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# ON LOCALIZATIONS AND SIMPLE $C^*$-ALGEBRAS

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**A method for associating $C^*$-algebras to inverse semigroups of partial homeomorphisms (termed localizations) is developed. Localizations which locally have the same structure yield $C^*$-algebras in the same strong Morita equivalence class (via the linking algebra characterization).**

**Free localizations are closely related to Renault's principal discrete groupoids, where the partial homeomorphisms are identified with open $G$-sets. The space on which a free localization is defined becomes the spectrum of a "Cartan" masa in the associated $C^*$-algebra (but this masa is not unique modulo conjugation by automorphisms).**

**It is shown that if $A$ is a simple unital AF algebra with comparability of projections (i.e. $K_0(A)$ can be embedded as an ordered subgroup of the reals) then $A$ embeds unitally in the transformation-group algebra associated to the action of a discrete subgroup of the unit circle.**

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**1. Introduction.** The convenient fiction indulged in by most practitioners in the field is that $C^*$-algebras are somehow to be conceived of as continuous functions on a "non-commutative" topological space. When the $C^*$-algebra is commutative then it is canonically isomorphic to the continuous functions (vanihsing at $\infty$) on its spectrum. Here the spectrum is locally compact and Hausdorff (as well as second countable if the algebra is separable). For an arbitrary $C^*$-algebra $A$, the primitive ideal space, $\mathrm{Prim}(A)$, is thought to be the most tenable generalization of the spectrum. For one thing the lattice of open sets on $\mathrm{Prim}(A)$ corresponds exactly to the lattice of closed ideals in $A$. For type I algebras, Prim($A$) parameterizes the unitary equivalence classes of irreducible representations. As a topological space, however, it is known to exhibit certain pathologies; further it reveals nothing of the structure of $A$ when $A$ is simple (and not the algebra of compact operators).

In recent years, there have arisen a number of fascinating examples of simple $C^*$-algebras including Bratelli's AF algebras, Cuntz' algebras generated by isometries, and those associated with topological dynamics. In **[22]**, Elliott derives a complete invariant for AF algebras (simple and non-simple), the so-called dimension range, a hereditary upward directed generating subset of the positive cone of an ordered abelian group, the Elliott group. The Elliott group turns out to be a complete invariant for the strong Morita equivalence class of an AF algebra (it is also the $K_0$ group), since hereditary subalgebras of an AF algebra are AF algebras as well **[24]**. Rieffel shows in Example 6.7 of **[41]** that a hereditary subalgebra is strong Morita equivalent to the closed two-sided ideal it generates (the bimodule implementing the equivalence is simply the left ideal associated to the hereditary subalgebra). This also means that a simple $C^*$-algebra is strong Morita equivalent to each hereditary subalgebra. (A consequence of this is that two simple $C^*$-algebras are strong Morita equivalent iff there is a hereditary subalgebra in one of them that is isomorphic to a hereditary subalgebra of the other.)

The notion of strong Morita equivalence is characterized by the existence of a linking algebra in which the two algebras are exhibited as complementary full corners **[10]**. The authors also show that two algebras are strong Morita equivalent iff they are stably isomorphic (when the algebras have strictly positive elements).

This notion has also proven to be fruitful in the study of certain transformation group algebras **[43]**. Further, application to the study of the irrational rotation algebra led Rieffel to the discovery of a countable number of inequivalent projections whose image under the unique normalized trace is $[0, 1] \cap \{n + m\alpha\colon n, m \in \mathbf{Z}\}$, where $\alpha$ is the irrational **[44]**. Pimsner and Voiculescu have shown, very recently, that this algebra embeds in an AF algebra whose Elliott group is $\{n + m\alpha\colon n, m \in \mathbf{Z}\}$ (with the order inherited from the reals).

Perhaps the most clear-sighted effort at deriving $C^*$-algebras from "non-commutative" topological spaces is to be found in Jean Renault's thesis on locally compact groupoids **[39]**. He credits the work of Strătilă and Voiculescu in diagonalizing AF algebras with providing the impetus of his study. The feature of interest is the so-called Cartan subalgebra, a maximal abelian subalgebra with total normalizer (span is dense). This and other considerations led to his study of representations of locally compact groupoids and their associated $C^*$-algebras with special emphasis on principal $r$-discrete groupoids. Renault evolves a unified approach to the study of AF algebras, transformation group algebras and the Cuntz algebras, $\mathcal{O}_n$. Renault's observation that a principal $r$-discrete groupoid $G$ has a basis consisting of open $G$-sets and that the $G$-sets are naturally equipped with the structure of an inverse semigroup is key for what follows. Countable inverse semigroups of partial homeomorphisms which come from a basis of a principal $r$-discrete groupoid are called free localizations in our terminology (§7).

Localizations will be defined as countable inverse semigroups of partial homeomorphisms, the idempotents of which may be identified with a basis for the underlying space. Localizations induce a partial ordering on the open subsets. Of chief concern will be those localizations, called simplifications, for which the order is trivial. A localization may profitably be viewed as a non-commutative analog of a countable basis; its affiliated inverse semigroup is to be viewed as the analog of a topology. There is no essential difference between localizations with the same affiliation. A normed *-algebra, $C_c(\Omega)$, is associated to a localization $\Omega$, so that localizations with the same affiliation yield canonically isomorphic normed *-algebras. If $\Omega$ is a simplification, the notion of $\Omega$-manifold provides a method for inducing simplifications to other spaces so that the enveloping $C^*$-algebras of the associated normed *-algebra are canonically strong Morita equivalent (the linking algebra characterization is key here).

More explicitly, the collection of partial homeomorphisms (both domain and range are open) on a space $X$ carries the structure of an inverse semigroup (ISG), denoted $S_0(X)$. The involution ($\sigma \to \sigma^*$) assigns the partial inverse, while composition is defined where it makes sense; the idempotents are simply restrictions (to open subsets) of the identity map and are implicitly identified with their domains. A localization $\Omega \subseteq S_0(X)$ is a countable ISG such that $\mathrm{id}(\Omega)$ forms a basis for $X$.

Not surprisingly, much of topological interest inheres in the algebraic structure of $S_0(X)$. A set, $\Sigma \subseteq S_0(X)$ is said to be left coherent if for every $\omega, \sigma \in \Sigma$, $\omega^*\sigma \in \mathrm{id}(S_0(X))$. Suppose $\Sigma$ is left coherent and $V = \bigcup_\Sigma d(\sigma)$, $U = \bigcup_\Sigma r(\sigma)$ then there is a unique local homeomorphism $\psi$: $U \to V$ such that if $\sigma \in \Sigma$ and $x \in d(\sigma)$ then $\psi(\sigma(x)) = x$. If, in addition, $\Sigma^*$ is left coherent then $\Sigma$ is said to be coherent; in this case $\psi \in S_0(X)$ (§3/2). The idea is that $\psi^*$ is to be viewed as the coherent union of the elements of $\Sigma$

(since $\psi^* \mid d(\sigma) = \sigma$ for each $\sigma \in \Sigma$). The affiliation of $\Omega$, denoted $\Omega^\alpha$, consists of coherent unions of coherent subsets of $\Omega$. Since the composition of two coherent subsets is coherent, we have that $\Omega^\alpha$ is an ISG.

Let $Y$ be a topological space and $X \vee Y$ denote the disjoint union of $X$ and $Y$. A countable collection $\Sigma \subseteq S_0(X \vee Y)$ with $\bigcup_\Sigma d(\sigma) = Y$ and $r(\sigma) \subseteq X$ for each $\sigma \in \Sigma$ is called an $\Omega$-atlas on $Y$ if $\Sigma\Sigma^* \subseteq \Omega^\alpha$; in this case $Y$ is termed an $\Omega$-manifold. The linking localization, $\Upsilon(\Sigma) \subseteq S_0(X \vee Y)$, is the localization generated by $\Omega \cup \Sigma$ (i.e., the smallest ISG containing $\Omega \cup \Sigma$). The induced localization, $\Omega(\Sigma) = \{\sigma \in \Upsilon(\Sigma)\colon d(\sigma), r(\sigma) \subseteq Y\}$ is generated by $\Sigma^*\Sigma \cup \Sigma^*\Omega\Sigma$. The $\Omega$-atlas is said to be full if $\bigcup_\Sigma r(\sigma) \sim X$ (in the sense of §3/4). To avoid unnecessary complications the notion of $\Omega$-manifold is developed in §4 only for $\Omega$, a simplification. The interested reader will perceive that the canonical imprimitivity bimodule (§6) exists in the general case so long as the $\Omega$-atlas is full (the linking algebra is $C^*(\Upsilon(\Sigma))$). Whence $C^*(\Omega)$ and $C^*(\Omega(\Sigma))$ are strong Morita equivalent.

