorname{ad} W = \psi$ for some unitary W in A_α, we write $\varphi \sim \psi$.

COROLLARY 4.3. *For θ, θ' in* $[0, 1]$, $\varphi_\theta \sim \varphi_{\theta'}$ *iff $e^{2\pi i\theta'} \in \{e^{2\pi i(\theta+n\alpha)} \mid n \in \mathbf{Z}\}$.*

As β_g (for g in $\mathrm{SL}(2, \mathbf{Z})$) is a *-automorphism of A_α, the states $\varphi_\theta \circ \beta_g$ are pure. We examine what these states look like on the dense subalgebra $\{\sum_{\text{finite}} a_{pn} V^p U^n \mid a_{pn} \in \mathbf{C}\}$ of A_α. Recall

$$\beta_g(V^p U^n) = \xi(p, n)^{-1} \xi(\tilde{p}, \tilde{n}) V^{\tilde{p}} U^{\tilde{n}},$$

where $(\tilde{p}, \tilde{n}) = g(p, n)$ and $\xi(p, n) = e^{\pi i \alpha p n}$ (p, n in $\mathbf{Z}$). Thus

$$\beta_g\left(\sum a_{pn} V^p U^n\right) = \sum a_{pn} \xi(p, n)^{-1} \xi(\tilde{p}, \tilde{n}) V^{\tilde{p}} U^{\tilde{n}} = \sum c_{\tilde{p}\tilde{n}} V^{\tilde{p}} U^{\tilde{n}},$$

where

$$c_{\tilde{p}, \tilde{n}} = a_{g^{-1}(\tilde{p}, \tilde{n})} \xi\left(g^{-1}(\tilde{p}, \tilde{n})\right)^{-1} \xi(\tilde{p}, \tilde{n}).$$

We have

$$E\left(\sum a_{pn} V^p U^n\right) = \sum a_{p,0} V^p,$$

so

$$E\left(\beta_g\left(\sum a_{pn} V^p U^n\right)\right) = E\left(\sum c_{\tilde{p}\tilde{n}} V^{\tilde{p}} U^{\tilde{n}}\right) = \sum c_{\tilde{p},\tilde{0}} V^{\tilde{p}}.$$

Thus

$$\varphi_\theta\left(\beta_g\left(\sum a_{pn} V^p U^n\right)\right) = \sum_{\tilde{p}} c_{\tilde{p},0} e^{2\pi \imath \tilde{p} \theta}.$$

For $g = \left[\begin{smallmatrix} h & r \\ b & d \end{smallmatrix}\right]$ in $\mathrm{SL}(2, \mathbf{Z})$,

$$g^{-1} = \begin{bmatrix} d & -r \\ -b & h \end{bmatrix} \quad \text{and} \quad c_{\tilde{p},0} = a_{(d\tilde{p}, -b\tilde{p})} \xi(d\tilde{p}, -b\tilde{p})^{-1}.$$

Thus

$$\varphi_\theta\left(\beta_g\left(\sum a_{pn} V^p U^n\right)\right) = \sum_k a_{(dk, -bk)} \xi(dk, -bk)^{-1} e^{2\pi i k \theta}.$$

Note that if g is in the subgroup $H = \{\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] \mid n \in \mathbf{Z}\}$ of $\mathrm{SL}(2, \mathbf{Z})$, then $\varphi_\theta \circ \beta_g = \varphi_\theta$ on $\{\sum_{\text{finite}} a_{pn} V^p U^n \mid a_{pn} \in \mathbf{C}\}$, so $\varphi_\theta \circ \beta_g = \varphi_\theta$ on A_α. Also if g is of the form $\left[\begin{smallmatrix} -1 & n \\ 0 & -1 \end{smallmatrix}\right]$ for some n in $\mathbf{Z}$ (so $-g \in H$), we have $\varphi_\theta \circ \beta_g = \varphi_{1-\theta}$.

PROPOSITION 4.4. *If $\theta, \theta' \in [0, 1]$, $g \in SL(2, \mathbf{Z})$ and $\varphi_{\theta'} \circ \beta_g$ is unitarily equivalent to φ_θ, then g or $-g$ is in $H = \{\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] \mid n \in \mathbf{Z}\}$.*

Proof. Choose $\{B_n\}$ a sequence in $C_c(\mathbf{Z}, C(\mathbf{T}))$ with $B_n \to W$ (in norm), where W is a unitary in A_α with $\varphi_{\theta'} \circ \beta_g = \varphi_\theta \circ \operatorname{ad} W$. Thus

$$\sum_m |B_n(m)|^2 = E(B_n B_n^*) \to E(WW^*) = E(\mathrm{Id}) = \mathrm{Id}_{C(\mathbf{T})}$$

in the norm of $C(\mathbf{T})$ as $n \to \infty$. Let $\varepsilon > 0$ be given. Choose n_0 in $\mathbf{N}$ with $\|E(B_n B_n^*) - \mathrm{Id}_{C(\mathbf{T})}\| < \varepsilon/4$ and $\|\varphi_\theta \circ \mathrm{ad}\, W - \varphi_\theta \circ \mathrm{ad}\, B_n\| < \varepsilon/4$ for $n \geq n_0$. As support B_{n_0} is finite and α is irrational, we can choose p in $\mathbf{Z}\backslash\{0\}$ with $|e^{2\pi i p\alpha m} - 1| < \varepsilon/4$ for all m in support B_{n_0}. Thus

$$\varphi_\theta\big(B_{n_0} V^p B_{n_0}^*\big) = \rho_\theta\big(E\big(B_{n_0} V^p B_{n_0}^*\big)\big) = \sum_m e^{2\pi i p(\theta + m\alpha)}\Big(\big|B_{n_0}(m)\big|^2 (e^{2\pi i\theta})\Big)$$

is within $\varepsilon/4(1 + \varepsilon/4)$ of $e^{2\pi i p\theta} E(B_{n_0} B_{n_0}^*)(e^{2\pi i\theta})$, which is within $\varepsilon/4$ of $e^{2\pi i p\theta}$. Thus φ_θ ad $W(V^p)$ is within ε of $e^{2\pi i p\theta}$. However,

$$\varphi_{\theta'}\beta_g(V^k) = \begin{cases} 0 & \text{if } \; bk \neq 0, \\ e^{2\pi i hk\theta'} & \text{if } \; bk = 0, \end{cases}$$

for k in $\mathbf{Z}$ and $g = \left[\begin{smallmatrix} h & r \\ b & d \end{smallmatrix}\right]$ in $\mathrm{SL}(2, \mathbf{Z})$. Choosing ε small enough and letting $k = p$ (so k is non-zero), we have $\varphi_{\theta'}\beta_g(V^p) = \varphi_\theta \circ \mathrm{ad}\, W(V^p)$ only if $b = 0$. Thus g or $-g$ is in H. □

COROLLARY 4.5. *For θ, θ' in* $[0, 1]$, *g in* $\mathrm{SL}(2, \mathbf{Z})$, *we have $\varphi_{\theta'} \circ \beta_g$ unitarily equivalent to φ_θ iff $g \in H$ and $e^{2\pi i\theta'} \in \{e^{2\pi i(\theta + n\alpha)} \mid n \in \mathbf{Z}\}$ or $-g \in H$ and $e^{-2\pi i\theta'} \in \{e^{2\pi i(\theta + n\alpha)} \mid n \in \mathbf{Z}\}$.*

Proof. If $\varphi_{\theta'} \circ \beta_g \sim \varphi_\theta$, we have g or $-g$ in H. If $g \in H$, $\varphi_{\theta'} \circ \beta_g = \varphi_{\theta'}$, so $\varphi_{\theta'} \sim \varphi_\theta$. We know this is equivalent to $e^{2\pi i\theta'} \in \{e^{2\pi i(\theta + n\alpha)} \mid n \in \mathbf{Z}\}$. If $-g \in H$, then $\varphi_{\theta'} \circ \beta_g = \varphi_{1-\theta'}$, so $\varphi_{1-\theta'} \sim \varphi_\theta$, which is equivalent to $e^{-2\pi i\theta'} \in \{e^{2\pi i(\theta + n\alpha)} \mid n \in \mathbf{Z}\}$. The reverse implication of the corollary is immediate. □

We now show that the set of pure states $\{\varphi_\theta \circ \beta_g \mid \theta \in [0, 1],\ g \in \mathrm{SL}(2, \mathbf{Z})\}$ separate the points of A_α, i.e., if x in A_α is non-zero, then $\varphi_\theta \circ \beta_g(x) \neq 0$ for some θ and g.