If $\Omega \subseteq S_0(X)$ is a localization consisting of idempotents and $\psi$: $Y \to X$ is a local homeomorphism then there is a countable left coherent family $\Sigma \subseteq S_0(X \vee Y)$ of local sections of $\psi$. This family constitutes a full $\Omega$-atlas on $Y$ and the induced localization $\Omega(\Sigma)$ is called a localization of imprimitivity related to $\psi$, the space $Y$ is called an imprimitive.

Now, if $\Omega$ is an arbitrary localization on $X$ and $\psi$: $X \to Y$ is a local homeomorphism so that $\Omega^\alpha$ contains a localization of imprimitivity related to $\psi$ (we also say $\psi$ is admitted by $\Omega$), then $Z$ is called a primitive. Then $Z$ naturally carries the structure of an $\Omega$-manifold; the induced localization is written $\Omega_\psi$. If $Z$ is compact, there is a projection $p \in C^*(\Omega)$ so that $C^*(\Omega_\psi) \cong pC^*(\Omega)p$ (§8/2). If there are "sufficiently" many compact primitives then $C^*(\Omega)$ has an approximate identity consisting of an increasing sequence of projections (§8/3). Rieffel's projections [**44**] provide the motivation for this construction.

Let $G_\alpha = \mathbf{Z} + \mathbf{Z}\alpha$; then $G_\alpha$ acts on $\mathbf{R}$ by translation and the associated transformation group algebra, $C^*(G_\alpha, \mathbf{R})$, is strong Morita equivalent to the irrational rotation algebra. Because $G_\alpha$ is dense, the localization $\Omega$ generated by a basis and translation by elements of $G_\alpha$ is a simplification (§3/4). Further the quotient map $\psi$: $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$ defines a primitive. In fact, $C^*(\Omega_\psi)$ is just the irrational rotation algebra.

The situation considered in §9 is slightly more general. Let $G$ be a countable dense subgroup of $\mathbf{R}$, and let $\Omega$ be "the" associated simplification. Suppose $A$ is a unital AF algebra with $K_0(A) = G$ (order inherited from $\mathbf{R}$). We will show (§9/9) that there is a projection $p \in C^*(\Omega)$ so that $A$ embeds unitally in $pC^*(\Omega)p$.

Further, to every open $U \subseteq \mathbf{R}$ there corresponds a hereditary subalgebra, written $C^*(\Omega_U)$. Then $C^*(\Omega_U)$ has an approximate identity consisting of an increasing sequence of projections. Further, if $V$ is another open set with the same Lebesgue measure as $U$ then $C^*(\Omega_V) \cong C^*(\Omega_U)$ (§9/6).

A localization $\Omega$ on $X$ is said to be free if given $\omega \in \Omega$ and $x \in d(\omega)$ so that $\omega(x) = x$ then $\omega$ is locally an idempotent (§7). If $\Omega$ is free then there is a unique principal $r$-discrete groupoid $R(\Omega)$ with unit space $X$ so that $\Omega^\alpha$ is naturally identified with the open $R(\Omega)$-sets [**39**] (conversely, principal $r$-discrete groupoids give rise to free localizations). A conditional expectation is defined on $C^*(\Omega)$ with range $C_0(X)$ (§7.4); there is some difficulty connected with determining when it is faithful (related to the question of amenability for groupoids). If $\Omega$ simplifies and $C^*(\Omega)$ has a finite trace then $C^*(\Omega)$ is simple.

The appendices include a description of two simplifications, each generated by a single local homeomorphism, the associated $C^*$-algebras of which are simple while the simplifications are not free. The first is generated by the one-sided shift related to the topological Markov chains considered by Cuntz and Krieger [**13**] and the second is generated by the two-fold covering of the circle and is related to an example of N. V. Pedersen [**35**].

Another question of interest is to what extent is the strong Morita equivalence class of $C^*(\Omega)$ exhausted by the algebras associated to full $\Omega$-manifolds. Further, when are there sufficiently many compact $\Omega$-manifolds so that any $\Omega$-manifold is weak equivalent to the disjoint sum of a countable sequence of them (§6/3)?

In what follows, all topological spaces are implicitly assumed to be second countable, locally compact and Hausdorff (and consequently paracompact and $\sigma$-compact); they are referred to simply as spaces. Let $X$ be a space; then $C(X)$ ($C_0(X)$, $C_c(X)$) denotes the algebra of complex valued continuous functions on $X$ (vanishing at $\infty$, with compact support). Further, all representations are assumed to be non-degenerate.

Some of the results of this study seem to point toward the following definition of a Cartan masa in a $C^*$-algebra. First, define the left normalizer of a subalgebra $B$ of a $C^*$-algebra $A$ to be:

$$N(B) = \{a \in A\colon a^*Ba \subseteq B\}.$$

Then if $B \cong C_0(X)$ is maximal abelian we say that it is Cartan whenever:

(i) Each $f \in C_0(X)$ with $f(x) > 0$ for each $x \in X$ is strictly positive in $A$.

(ii) $N(B) \cap N(B)^*$ is total.

(iii) If $I \subseteq A$ is an ideal such that $I \cap B = \{0\}$ then $I = \{0\}$.

It is worth noting that the results of §9 imply that Cartan subalgebras need not be isomorphic. Some of the motivation for considering localizations comes from the observation that if $a \in N(B) \cap N(B)^*$ then $a^*a$, $aa^* \in B$. Further, there is a partial homeomorphism $\sigma \in S_0(X)$ so that if $f \in C_0(X)$ then

$$a^*fa = (a^*a)(f \cdot \sigma) \qquad (\S 5/2).$$

So in some sense a localization may be associated to the Cartan pair $(A, B)$; a cohomological obstacle, however, remains to be unravelled.

**2. Inverse semigroups.**

1. We give the standard definition of an inverse semigroup with the exception that we require it to always have a trivial element $\theta$ [**5, 38**].

DEFINITION. An inverse semigroup (ISG) is a triple $(S, \theta, {}^*)$ where $S$ is a semigroup (a set with an associative binary law of composition) and $\theta \in S$ is a trivial element, that is, $s\theta = \theta s = \theta$ for each $s \in S$. The involution $s \to s^*$ is a bijection on $S$ so that:

(i) $s^{**} = s$, each $s \in S$,

(ii) $(st)^* = t^*s^*$, each $s, t \in S$,

(iii) $ss^*s = s$, each $s \in S$.

*Facts.* $\theta$ is the unique trivial element, so $\theta^* = \theta$. Clearly $s^*ss^* = s^*$.

DEFINITION. An idempotent is an element $e \in s$ with $e^*e = e$. Write $\mathrm{id}(S) = \{e \in S\colon e^*e = e\}$. Note that $\theta \in \mathrm{id}(S)$ and if $e \in \mathrm{id}(S)$ then $e = e^* = e^2$.

DEFINITION. $d\colon S \to \mathrm{id}(S)$, $r\colon S \to \mathrm{id}(S)$ by $d(s) = s^*s$, $r(s) = ss^*$. Note that $s = sd(s) = r(s)s$, $d(s^*) = r(s)$, $r(s^*) = d(s)$. Also $s \in \mathrm{id}(S)$ iff $r(s) = s$ iff $d(s) = s$. For $s \in S$, $d(s)$ is a right unit and $r(s)$ is a left unit while $s^*$ is a (partial) inverse with respect to these units.