We proceed with some preliminary calculations. For $x = \sum_{\text{finite}} a_{rs} V^r U^s$ in A_α, τ the trace on A_α, and m, n in $\mathbf{Z}$, we have

$$|\tau(xV^{-m}U^{-n})| = \Big|\tau\Big(\sum a_{rs} V^{r-m} U^{s-n}\Big)\Big| = |a_{mn}|.$$

If $s = \text{g.c.d.}(m, n)$, choose $g = \left[\begin{smallmatrix} h & r \\ b & d \end{smallmatrix}\right]$ in $\mathrm{SL}(2, \mathbf{Z})$ with $g(m, n) = (s, 0)$. We have $g^{-1} = \left[\begin{smallmatrix} d & -r \\ -b & h \end{smallmatrix}\right]$ and $g^{-1}(s, 0) = (m, n)$, so $ds = m$ and $-bs = n$. We saw (in computing $\varphi_\theta(\beta_g(x))$ that

$$E\big(\beta_g(x)\big) = E\Big(\beta_g\Big(\sum a_{rs} V^r U^s\Big)\Big) = \sum_k a_{(dk, -bk)} \xi(dk, -bk)^{-1} V^k,$$

so

$$\left|\tau\left(E\left(\beta_g(x)\right)V^{-s}\right)\right| = |a_{(ds,-bs)}| = |a_{(m,n)}|.$$

The algebra $\{\Sigma_{\text{finite}}\, a_{rs}V^rU^s \mid a_{rs} \in \mathbf{C}\}$ is dense in A_α so continuity yields

$$|\tau(xV^{-m}U^{-n})| = \left|\tau\left(E\left(\beta_g(x)\right)V^{-s}\right)\right|$$

for x in A_α.

PROPOSITION 4.6. *The set of pure states* $\{\varphi_\theta\beta_g \mid \theta \in [0,1], g \in \mathrm{SL}(2,\mathbf{Z})\}$ *separate the elements of* A_α.

Proof. If $\delta_{(r,j)}$ denotes the element of $L^2(\mathbf{Z}\times\mathbf{Z})$ which is one at (r, j), zero elsewhere, we define unitaries $\pi_\tau(U)$, $\pi_\tau(V)$ on $L^2(\mathbf{Z}\times\mathbf{Z})$ by

$$\pi_\tau(V)\colon \delta_{(r,j)} \to \delta_{(r+1,j)},$$
$$\pi_\tau(U)\colon \delta_{(r,j)} \to e^{2\pi i r\alpha}\delta_{(r,j+1)}.$$

As $\pi_\tau(U)\pi_\tau(V) = e^{2\pi i\alpha}\pi_\tau(V)\pi_\tau(U)$, A_α is isomorphic to the C^*-algebra generated by $\pi_\tau(V)$, $\pi_\tau(U)$.

Denoting this isomorphism by π_τ we note that π_τ with cyclic vector $\delta_{(0,0)}$ is the GNS representation of A_α associated with the trace τ. For x non-zero in A_α, $\pi_\tau(x) \neq 0$, and we have p, q, r, j in $\mathbf{Z}$ with $\langle \pi_\tau(x)\delta_{(p,q)}, \delta_{(r,j)}\rangle \neq 0$. Now

$$\begin{aligned}
&\left|\left\langle \pi_\tau(x)\pi_\tau(V^pU^q)\delta_{(0,0)}, \pi_\tau(V^rU^j)\delta_{(0,0)}\right\rangle\right| \\
&\quad = \left|\left\langle \pi_\tau(U^{-j}V^{-r}xV^pU^q)\delta_{(0,0)}, \delta_{(0,0)}\right\rangle\right| \\
&\quad = \left|\tau(U^{-j}V^{-r}xV^pU^q)\right| = \left|\tau(xV^pU^qU^{-j}V^{-r})\right| = \left|\tau(xV^{p-r}U^{q-j})\right|,
\end{aligned}$$

so there are m, n in $\mathbf{Z}$ with $|\tau(xV^{-m}U^{-n})| \neq 0$. The preliminary calculation shows there is a g in $\mathrm{SL}(2,\mathbf{Z})$, s in $\mathbf{N}$ with $\tau(E(\beta_g(x))V^{-s}) \neq 0$. Thus $E\beta_g(x)V^{-s} \neq 0$ and $E\beta_g(x) \neq 0$. For some θ in $[0,1]$, $0 \neq \rho_\theta(E(\beta_g(x))) = \varphi_\theta(\beta_g(x))$. $\square$

5. Irreducible representations of A_α and ergodic actions. Given a factor representation π of A_α, we associated with it a cardinal m, namely the multiplicity of the representation restricted to the C^*-algebra generated by V. We can, of course, associate a second cardinal m' (and a second ergodic quasi-invariant regular Borel measure ν' on $\mathbf{T}$) with π, where the restriction of π to the C^*-algebra generated by U (which is isomorphic to $C(\mathbf{T})$) is unitarily equivalent via a unitary W: $\oplus^m L^2(\mathbf{T}, \nu) \to \oplus^{m'} L^2(\mathbf{T}, \nu')$ to the representation $g \to \oplus^{m'} M_g$ of $C(\mathbf{T})$ of uniform multiplicity m' on $\oplus^{m'} L^2(\mathbf{T}, \nu')$. If (m, ν, b) is the triple associated with π and f_n the function $t \to t^n$ on $\mathbf{T}$, then $b_n = W^* \oplus^m M_{f_n} W\tilde{U}_n^{-1}$ $(n \in \mathbf{Z})$ (as $\pi(U^n) = b_n\tilde{U}_n$).

In the proposition that follows, we describe a sufficient condition for factor representations of A_α to be irreducible.

PROPOSITION 5.1. *If π is a factor representation of A_α and the restriction of π to the abelian C^*-subalgebras generated by V and U are of uniform finite multiplicities m and m', respectively, and m and m' are relatively prime, then π is an irreducible representation.*

Proof. Let A_1 and A_2 be the abelian C^*-subalgebras of A_α generated by V and U, respectively. By assumption, $\pi(A_1)'$ and $\pi(A_2)'$ are von Neumann algebras by types I_m and $\mathrm{I}_{m'}$, respectively. Each contains $\pi(A_\alpha)'$, and $\pi(A_\alpha)'$ is a factor by assumption. From Proposition 2.2., as m (or m') $\in \mathbf{N}$, $\pi(A_\alpha)'$ is type I. Suppose $\pi(A_\alpha)'$ is of type I_n. Then there are n orthogonal equivalent projections $E_1, \ldots, E_n$ in $\pi(A_\alpha)'$ with sum I. This equivalence persists in $\pi(A_1)'$. Since $\Sigma E_k = \mathrm{I}$, each E_k has central carrier I in $\pi(A_1)'$. From **[10]** there is a non-zero central projection Q in $\pi(A_1)'$ such that QE_1 is the sum of j equivalent abelian projections in $\pi(A_1)'$. Since QE_1 and QE_k are equivalent in $\pi(A_1)'$, QE_k is the sum of j equivalent abelian projections in $\pi(A_1)'$ for each k in $\{1, \ldots, n\}$. But $Q = \Sigma QE_k$, so Q is the sum of nj equivalent abelian projections in $\pi(A_1)'$. Now $\pi(A_1)'Q$ is of type I_m, so $m = nj$ and n divides m. This same argument applied to $\pi(A_2)'$ and m' yields the fact that n divides m'. By hypothesis, m and m' are relatively prime. Thus $n = 1$; $\pi(A_\alpha)'$ is a factor of type I_1, so $\pi(A_\alpha)'$ consists of scalars and π is irreducible. □

We shall see examples of irreducible representations of A_α for which the finite multiplicities m and m' above are any given pair of non-zero relatively prime numbers.