2. EXAMPLE. Let $I_n$ be a set of cardinality $n$ (finite or countably infinite). Let $S_n = I_n \times I_n \cup \{\theta\}$; define composition:

$$(i, j)(k, l) = \begin{cases} (i, l) & \text{if } j = k, \\ \theta & \text{if } j \neq k, \end{cases} \qquad i, j, k, l \in I_n,$$

and involution

$$(i, j)^* = (j, i), \qquad i, j \in I_n.$$

Then $S_n$ is an ISG, $\mathrm{id}(S_n) = \{(i, i): i \in I_n\} \cup \{\theta\}$, and $d(i, j) = (j, j)$, $r(i, j) = (i, i)$.

3. If $S$, $T$ are ISG's put $S \times T = (S \backslash \{\theta\}) \times (T \backslash \{\theta\}) \cup \{\theta\}$, that is, in the cartesian product of $S$ and $T$ the elements $(s, \theta)$, $(\theta, t)$ (each $s \in S$, $t \in T$) are identified with the trivial element $\theta \in S \times T$. Composition is defined by

$$(s_1, t_1)(s_2, t_2) = \begin{cases} (s_1 s_2, t_1 t_2) & \text{if } s_1 s_2 \neq \theta,\ t_1 t_2 \neq \theta, \\ \theta & \text{otherwise;} \end{cases}$$

involution by

$$(s, t)^* = (s^*, t^*).$$

Then the idempotents $\mathrm{id}(S \times T) = \{(e, f): e \in \mathrm{id}(S), f \in \mathrm{id}(T), e, f \neq \theta\} \cup \{\theta\}$. Note that $S_n \times S_m \cong S_{nm}$.

4. If $S$, $T$ are ISG's let $S + T = S \vee T$ with the trivial elements identified. For $s, t \in S + T$ with $s \in S$, $t \in T$, define $st = \theta$ and $ts = \theta$. Otherwise composition and involution are inherited from $S$ and $T$. Note that $e \in \mathrm{id}(S + T)$ iff $e \in \mathrm{id}(S)$ or $e \in \mathrm{id}(T)$. If $\nu = (n_1, \ldots, n_k)$ is a $k$-tuple of positive integers with

$$|\nu| = \sum_{i=1}^{k} n_i,$$

put

$$S_\nu = S_{n_1} + S_{n_2} + \cdots + S_{n_k}.$$

$S_\nu$ is called a minimal finite localization. Note that there are exactly $|\nu|$ non-trivial idempotents in $S_\nu$. A bijection between $\mathrm{id}(S_\nu)$ and $\mathrm{id}(S_{|\nu|})$ induces an injection $S_\nu \to S_{|\nu|}$. Minimal finite localizations are closely related to the unit systems of Krieger **[30]**.

**3. Localizations and simplifications.**

1. In what follows we shall concern ourselves with concrete ISG's of partial homeomorphisms for which the idempotents are realized as restrictions of the identity (qua homeomorphism) to open subsets.

DEFINITION. If $X$ and $Y$ are spaces, then a partial homeomorphism from $X$ to $Y$ is a homeomorphism $\sigma$ defined on an open subset $U$ of $X$ with range an open subset $V$ of $Y$. Write $\sigma: U \to V$ and let $S_0(X, Y)$ denote the collection of all such partial homeomorphisms. Write $\sigma \to \sigma^*$

for the bijection ($S_0(X, Y) \to S_0(Y, X)$) which sends a partial homeomorphism to its partial inverse. Write $S_0(X) = S_0(X, X)$.

DEFINITION. A law of composition is defined on $S_0(X)$, as follows: If $\sigma_1$: $U_1 \to V_1$, $\sigma_2$: $U_2 \to V_2$ are partial homeomorphism on $X$, and if $U_1 \cap V_2 = \varnothing$ then put $\sigma_1\sigma_2 = \theta$ where $\theta$ is the unique partial homeomorphism on $X$ with empty domain. If $U_1 \cap V_2 \neq \varnothing$ put $U_3 = \sigma_2^{-1}(U_1 \cap V_2)$ and $V_3 = \sigma_1(U_1 \cap V_2)$ and put $\sigma_3$: $U_3 \to V_3$ by $\sigma_3(x) = \sigma_1(\sigma_2(x))$; write $\sigma_3 = \sigma_1\sigma_2$. Note that if $\sigma \in S_0(X)$ then $\sigma$: $U \to V$ is a homeomorphism while $\sigma^*$: $V \to U$ is its inverse; and with the above composition $\sigma^*\sigma$: $U \to U$, $\sigma\sigma^*$: $V \to V$ are identity maps (restricted to their domains of definition).

*Facts*. With the above composition and involution ($\sigma \to \sigma^*$) $S_0(X)$ is an ISG. Clearly $\mathrm{id}(S_0(X))$ may be identified with the open subsets of $X$; this identification will be assumed implicitly in what follows. Using notation from §2 if $\sigma \in S_0(X)$ then $\sigma$: $d(\sigma) \to r(\sigma)$ is a homeomorphism.

2. DEFINITION. A countable sub-ISG $\Omega \subseteq S_0(X)$ is said to be a localization on $X$ if $\mathrm{id}(\Omega)$ forms a basis for $X$. (When notationally convenient we shall write $\hat{\Omega} = X$, or, more briefly, $\Omega$ localizes $X$.)

EXAMPLES. (i) The minimal finite localizations defined in §2 are in fact localizations. If $\nu = (n_1, \ldots, n_k)$ is as above, then $I_{|\nu|}$ is a finite discrete space. $S_\nu$ localizes $I_{|\nu|}$ because $\mathrm{id}(S_\nu)$ forms a basis; each $\sigma \in S_\nu$ transposes a single point in $I_{|\nu|}$.

(ii) If $X$ is a space and $G$ is a countable group of homeomorphisms acting freely on $X$, then $G \subseteq S_0(S)$ (and $g^{-1} = g^*$); if $\Lambda$ is any countable basis for $X$ then the ISG $\Omega$ generated by $G$ and $\Lambda$ localizes.

DEFINITION. Let $\psi$: $X \to Y$ be an open surjection; then $\psi$ is said to be a local homeomorphism if for every $x \in X$ there is an open set $U$, $x \in U$, so that $\psi|_U$ is a homeomorphism onto its range. Evidently $X$ has a countable covering of open sets $\Lambda$ so that $\psi|_\lambda$ is a homeomorphism onto its range for each $\lambda \in \Lambda$.

DEFINITION. If $\psi$ is as above, then $\sigma \in S_0(Y, X)$ is called a local section for $\psi$, if for every $y \in d(\sigma)$, $\psi(\sigma(y)) = y$ ($S_0(Y, X)$ is identified with the set $\{\sigma \in S_0(Y \vee X)$: $d(\sigma) \subseteq Y$ and $r(\sigma) \subseteq X\}$). A collection $\Sigma = \{\sigma_i\colon i\} \subseteq S_0(Y, X)$ of local sections is said to be complete if $X = \bigcup_i r(\sigma_i)$. Since $X$ has a covering of open sets, $\Lambda$, for which the restrictions of $\psi$ yield homeomorphisms it is evident that $\Sigma = \{(\psi|_\lambda)^*\colon \lambda \in \Lambda\}$ forms a complete family of local sections for $\psi$.

DEFINITION. With $\psi$ as above and $\Sigma = \{\sigma_i\colon i\}$ a complete family of local sections then the localization $\Omega$ generated by a basis and the set $\{\sigma_i\sigma_j^*\colon i, j\}$ is called a localization of imprimitivity related to $\psi$.

3. Observe that if $\sigma_1, \sigma_2$ are local sections for $\psi$ then $\sigma_1^*\sigma_2 \in \mathrm{id}(S_0(Y))$ since if their ranges intersect then their partial inverses must agree on the intersection (since they both agree with $\psi$).

DEFINITION. A countable collection $\Sigma = \{\sigma_j\colon j \in J\} \subseteq S_0(X)$ is said to be left coherent if $\sigma_i^*\sigma_j \in \mathrm{id}(S_0(X))$ each $i, j \in J$. The next proposition clarifies the relationship between the notions of left coherence and local sections.