Let (ρ_m, T) be the covariant representation associated to a factor representation π of A_α. Thus ρ_m is a representation of $C(\mathbf{T})$ with uniform multiplicity $m \in \mathbf{N} \cup \{\infty\}$ on $\oplus^m L^2(\mathbf{T}, \nu)$ with ν a quasi-invariant ergodic regular Borel measure on $\mathbf{T}$ and $T_n = h_n\tilde{U}_n$, where h_n is a unitary cocycle in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$.

PROPOSITION 5.2. *Let π, (π_m, T) be as above. The representation π is irreducible iff* ad T *is an ergodic automorphism on $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$.*

Proof. Note $C \in \pi(A_\alpha)'$ iff $C \in \rho_m(C(\mathbf{T}))' = \mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$ and $TCT^* = C$. Thus $\pi(A_\alpha)' = \mathbf{C}\,\mathrm{Id}$ iff for C in $\mathfrak{M}_m(L^\infty(\mathbf{T}, \nu))$, ad $T(C) = C$ implies $C \in \mathbf{C}\,\mathrm{Id}$. □

Note that if $m = 1$, then ad T is the ergodic automorphism α on $L^\infty(\mathbf{T}, \nu)$, so π is an irreducible representation. In the next paragraphs, explicit irreducible representations of A_α are given for all m in $\mathbf{N}$. These examples also serve to show that there are irreducible representations of A_α for which any pair of non-zero relatively prime numbers m and m' can occur as the multiplicities (obtained by restricting the representation to the C^*-algebras generated by V and U respectively). Choose b, d in $\mathbf{N}\backslash\{0\}$ with g.c.d.$(b, d) = 1$. There are z, k in $\mathbf{Z}$ such that $g = [\begin{smallmatrix} z & k \\ b & d \end{smallmatrix}] \in \mathrm{SL}(2, \mathbf{Z})$. For $\theta \in [0, 1]$ the (necessarily) irreducible representation associated with the pure state $\varphi_\theta \circ \beta_g$ is a representation of A_α whose restriction to $C(\mathbf{T})$ has multiplicity b. It will also be clear that the uniform multiplicity of this irreducible representation restricted to the C^*-algebra generated by U is d. Note that given b, d in $\mathbf{N}\backslash\{0\}$, there are many elements $g = [\begin{smallmatrix} z & d \\ b & d \end{smallmatrix}]$ in $\mathrm{SL}(2, \mathbf{Z})$, however, given two such $g_1 = [\begin{smallmatrix} z & k \\ b & d \end{smallmatrix}]$ and $g_2 = [\begin{smallmatrix} z' & k' \\ b & d \end{smallmatrix}]$, we have

$$g_2 g_1^{-1} \in H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \,\middle|\, n \in \mathbf{Z} \right\}$$

Thus $\varphi_\theta \circ \beta_{g_1}$ is equivalent to $\varphi_\theta \circ \beta_{g_2}$.

We first find a representation $\tilde{\pi}$ of A_α associated with the pure state $\varphi_\theta \circ \beta_g$. Define δ in $L^2(\mathbf{Z})$ by

$$\delta_n(m) = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

Define unitaries $\tilde{\pi}(V)$, $\tilde{\pi}(U)$ on $L^2(\mathbf{Z})$ by describing their effect on the basis $\{\delta_n \mid n \in \mathbf{Z}\}$ of $L^2(\mathbf{Z})$.

$$\tilde{\pi}(V)\colon \delta_r \to e^{2\pi i d^{-1}\theta} e^{-2\pi i d^{-1} r\alpha} \delta_{r+b},$$

$$\tilde{\pi}(U)\colon \delta_r \to e^{\pi i \alpha} \delta_{r+d}.$$

We have $\tilde{\pi}(U)\tilde{\pi}(V) = e^{2\pi i\alpha}\tilde{\pi}(V)\tilde{\pi}(U)$. Thus A_α is *-isomorphic to the C^*-algebra generated by $\tilde{\pi}(V)$, $\tilde{\pi}(U)$, the *-isomorphism (denoted $\tilde{\pi}$) mapping V to $\tilde{\pi}(V)$ and U to $\tilde{\pi}(U)$.

If $p, n \in \mathbf{Z}$ then

$$\tilde{\pi}(V^p U^n)\colon \delta_r \to e^{\pi i n\alpha} e^{2\pi i p d^{-1}\theta} \times e^{-2\pi i d^{-1}\alpha[p(r+nd)+p(p-1)b/2]} \delta_{r+nd+pb},$$

so

$$\langle \tilde{\pi}(V^pU^n)\delta_0, \delta_0 \rangle = \begin{cases} e^{2\pi i p d^{-1}\theta} e^{-\pi i p n \alpha} & \text{if } nd + pb = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is equal to $\varphi_\theta \circ \beta_g(V^pU^n)$. As g.c.d.$(b, d) = 1$, we can choose p, n such that $nd + pb$ is any preassigned integer. Thus δ_0 is a cyclic vector for the representation $\tilde{\pi}$ of A_α. Linearity and continuity imply $\langle \tilde{\pi}(x)\delta_0, \delta_0 \rangle = \varphi_\theta \circ \beta_g(x)$ for all $x \in A_\alpha$. The representation $\tilde{\pi}$ of A_α is associated with the pure state $\varphi_\theta \circ \beta_g$, so it is irreducible.

We show that $\tilde{\pi}|_{C(\mathbf{T})}$ is unitarily equivalent to the multiplicity b representation ρ_b with Haar measure on $\mathbf{T}$. Investigating $\tilde{\pi}|_{C(\mathbf{T})}$, where $C(\mathbf{T})$ is the C^*-algebra generated by V, we see that $L^2(\mathbf{Z})$ is the direct sum of b subspaces $\{H_j\}_{j=0}^{b-1}$, where H_j is the $\tilde{\pi}|_{C(\mathbf{T})}$ invariant subspace generated by $\{\delta_{qb+j} \mid q \in \mathbf{Z}\}$, $j = 0,\ldots,b-1$. If W_j is the partial isometry on $L^2(\mathbf{Z})$ with initial space H_j and final space H_0 determined by

$$W_j\colon \delta_{qb+j} \to e^{2\pi i d^{-1}\alpha q j}\delta_{qb} \qquad (q \in \mathbf{Z}),$$

then $W = \oplus_{j=0}^{b-1} W_j$ is a unitary transformation from $\oplus_0^{b-1} H_j$ onto $\oplus_0^{b-1} H_0$, with $W(\tilde{\pi}|_{C(\mathbf{T})}(x)) = (\oplus_0^{b-1} \pi(x))W$ (as $W_j\tilde{\pi}(V)\delta_{qb+j} = \tilde{\pi}(V)W_j\delta_{qb+j}$), where π is $\tilde{\pi}|_{C(\mathbf{T})}$ restricted to the subspace H_0 and $x \in C(\mathbf{T})$.

We show π is equivalent to the multiplication representation of $C(\mathbf{T})$ on $L^2(\mathbf{T}, \mu)$ where μ is Haar measure on $\mathbf{T}$. If f is an element of $B = \{\sum_{\text{finite}} a_n V^n \mid a_n \in \mathbf{C}\}$, then

$$\|\pi(f)\delta_0\|^2 = \langle \pi(f^*f)\delta_0, \delta_0 \rangle = \sum_{\text{finite}} |a_n|^2 = \int_{\mathbf{T}} |f|^2\, d\mu.$$