PROPOSITION. *Suppose $\Sigma = \{\sigma_j\colon j \in J\} \subseteq S_0(X)$ is left coherent with $U = \bigcup_j r(\sigma_j)$ and $V = \bigcup_j d(\sigma_j)$. Then there is a unique local homeomorphism $\psi\colon U \to V$ for which $\Sigma$ forms a complete family of local sections.*

*Proof.* For $x \in r(\sigma_j)$ put $\psi(x) = \sigma_j^*(x)$; we must check that $\psi$ is well defined. Suppose $x \in r(\sigma_i)r(\sigma_j)$ $(i \neq j)$ and $\sigma_i^*(x) \neq \sigma_j^*(x)$. Evidently $\sigma_j^*(x) \in d(\sigma_i^*\sigma_j)$; then $\sigma_i^*\sigma_j(\sigma_j^*(x)) = \sigma_i^*(x)$, but by left coherence $\sigma_i^*\sigma_j$ must be an idempotent so $\sigma_i^*(x) = \sigma_j^*(x)$ and $\psi$ is well defined. The rest follows by definition. □

DEFINITION. A countable collection $\Sigma = \{\sigma_j\colon j \in J\} \subseteq S_0(X)$ is said to be coherent if both $\Sigma$ and $\Sigma^*$ are left coherent. The next proposition is crucial for what follows:

PROPOSITION. *Suppose $\Sigma = \{\sigma_j\colon j \in J\} \subseteq S_0(X)$ is coherent with $U = \bigcup_j d(\sigma_j)$ and $V = \bigcup_j r(\sigma_j)$. Then there is a unique $\sigma \in S_0(X)$ with $d(\sigma) = U$ and $r(\sigma) = V$ so that $\sigma_j = \sigma d(\sigma_j) = r(\sigma_j)\sigma$, each $j \in J$.*

*Proof.* For $x \in d(\sigma_j)$ put $\sigma(x) = \sigma_j(x)$. Then $\sigma$ is well defined because $\Sigma^*$ is left coherent (as in the above proposition). The inverse of $\sigma$, $\sigma^*$, is well defined because $\Sigma$ is left-coherent. So $\sigma$ is a bijection which is continuous at every point; $\sigma^*$ is continuous as well, hence $\sigma \in S_0(X)$ has the prescribed properties. □

Suppose $\Omega$ localizes $X$.

DEFINITION. An element $\sigma \in S_0(X)$ is said to be affiliated to $\Omega$ if for every $x \in d(\sigma)$, there is $\omega \in \Omega$ so that $x \in d(\omega) \subseteq d(\sigma)$ and $\omega = \sigma d(\omega)$. Put $\Omega^\alpha = \{\sigma \in S_0(X)\colon \sigma$ is affiliated to $\Omega\}$; $\Omega^\alpha$ is called the affiliation of $\Omega$. Note that $\sigma \in \Omega^\alpha$ iff there is a coherent family $\Sigma = \{\omega_j\colon j \in J\} \subseteq \Omega$ to which $\sigma$ is associated by the last proposition. Evidently $\sigma \in \Omega^\alpha$ iff $\sigma^* \in \Omega^\alpha$. The fact that $\Omega^\alpha$ is an ISG follows from the observation that if $\lambda \in \mathrm{id}(S_0(X))$ then $\sigma\lambda\sigma^* \in \mathrm{id}(S_0(X))$ for any $\sigma \in S_0(X)$.

DEFINITION. Two localizations $\Omega$, $\Lambda$ on $X$ are called equivalent; write $\Omega \sim \Lambda$ if $\Omega^\alpha = \Lambda^\alpha$.

4. Suppose $\Omega$ localizes $X$. Then we will show that $\Omega$ orders the open subsets of $X$; if $\Omega \subseteq \mathrm{id}(S_0(X))$ then the ordering is the usual inclusion ordering.

DEFINITION. For $U$, $V$ open in $X$ write $U \geq V$ if there are $\omega_i \in \Omega$ with $d(\omega_i) \subseteq U$ each $i$ and $V \subseteq \bigcup_i r(\omega_i)$. Transitivity must be established: Suppose $U \geq V \geq W$, then there are $\{\omega_i\}$, $\{\sigma_j\} \subseteq \Omega$, $d(\omega_i) \subseteq U$ each $i$, $d(\sigma_j) \subseteq V$ each $j$, and $V \subseteq \bigcup_i r(\omega_i)$ while $W \subseteq \bigcup_j r(\sigma_j)$. Take $\{\sigma_j\omega_i\}$; since $d(\sigma_i\omega_j) \subseteq d(\omega_i) \subseteq U$ each $i$, $j$ and $W \subseteq \bigcup_{i,j} r(\sigma_j\omega_i)$ $(r(\sigma_j) \subseteq \bigcup_i r(\sigma_j\omega_i))$, the result follows.

DEFINITION. Write $U \sim V$ if $U \geq V$ and $V \geq U$.

PROPOSITION. *$U \sim V$ iff there are $\omega_i \in \Omega$ with $U = \bigcup_i d(\omega_i)$, $V = \bigcup_i r(\omega_i)$.*

*Proof.* Suppose there are $\sigma_j, \gamma_k \in \Omega$ with $d(\sigma_j) \subseteq U$, $d(\gamma_k) \subseteq V$ while $V \subseteq \bigcup_j r(\sigma_j)$ and $U \subseteq \bigcup_k r(\gamma_k)$. There are $\lambda_l, \mu_m \in \mathrm{id}(\Omega)$ with $U = \bigcup_l \lambda_l$ and $V = \bigcup_m \mu_m$. Put $\{\omega_i\} = \{\mu_m\sigma_j\} \cup \{\gamma_k * \lambda_l\}$ (after reindexing); these $\omega_i$ satisfy the prescribed conditions. The converse is obvious. □

COROLLARY. *If $\Omega$ localizes $X$, the following are equivalent:*

(i) *For each open $U \neq \varnothing$, $U \sim X$.*

(ii) *For each open $U \neq \varnothing$, there are $\omega_i \in \Omega$ with $d(\omega_i) \subseteq U$ each $i$ and $X = \bigcup_i r(\omega_i)$.*

(iii) *For each pair of open sets $U$, $V$ there are $\omega_i \in \Omega$ so that $U = \bigcup_i d(\omega_i)$ and $V = \bigcup_i r(\omega_i)$.*

This follows immediately from the definitions and the proposition.

DEFINITION. Such a localization is called a simplification, or briefly, say $\Omega$ simplifies $X$.

EXAMPLE. Suppose $\Omega$ is a localization associated with a free group action as in example (ii) of 2. Claim: $\Omega$ simplifies iff the action is minimal (i.e., each orbit is dense). If the action is minimal, and $U$ is a non-empty open set then every orbit meets $U$. So for each $x \in X$ there is a $g \in G$ so that $g(x) \in U$. Hence $X = \bigcup_g g^{-1}(U)$. If the action is not minimal, there is an orbit for which the complement has non-void interior $U$. $U$ is invariant under the action of $G$ so $U \not\geq X$.

PROPOSITION. *The partial orders induced by equivalent localizations are identical.*

*Proof.* Suppose $\Omega$, $\Pi$ localize $X$ and $\Omega^\alpha = \Pi^\alpha$. Let $U$, $V$ be non-empty open sets. Suppose there are $\omega_i \in \Omega$ with $d(\omega_i) \subseteq U$ each $i$ and $V \subseteq \bigcup_i r(\omega_i)$. Since $\Omega \subseteq \Pi^\alpha$, there is for each $i$ a coherent collection $\{\sigma_{ij}\} \subseteq \Pi$ and $d(\omega_i) = \bigcup_j d(\sigma_{ij})$ and $r(\omega_i) = \bigcup_j r(\sigma_{ij})$. Consequently $U \geq V$ with respect to $\Pi$. By symmetry the induced partial orders are the same. $\square$

COROLLARY. *If $\Omega$ and $\Pi$ are localizations on $X$ with $\Omega \sim \Pi$. Then $\Omega$ is a simplification iff $\Pi$ is.*

5. Let $\Omega$ localize $X$. Then for $U \subseteq X$ open, put $\Omega_U = \{\omega \in \Omega$: $d(\omega), r(\omega) \subseteq U\}$. Note that $\Omega_U$ localizes $U$.

DEFINITION. $\Omega_U$ is called the reduction of $\Omega$ to $U$.

DEFINITION. If $\Lambda$ is a countable open covering of $X$, then a localization $\Omega$ on $X$ is said to be subordinate to $\Lambda$ if $\mathrm{id}(\Omega)$ refines $\Lambda$ (i.e., for $\omega \in \mathrm{id}(\Omega)$ there is $\lambda \in \Lambda$ so that $\lambda\omega = \omega$).

PROPOSITION. *If $\Lambda$ is a countable open covering of $X$ and $\Omega$ is a localization, then there is a sublocalization $\Pi \subseteq \Omega$ so that $\Pi$ is subordinate to $\Lambda$ and $\Pi \sim \Omega$.*

*Proof.* Let $\Pi = \{\omega \in \Omega$: there is $\lambda$, $\mu \in \Lambda$, $\lambda d(\omega) = d(\omega)$ and $\mu r(\omega) = r(\omega)\}$. Evidently $\Pi$ is an ISG and $\mathrm{id}(\Pi)$ forms a basis. It remains to show $\Omega \subseteq \Pi^\alpha$. Fix $\omega \in \Omega$; choose $\mu_m, \nu_n \in \mathrm{id}(\Pi)$ so that $d(\omega) = \bigcup_m \mu_m$, $r(\omega) = \bigcup_n \nu_n$ then $\{\nu_n \omega \mu_m: n, m\} \subseteq \Pi$. So $\omega \in \Pi^\alpha$. $\square$

COROLLARY. *Suppose $\Omega$ localizes $X$ and $U \subseteq X$ is open. If there is $\{\omega_i: i \in I\} \subseteq \Omega$ so that $d(\omega_i) \subseteq U$ each $i$ and $X = \bigcup_i r(\omega_i)$ then $\Pi = \{\omega_i \omega \omega_j^*: \omega \in \Omega_U, i, j \in I\}$ is a localization; further $\Pi \sim \Omega$.*

PROPOSITION. *Let $\Omega$, $U$, and $\{\omega_i: i \in I\}$ be as above. Then $\Omega$ simplifies $X$ iff $\Omega_U$ simplifies $U$.*

*Proof.* If $\Omega$ simplifies then $\Omega_U$ simplifies because the induced trivial ordering is inherited. Suppose $\Omega_U$ simplifies $U$; if $V$ is open ($\neq \varnothing$) there is $\omega_{i_0}$ and $\lambda \in \mathrm{id}(\Omega_U)$ so that $r(\omega_{i_0}\lambda) \subseteq V$ and $\lambda d(\omega_{i_0}) = \lambda$. There are $\sigma_j \in \Omega_U$ so that $\lambda d(\sigma_j) = d(\sigma_j)$ and $U = \bigcup_j r(\sigma_j)$ (because $\Omega_U$ simplifies). Let $\{\gamma_l\}$ be a reindexing of the set $\{\omega_k \sigma_j \omega_{i_0}^*: k, j\}$. Then $d(\gamma_k) \subseteq V$ each $k$ while $X = \bigcup_k r(\gamma_k)$. $\square$

DEFINITION. Suppose $\Omega$ simplifies $X$, $\Pi$ simplifies $Y$. Then they are said to belong to the same texture if there is $\sigma \in S_0(X, Y)$ so that $\Omega^{\alpha}_{d(\sigma)} = \sigma^*(\Pi^{\alpha}_{r(\sigma)})\sigma$. We return to this notion in §4.

6. We consider a class of localizations on compact zero-dimensional spaces, which may, in some sense, be viewed as inductive limits of finite localizations and are termed AF localizations. This construction is suggested in Remark 2.16 of **[13]**.

Choose a function $n: \mathbf{Z}^+ \to \mathbf{Z}^+$ so that $n(0) = 1$ and $n(i) \geq 1$ for all $i$. Choose multiplicity indices $m(i, j, k) \in \mathbf{Z}^+$ for $i \geq 0$, $0 \leq j < n(i)$, $0 \leq k < n(i+1)$ so that given $i, j$ there is $k$ with $m(i, j, k) > 0$ and given $i$, $k$ there is $j$ with $m(i, j, k) > 0$.

Let $X(m)$ denote the collection of ordered pairs $(p, q)$ with $p, q: \mathbf{Z}^+ \to \mathbf{Z}^+$ satisfying:

(i) $p(0) = q(0) = 0$,

(ii) $0 \leq q(i) < n(i)$, $i \geq 0$,

(iii) $m(i, q(i), q(i+1)) > 0$, $i \geq 0$,

(iv) $0 \leq p(i+1) < m(i, q(i), q(i+1))$, $i \geq 0$.

For $l \in \mathbf{Z}^+$ let $E(l)$ denote the collection of pairs $(s, t)$ with $s$, $t$: $\{0, 1, \ldots, l\} \to \mathbf{Z}^+$ for which there is $(p, q) \in X(m)$ such that:

$$s(i) = p(i),$$
$$t(i) = q(i) \quad \text{for } 0 \leq i \leq l.$$

Set $K(s, t) = \{(p, q) \in X(m): p(i) = s(i),\ q(i) = t(i),\ 0 \leq i \leq l\}$. Observe that if $(s, t) \in E(l)$ and $(u, v) \in E(l+1)$, then $K(s, t) \cap K(u, v) = \varnothing$ or $K(u, v)$. The collection $K(E) = \{K(s, t): (s, t) \in E(l),\ l \geq 0\}$

forms a basis for a topology on $X(m)$ so that each $K(s, t)$ is compact-open (observe that for each $l > 0$, the collection

$$\{K(s, t) \colon (s, t) \in E(l)\}$$

is a finite disjoint covering of $X(m)$).

Suppose $(s, t)$, $(u, v) \in E(l)$ and $t(l) = v(l)$. Then for every $(p, q) \in K(s, t)$, there is a unique $(x, y) \in K(u, v)$ so that:

$$x(i) = p(i), \quad y(i) = q(i) \quad \text{all } i > l.$$

Let $\sigma[(u, v), (s, t)]$ denote this assignment and observe that

$$\sigma[(u, v), (s, t)] \in S_0(X(m)).$$

Evidently

$$d(\sigma[(u, v), (s, t)]) = K(s, t)$$

and

$$\sigma[(u, v), (s, t)]^* = \sigma[(s, t), (u, v)]$$

and if

$$(e, f) \in E(l), f(l) = t(l) = v(l)$$

then

$$\sigma[(u, v), (s, t)]\sigma[(s, t), (e, f)] = \sigma[(u, v), (e, f)].$$

Let

$$\Omega(m) = \{\sigma[(u, v), (s, t)] \colon (u, v), (s, t) \in E(l), v(l) = t(l), l \geq 0\}.$$

Then $\Omega(m)$ is called an AF localization (on $X(m)$). For each $i \geq 0$ let $\nu_i$: $\{0, \ldots, n(i) - 1\} \to \mathbf{Z}^+$ be defined inductively by $\nu_0(0) = 1$ and

$$\nu_{i+1}(k) = \sum_j m(i, j, k)\nu_i(j).$$

For $l \geq 0$, $0 \leq k < n(l)$:

$$\nu_l(k) = \#\{(s, t) \in E(l) \colon t(l) = k\}.$$

Whence,

$$S_{\nu_l} \cong \{\sigma[(u, v), (s, t)] \colon (u, v), (s, t) \in E(l), v(l) = t(l)\}$$

(i.e., they are isomorphic as ISG's).

If $\nu = (n_1, \ldots, n_k)$ is a multi-index with $n_\iota > 0$, then $S_\nu$ localizes $I_{n_1} \vee \cdots \vee I_{n_k}$ so that each summand $I_{n_\iota}$ is left invariant (§2/4). The affiliation $S_\nu^\alpha$ consists of all partial bijections on $I_{n_\iota} \vee \cdots \vee I_{n_k}$ leaving the summand $I_{n_l}$ invariant. Then, identifying $S_{\nu_l}$ with its image in $S_0(X(m))$, it is clear that $S^\alpha_{\nu_l} \subseteq S^\alpha_{\nu_{l+1}}$, and that

$$\Omega'(m) = \bigcup_l S^\alpha_{\nu_l}$$

is equivalent to $\Omega(m)$. (Also see Strătilă-Voiculescu diagonalization of an AF algebra [**46**], p. 19.)