The linear map Y defined on B, a norm dense subspace of $L^2(\mathbf{T}, \mu)$, to H_0 by $f \to \pi(f)\delta_0$ is therefore an isometry onto a dense subspace of H_0. The map Y extends to a unitary map (also called Y) of $L^2(\mathbf{T}, \mu)$ onto H_0. For x in B,

$$Y^*\pi(V)Y(x) = Y^*\pi(V)\pi(x)\delta_0 = Y^*\pi(Vx)\delta_0 = Vx,$$

so continuity ensures that $Y^*\pi(V)Y$ is multiplication by V on $L^2(\mathbf{T}, \mu)$. Thus π is unitarily equivalent via the unitary Y^* to the multiplication representation of $C(\mathbf{T})$ on $L^2(\mathbf{T}, \mu)$, and $\tilde{\pi}|_{C(\mathbf{T})}$ is unitarily equivalent to the multiplicity b representation ρ_b on $\oplus_0^{b-1} L^2(\mathbf{T}, \mu)$ via the unitary $(\oplus_0^{b-1} Y^*)W$. In an analogous manner we note that $\tilde{\pi}$ restricted to the C^*-algebra $C^*(U)$ generated by U is (unitarily equivalent to) a multiplicity d representation on $\oplus_0^{d-1} L^2(\mathbf{T}, \mu)$.

To determine the unitary cocycle h_1 for the representation $\tilde{\pi}$ of A_α, we examine what $\tilde{\pi}(U)$ looks like on $\oplus_0^{b-1} L^2(\mathbf{T}, \mu)$, i.e., we compute $(\oplus_0^{b-1} Y^*)W\tilde{\pi}(U)W^*(\oplus_0^{b-1} Y)$.

There are a, s in $\mathbf{N}$ with $0 < s < b$, $d = ab + s$ and g.c.d.$(b, s) =$ g.c.d.$(b, d) = 1$. The unitary $\tilde{\pi}(U)$ maps H_j isometrically onto H_{j+s} if $0 \leq j < b - s$ and onto $H_{j-(b-s)}$ if $b - s \leq j \leq b - 1$. Note that $Y(\gamma(n)V^n) = \delta_{nb}$ with

$$\gamma(n) = e^{-2\pi i n d^{-1}\theta} e^{\pi i d^{-1}\alpha b(n(n-1))},$$

so (for $x, y \in \{0, \ldots, b-1\}$)

$$\begin{aligned} &Y^* W_x \tilde{\pi}(U) W_y^* Y(V^n) \\ &= \begin{cases} \gamma(n+a)\overline{\gamma(n)} e^{\pi i \alpha} e^{2\pi i d^{-1}\alpha[ns+ax]} V^{n+a} \\ \qquad \text{if } s \leq x \leq b-1 \text{ and } x - y = s, \\ \gamma(n+a+1)\overline{\gamma(n)} e^{\pi i \alpha} e^{2\pi i d^{-1}\alpha[n(s-b)+(a+1)x]} V^{n+a+1} \\ \qquad \text{if } 0 \leq x < s \text{ and } y - x = b - s, \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned}$$

Letting

$$g(x) = \begin{cases} e^{-2\pi i d^{-1}\theta a} e^{\pi i d^{-1}\alpha[2ax+s+ba^2]} V^a \quad \text{if } s \leq x \leq b-1, \\ e^{-2\pi i d^{-1}\theta(a+1)} e^{\pi i d^{-1}\alpha[2(a+1)x+2ab+s+ba^2]} V^{a+1} \\ \qquad\qquad\qquad\qquad \text{if } 0 \leq x < s, \end{cases}$$

we have

$$Y^* W_x \tilde{\pi}(U) W_y^* Y(V^n) = e^{2\pi i n \alpha} g(x) V^n = g(x)\alpha(V^n),$$

where $x - y = s$ if $s \leq x \leq b - 1$ and $y - x = b - s$ if $0 \leq x < s$.

Thus the unitary cocycle h_1 in $\mathfrak{M}_b(L^\infty(\mathbf{T}, \mu))$ associated with $(\oplus_0^{b-1} Y^*)W\tilde{\pi}(U)W^*(\oplus_0^{b-1} Y)$ is the following matrix on $\oplus_0^{b-1} L^2(\mathbf{T}, \mu)$:

$$\left[\begin{array}{ccc|ccc} & & & g(0) & & \\ & & & & g(1) & \\ & & & & & g(s-1) \\ \hline g(s) & & & & & \\ & g(s+1) & & & & \\ & & g(b-1) & & & \end{array}\right].$$

Noticing that the entries of this matrix are continuous functions on $\mathbf{T}$, we have the additional information that $\operatorname{ad} T = \operatorname{ad} h_I \tilde{U}$ (see Proposition 5.2) is actually an ergodic automorphism of the C^*-algebra $\mathfrak{M}_m(C(\mathbf{T}))$. We have thus found explicit examples of cocycles which perturb the non-ergodic action $[a_{ij}] \to [\alpha(a_{ij})]$ on $\mathfrak{M}_m(C(\mathbf{T}))$ $(m > 1)$ into an ergodic action.