7. Definition. If $\Omega$ localizes $X$ then define the ample group of $\Omega$, $\Gamma(\Omega)$, to be the group of homeomorphisms in $\Omega^\alpha$, more explicitly $\Gamma(\Omega) = \{\gamma \in \Omega^\alpha\colon d(\gamma) = r(\gamma) = X\}$. Then $\Omega$ is termed dynamical if for every $\omega \in \Omega$, there is $\gamma \in \Gamma(\Omega)$ such that $\omega = \gamma d(\omega)$.

Examples.

(i) If $S_\nu$ is a finite localization ($\nu = (n_1, \ldots, n_k)$) then $\Gamma(S_\nu) = S(n_1) \times \cdots \times S(n_k)$ where $S(n_i)$ is the symmetric group on $n_i$ letters.

(ii) If $\Omega$ is an AF localization, as above, then $\Gamma(\Omega)$ is the inductive limit of the ample groups for the finite localizations $S_{\nu_l}$. Then $\Omega$ is dynamical, since each $S_{\nu_l}$ is.

(iii) Let $G$ be a countable dense subgroup of the additive reals. Let $\Omega$ be a localization on $\mathbf{R}$ generated by a basis and the translations by elements of $G$. Since $G$ is dense, $\Omega$ simplifies. Evidently $G \subseteq \Gamma(\Omega)$. If $\gamma \in \Gamma(\Omega)$, there is $\lambda \in \mathrm{id}(\Omega)$ and $g \in G$ so that $\gamma\lambda = g\lambda$. Since $\mathbf{R}$ is connected, $\gamma = g$; hence $\Gamma(\Omega) = G$. Also, $\Omega$ is dynamical.

## 4. Induced simplifications.

1. Keeping in mind that the notion of strong Morita equivalence (SME) is a powerful classifying tool when applied to simple $C^*$-algebras (since every hereditary subalgebra must belong to the same SME class), and that the goal is to evolve a unifying approach to a large class of simple $C^*$-algebras via the device of simplifications, it would be desirable to transplant the notion to this situation. Recall that two simple $C^*$-algebras are SME iff there is a hereditary algebra in one isomorphic to a hereditary subalgebra of the other; this suggests that the relationship of belonging to the same texture, among simplifications, be explored (it is certainly a transitive relationship).

2. There are three sorts of texture preserving operations that can be performed in which simplifications are induced to other spaces in a canonical way and which when combined exhaust the texture class. Suppose $\Omega$ simplifies $X$.

(i) Reductions: If $U \subseteq X$ is open ($\neq \varnothing$) then the reduction $\Omega_U$ simplifies $U$ and trivially belongs to the same texture as $\Omega$.

(ii) Ampliations: $S_n \times \Omega$ simplifies $I_n \times X$. In this case write $\Omega^n = S_n \times \Omega$; $\Omega^n$ is called the $n$-fold ampliation of $\Omega$. $\Omega^\infty$ is simply called the ampliation of $\Omega$. Then $\Omega^n$ clearly belongs to the same texture as $\Omega$.

(iii) Primitives: Suppose $\psi$: $X \to Y$ is a local homeomorphism; then $\psi$ is said to be admitted by $\Omega$ if for every $x_1, x_2 \in X$ with $\psi(x_1) = \psi(x_2)$ there is $\omega \in \Omega$, $x_1 \in d(\omega)$, $\omega(x_1) = x_2$ and for each $x \in d(\omega)$, $\psi(\omega(x)) = \psi(x)$. (Otherwise put: If $\Pi$ is a localization of imprimitivity related to $\psi$ then $\Pi \subseteq \Omega^\alpha$.) Put $\Sigma = \{\sigma \in S_0(Y, X)$: $\sigma\sigma^* \in \mathrm{id}(\Omega)$, $\psi(\sigma(y)) = y$ each $y \in d(\sigma)\}$, then $\Sigma$ is countable and so forms a complete family of local sections for $\psi$. Let $\Omega_\psi$ be the localization on $Y$ generated by $\{\sigma_1^*\omega\sigma_2$: $\omega \in \Omega, \sigma_1, \sigma_2 \in \Sigma\}$. Then $\Omega_\psi$ simplifies and is called the primitive induced by $\psi$. $\Omega_\psi$ is in the same texture class as $\Omega$.

3. If $\Omega$ simplifies $X$, and $Y$ is a space, then we would like to know when $Y$ admits a simplification of the same texture class.

DEFINITION. $\Sigma = \{\sigma_i\colon i\} \subseteq S_0(Y, X)$ is called an $\Omega$-atlas for $Y$ if $Y = \bigcup_i d(\sigma_i)$ and $\Sigma\Sigma^* \subseteq \Omega^\alpha$. We say that two $\Omega$-atlases are equivalent if their union is an $\Omega$-atlas as well. Then $Y$ with an equivalence class of $\Omega$-atlases is called an $\Omega$-manifold. If $\Sigma = \{\sigma_i\colon i\}$ is an $\Omega$-atlas for $Y$ we define the simplification induced via $\Sigma$ (and write $\Omega(\Sigma)$) to be

$$\Omega(\Sigma) = \mathrm{ISG}\{\sigma_i^*\omega\sigma_j\colon \omega \in \Omega, i, j\} \subseteq S_0(Y).$$

Alternately let $\Upsilon(\Sigma) \subseteq S_0(X \vee Y)$ be the localization generated by $\Omega$ and $\Sigma$. Then $\Upsilon(\Sigma)$ simplifies and is called the linking simplification. Observe that $\Omega \sim \Upsilon(\Sigma)_X$ and $\Omega(\Sigma) \sim \Upsilon(\Sigma)_Y$. Certainly $\Omega$ and $\Omega(\Sigma)$ belong to the same texture class (and if $\Lambda$ is equivalent to $\Sigma$ as $\Omega$-atlas then $\Omega(\Lambda) \sim \Omega(\Sigma)$).

PROPOSITION. *If $\Pi$ simplifies $Y$ and belongs to the same texture class as $\Omega$, then there is an $\Omega$-atlas $\Sigma$ on $Y$ so that $\Sigma^*$ is a $\Pi$-atlas on $X$. Moreover $\Pi \sim \Omega(\Sigma)$ while $\Omega \sim \Pi(\Sigma^*)$.*

*Proof.* By definition, $\Pi$ and $\Omega$ each have reductions with identical affiliations. □

PROPOSITION. *If $\Sigma$ is an $\Omega$-atlas on $Y$, then $\Omega(\Sigma)$ is expressible as the primitive of a reduction of an ampliation.*

*Proof*. Put $\Sigma = \{\sigma_i : i\} \subseteq S_0(Y, X)$ and let $\Omega^\infty$ be the ampliation of $\Omega$ on $I_\infty \times X$; let $U = \bigcup_i \{i\} \times r(\sigma_i)$; then $U$ is open in $I_\infty \times X$. We define a local homeomorphism $\psi\colon U \to Y$ by $\psi((i, x)) = \sigma_i^*(x)$; then $\psi$ is admitted by $(\Omega^\infty)_U$ exactly because $\Sigma$ is an $\Omega$-atlas. Further $\Omega(\Sigma) \sim ((\Omega^\infty)_U)_\psi$. □

4. There are two other methods for constructing $\Omega$-manifolds which will prove to be useful in the sequel. The first is a synthesis of disjoint $\Omega$-manifolds and the second is a reciprocal of the primitive construction.

*Fact*. Suppose $\{Y_i\}$ is a countable (finite or infinite) collection of spaces with $\Omega$-atlases $\{\Sigma_i\}$. Then $\bigcup_i \Sigma_i \subseteq S_0(\bigvee_i Y_i, X)$ forms an $\Omega$-atlas on $\bigvee_i Y_i$.

DEFINITION. The resulting $\Omega$-manifold is written $\bigvee_i Y_i$ and is referred to as the disjoint sum (or disjoint union) of the collection of $\Omega$-manifolds, $\{Y_i\}$.

*Fact*. Suppose $\psi\colon Y \to X$ is a local homeomorphism; let $\Sigma \subseteq S_0(X, Y)$ be the collection of local sections of $\psi$ such that $\sigma \in \Sigma$ iff $d(\sigma) \in \mathrm{id}(\Omega)$. Then $\Sigma^*$ is an $\Omega$-atlas on $Y(\Sigma^*\Sigma \subseteq \mathrm{id}(\Omega^\alpha))$.

DEFINITION. The induced simplification is written $\Omega^\psi$ and is termed the simplification of imprimitivity, or simply the imprimitive (note that this notion generalizes the notion of ampliation).

REMARK. Evidently, $\Omega^\psi$ admits $\psi$ and $(\Omega^\psi)_\psi \sim \Omega$.

5. EXAMPLES. (i) Let $X$ be the Cantor set with the relative topology. then $X \cong \{x = (x_n) : x_n = 0 \text{ or } 1, n \geq 0\}$. Let $E = \{(e_0, e_1, \ldots, e_n) : e_i = 0$ or $1, n \geq 0\}$, the collection of finite strings in $\{0, 1\}$, if $e = (e_0, e_1, \ldots, e_n) \in E$, write $l(e) = n$, and set $K(e) = \{x \in X : x_j = e_j \text{ for } 0 \leq j \leq l(e)\}$. Then $\{K(e) : e \in E\}$ is a countable basis for $X$ consisting of compact open sets. For $e, f \in E$ with $l(e) = l(f)$ define $\sigma(e, f)\colon K(f) \to K(e)$ by $(\sigma(e, f)(x))_n = x_n$, $n > l(f)$. Then $\sigma(e, f) \in S_0(X)$, $d(\sigma(e, f)) = \sigma(f, f) = K(f)$, $r(\sigma(e, f)) = \sigma(e, e) = K(e)$, and $\sigma(e, f)^* = \sigma(f, e)$. Then $\Omega = \{\sigma(e, f) : e, f \in E,\ l(e) = l(f)\}$ simplifies $X$. Also, $\Omega$ is dynamical; in fact $\Omega$ is equivalent to the simplification generated by the action of $\oplus \mathbf{Z}_2$. If $Y$ is an $\Omega$-manifold then it has a basis of compact open sets. Consequently, $Y$ may be reconstituted as the countable disjoint union

of reductions to compact open subsets of $X$ (this is also true for arbitrary AF localizations).

(ii) If $G$ is a countable dense subgroup of $\mathbf{R}$ then choose a simplification on $\mathbf{R}$ generated by translation by elements of $G$ and a basis. If $g \in G$, $g > 0$, define a local homeomorphism $\psi_g$: $\mathbf{R} \to \mathbf{T}$ by $\psi_g(x) = x \pmod g$. Then $\psi_g$ defines a primitive $\Omega_g$ on $\mathbf{T}$ because $\psi_g(x) = \psi_g(y)$ iff $y = x + ng$ for some $n \in \mathbf{Z}$. Write $X_g$ for the $\Omega$-manifold structure assigned to $\mathbf{T}$ by $\psi_g$. Observe that $\Gamma(\Omega_g) = G/\mathbf{Z}g$ and that every $\omega \in \Omega_g$ is the restriction of some $\gamma \in \Gamma(\Omega_g)$ (the ample group). Observe that $\Omega$ may be recovered as a simplification of imprimitivity: more explicity, $\Omega \sim (\Omega_g)^{\psi_g}$.

(iii) We consider induced flows as defined in [**34**]. Suppose $X$ is compact and $T$: $X \to X$ is a homeomorphism so that the associated action of the integers is free (i.e., for each $x$, $T^n x = T^m x$ iff $m = n$) and minimal. Suppose $u$: $X \to \{1, 2, \ldots, n\}$ is continuous; set $X_k = u^{-1}(k)$. Then put $X^u = \bigvee_{i=1}^{n} I_i \times X_i$ and define a homeomorphism $T^u$: $X^u \to X^u$ as follows:

$$T^u x = \begin{cases} (i+1, y) & \text{if } x = (i, y) \in I_k \times X_k \quad \text{and} \quad i < k, \\ (1, Ty) & \text{if } x = (k, y) \in I_k \times X_k, \end{cases}$$

where here $I_k = \{1, \ldots, k\}$. The authors call the pair $(X^u, T^u)$ a primitive of the flow $(X, T)$. If $\Omega$ is a simplification on $X$ associated to $T$ then the local homeomorphism $\psi$: $X^u \to X$ by $\psi(i, y) = y$ yields the imprimitive $\Omega^\psi$ on $X^u$. It's not difficult to see that $T^u$ induces a map $\mathbf{Z} \to \Gamma(\Omega^\psi)$ and that $\Omega^\psi$ is dynamical (since if $\Pi$ is a simplification on $X^u$ associated to the action of $T^u$ then $\Pi \sim \Omega^\psi$). A derivative of $(X, T)$ is also defined; if $A \subseteq X$ is compact open define $T_A$: $A \to A$ by $T_A x = T^n x$ where $n = \min\{k > 0\colon T^k x \in A\}$. The authors show that $T_A$ is a homeomorphism and call the pair $(A, T_A)$ a derivative flow of $(X, T)$. Now, if $\Xi$ is a simplification on $A$ associated to the action of $T_A$, then $\Xi \sim \Omega_A$, the reduction. Additionally, let $\beta$: $X \to A$ be defined by $\beta(x) = T^n x$ where $n = \min\{k \geq 0\colon T^k x \in A\}$. Then $\beta$ is a local homeomorphism admitted by $\Omega$. Let $\Omega_\beta$ be the primitive on $A$; then $\Omega_\beta \sim \Omega_A \sim \Xi$.

## 5. Associated $C^*$-algebras.

1. Let $\Omega$ localize $X$. In this section a normed $*$-algebra is associated to $\Omega$, denoted $C_c(\Omega)$, so that if $\Pi \sim \Omega$ then $C_c(\Omega)$ and $C_c(\Pi)$ are canonically isomorphic. The subalgebra $C_c(X)$ is maximal abelian (in fact $C_c(X) \cong C_c(\mathrm{id}(\Omega))$), and if $\Omega$ simplifies, then $C_c(\Omega)$ is algebraically simple. The bounded non-degenerate $*$-representations yield the enveloping $C^*$-algebra, denoted $C^*(\Omega)$. A one-to-one correspondence between the representations of $C^*(\Omega)$ and the $*$-representations of $\Omega$ is established.

2. NOTATION. For $U$ open in $X$, put $C_c(U) = \{f \in C_c(X)\colon \operatorname{supp}(f) \subseteq U\}$. For $f \in C(X)$, $\sigma \in S_0(X)$ define $f \cdot \sigma \in C(d(\sigma))$ by $(f \cdot \sigma)(x) = f(\sigma(x))$; if $f \in C_c(r(\sigma))$, $f \cdot \sigma \in C_c(d(\sigma))$. For $\Omega$ a localization on $X$, put $E(\Omega) = \{(\omega, f)\colon\ \omega \in \Omega,\ f \in C_c(d(\omega))\}$. Then $E(\Omega)$ is a semigroup if composition is defined:

$$(\omega, f)(\sigma, g) = (\omega\sigma, (f \cdot \sigma)g).$$

(Observe that $(f \cdot \sigma)g = (f(g \cdot \sigma^*)) \cdot \sigma \in C_c(d(\omega\sigma))$.) Composition is associative since

$$(\omega_1, f_1)(\omega_2, f_2)(\omega_3, f_3) = (\omega_1\omega_2\omega_3, (f_1 \cdot \omega_2\omega_3)(f_2 \cdot \omega_3)f_3).$$

For $(\omega, f) \in E(\Omega)$, put $(\omega, f)^* = (\omega^*, \bar{f} \cdot \omega^*)$. Then it is easily checked that $(\omega, f)^{**} = (\omega, f)$, and if $(\sigma, g) \in E(\Omega)$ then $((\omega, f)(\sigma, g))^* = (\sigma, g)^*(\omega, f)^*$; $E(\Omega)$ is not usually an ISG since $(\theta, 0)$ is in general the only idempotent.