We have inequivalent pure states with the same pair of relatively prime numbers as their multiplicities. For example, with $g = \left[\begin{smallmatrix} z & k \\ b & d \end{smallmatrix}\right]$ as above, $\varphi_{\theta'} \circ \beta_g \nsim \varphi_\theta \circ \beta_g$, with θ' not in the orbit of θ. Also if $g' = \left[\begin{smallmatrix} -z & k \\ b & -d \end{smallmatrix}\right]$, then $g'g^{-1} \notin H$, so $\varphi_{\theta'} \circ \beta_g \nsim \varphi_\theta \circ \beta_{g'}$ for all θ in $[0, 1]$. One sees, however, that $\varphi_\theta \circ \beta_{g'}$ has b, d as its associated multiplicities (as does $\varphi_{\theta'} \circ \beta_g$).

Remark added in proof. Y. Watatani has also considered a representation of $\mathrm{SL}(2, \mathbf{Z})$ as automorphisms of A_α in *Toral automorphisms on irrational rotation algebras*, Math. Japonica, **26**, No. 4 (1981).

REFERENCES

[1] L. Baggett, *Representations of the Mautner Group, I* Pacific J. Math., **77** (1978).

[2] B. Brenken and R. Kadison to appear.

[3] A. Connes, *A factor not antiisomorphic to itself*, Annals of Math., **101** (1975), 536–554.

[4] J. Dixmier, *Les C*-algebras et Leurs Representations*, Bordas, (Gauthier-Villars), Paris, 1969.

[5] S. Doplicher, D. Kastler and D. Robinson, *Covariance algebras in field theory and statistical mechanics*, Comm. Math. Phys., **3** (1966), 1–28.

[6] P. Green, *The local structure of twisted covariance algebras*, Acta Math., **140** (1978), 191–250.

[7] R. Herman and V. Jones, *Period two automorphisms of U.H.F. C*-algebras*, J. Functional Anal., **45** (1982), 169–176.

[8] R. Kadison, *Representations of Matricial Operator Algebras*, (to appear, Proc. Neptun Conf. on Op. Alg. and Group Rep. Pittman, London).

[9] ______, *Diagonalizing Matrices*, (to appear).

[10] R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Vols. I and II, Academic Press, New York (1982–83).

[11] G. Mackey, *The theory of Unitary Group Representations*, University of Chicago Press, 1976.

[12] G. Pedersen, *C*-Algebras and their Automorphism Groups*, Academic Press, London, 1979.

[13] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups*, J. Operator Theory, **4** No. 1 (1980), 93–118.

[14] ______, *Imbedding the irrational rotation C*-algebra into an AF algebra*, J. Operator Theory, **4** (1980).

[15] S. Popa and M. Reiffel, *The Ext groups of the C*-algebras associated with irrational rotations*, J. Operator Theory, **3** No. 2 (1980).

[16] M. Reiffel, *C*-algebras associated with an irrational rotation*, Pacific J. Math., **93** No. 2 (1981), 415–429.

[17] S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer Verlag, 1970.
[18] J. Taylor, *Banach Algebras and Topology in "Algebras in Analysis"*, Academic Press, 1975, 118–186.
[19] V. Varadarajan, *Geometry of Quantum Theory*, Vol. 2, Van Nostrand 1970.
[20] G. Zeller-Meier, *Produits Croises d'une C^*-algebra par un Groupes d'automorphismes*, J. Math. Pures et Appl., **47** (1968), 101–239.

Received July 9, 1982 and in revised form November 30, 1982.

DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
CANADA, B3H-4H8