For $\sigma \in \Omega$, put $E(\sigma) = \{(\omega, f) \in E(\Omega)\colon\ \omega = \sigma\}$ and observe that $E(\sigma)$ may be viewed as a vector space ($\simeq C_c(d(\sigma))$); now put $D(\Omega) = \bigoplus_{\omega \in \Omega} E(\omega)$. We obtain a *-algebra when the operations on $E(\Omega)$ are extended linearly to $D(\Omega)$, viz.:

$$\Big(\sum_i (\omega_i, f_i)\Big)\Big(\sum_j (\sigma_j, g_j)\Big) = \sum_{ij} (\omega_i\sigma_j, (f_i \cdot \sigma_j)g_j),$$

$$\Big(\sum_i (\omega_i, f_i)\Big)^* = \sum_i (\omega_i^*, \bar{f}_i \cdot \omega_i^*),$$

where $(\omega_i, f_i)$, $(\sigma_j, g_j) \in E(\Omega)$ and $1 \le i \le n$, $1 \le j \le m$. (NB: $E(\theta) \simeq C_c(\varnothing) = \{0\}$, the trivial vector space.)

This *-algebra is too large for our purposes. If, for instance, $\lambda_i, \mu_j \in \operatorname{id}(\Omega)$ and $\Sigma(\lambda_i, f_i)$, $\Sigma(\mu_j, g_j) \in D(\Omega)$ and $\Sigma_i f_i = \Sigma_j g_j$, then it would be desirable to regard these two elements of $D(\Omega)$ as the same. Further, if $\omega, \sigma \in \Omega$ and there is an open set $U \subseteq d(\omega)d(\sigma)$ with $\omega|_U = \sigma|_U$, then for $f \in C_c(U)$ it would be desirable to regard $(\omega, f)$ and $(\sigma, f)$ as the same. We define the ideal of coherence, $I(\Omega)$, towards this end.

DEFINITION. $\{(\omega_i, f_i) \in E(\Omega)\colon i\}$ is said to be coherent if there are open sets $U_i \subseteq X$, with $\operatorname{supp}(f_i) \subseteq U_i \subseteq d(\omega_i)$ and $\{\omega_i U_i\}$ is coherent in $S_0(X)$. Put $I(\Omega) = \operatorname{sp}\{\Sigma(\omega_i, f_i)\colon \{(\omega_i, f_i)\}$ coherent and $\Sigma f_i = 0\}$.

We will define a norm $|\cdot|_0$ on the quotient $D(\Omega)/I(\Omega)$ so that if $\{(\omega_i, f_i)\}$ is coherent then $|\Sigma(\omega_i, f_i)|_0 = |\Sigma f_i|_\infty$ and $|\cdot|_0$ is the largest norm with this property.

PROPOSITION. *$I(\Omega)$ is a *-ideal in $D(\Omega)$.*

*Proof.* If $\{(\omega_i, f_i)\}$ is coherent then $\{(\omega_i^*, \bar{f}_i \cdot \omega_i^*)\}$ is coherent as well, and $\Sigma f_i = 0$ iff $\Sigma(f_i \cdot \omega_i^*) = 0$. Further, if $(\sigma, g) \in E(\Omega)$ and $\{(\omega_i, f_i)\}$ is coherent then $\{(\sigma\omega_i, (g \cdot \omega_i)f_i)\}$ is coherent and

$$\Sigma(g \cdot \omega_i)f_i = \Sigma(g(f_i \cdot \omega_i^*))\omega_i = (g\Sigma(f_i \cdot \omega_i^*))\omega_i = 0,$$

whence $I(\Omega)$ is a two-sided $*$-ideal in $D(\Omega)$. □

DEFINITION. A coherent partition for $a \in D(\Omega)$ is a finite collection $\{(\omega_{ij}, f_{ij}): i, j\}$ so that for fixed $i$, $\{(\omega_{ij}, f_{ij}): j\}$ is coherent and

$$a - \sum_{i,j} (\omega_{ij}, f_{ij}) \in I(\Omega).$$

Put

$$|a|_0 = \inf\left\{ \sum_i \left| \sum_j f_{ij} \right|_\infty : \{(\omega_{ij}, f_{ij})\} \text{ a coherent partition for } a \right\}.$$

If $a \in I(\Omega)$ then $|a|_0 = 0$. For $a \in D(\Omega)$, $|a|_0 = |a^*|_0$ since $\{(\omega_{ij}^*, \bar{f}_{ij} \cdot \omega_{ij}^*)\}$ is a coherent partition for $a^*$ whenever $\{(\omega_{ij}, f_{ij} \cdot \omega_{ij})\}$ is a coherent partition for $a$.

PROPOSITION. *For* $a, b \in D(\Omega)$, $|ab|_0 \leq |a|_0 \cdot |b|_0$ *and* $|a + b|_0 \leq |a|_0 + |b|_0$.

*Proof.* Let $\{(\omega_{ij}, f_{ij})\}$, $\{(\sigma_{kl}, g_{kl})\}$ be coherent partitions for $a$ and $b$ respectively. Then for each $i$, $k$ the collection $\{(\omega_{ij}\sigma_{kl}, (f_{ij} \cdot \sigma_{kl})g_{kl}): j, l\}$ is coherent (being the composition of two coherent collections). We have

$$\left| \sum_{j,l} (f_{ij} \cdot \sigma_{kl})g_{kl} \right|_\infty \leq \left| \sum_j f_{ij} \right|_\infty \left| \sum_l g_{kl} \right|_\infty \quad \text{for each } i, k.$$

Let $\{(\gamma_{st}, h_{st})\}$ be a reindexing of $\{(\omega_{ij}\sigma_{kl}, (f_{ij} \cdot \sigma_{kl})g_{kl})\}$ ($s$ is a reindexing for $i$, $k$ and $t$ is a reindexing for $j$, $l$). Then $\{(\gamma_{st}, h_{st})\}$ is a coherent partition for $ab$. Further,

$$|ab|_0 \leq \sum_s \left| \sum_t h_{st} \right|_\infty \leq \left( \sum_i \left| \sum_j f_{ij} \right|_\infty \right) \left( \sum_k \left| \sum_l g_{kl} \right|_\infty \right).$$

Since $\{(\omega_{ij}, f_{ij})\}$, $\{(\sigma_{kl}, g_{kl})\}$ were arbitrary coherent partitions for $a$ and $b$ the first inequality is established. Since $\{(\omega_{ij}, f_{ij})\} \cup \{(\sigma_{kl}, g_{kl})\}$ is a coherent partition for $a + b$ it follows that

$$|a + b|_0 \leq \sum_i \left| \sum_j f_{ij} \right|_\infty + \sum_k \left| \sum_l g_{kl} \right|_\infty.$$

Again, since the coherent partitions were arbitrary the second inequality is established. □

COROLLARY. *$I_0(\Omega) = \{a \in D(\Omega)\colon |a|_0 = 0\}$ is a two-sided ideal in $D(\Omega)$.*

*Fact.* Suppose $a = \Sigma_i(\lambda_i, f_i) \in D(\Omega)$ with $\lambda_i \in \mathrm{id}(\Omega)$. Then $|a|_0 = |\Sigma f_i|_\infty$. Further, if $b = \Sigma_j(\mu_j, g_j) \in D(\Omega)$ with $\mu_j \in \mathrm{id}(\Omega)$ so that $\Sigma g_j = \Sigma f_i$, then $|a - b|_0 = 0$.

DEFINITION. Let $C_c(\Omega)$ denote the quotient $D(\Omega)/I_0(\Omega)$.

PROPOSITION. *The quotient $C_c(\Omega)$ is a normed *-algebra with maximal abelian subalgebra $C_c(X)$ ($\cong C_c(\mathrm{id}(\Omega))$). Further, there is $\{f_i\} \subseteq C_c(X)$ so that $\{f_i\}$ forms an increasing approximate identity for $C_c(\Omega)$.*

*Proof.* If $f \in C_c(X)$, there are $\lambda_1, \ldots, \lambda_n \in \mathrm{id}(\Omega)$ so that $\mathrm{supp}(f) \subseteq \bigcup_i \lambda_i$ and $g_i \in C_c(\lambda_i)$ with $f = \Sigma g_i$. Whence, $C_c(X) \subseteq C_c(\Omega)$.

Suppose $a \in D(\Omega)$; thus $a = \Sigma(\omega_i, h_i)$. Let

$$K = \Big(\bigcup \mathrm{supp}(h_i)\Big) \cup \Big(\bigcup \mathrm{supp}(h_i \cdot \omega_i^*)\Big).$$

Then there is $f \in C_c(X)$ so that $f(x) = 1$ for every $x \in K$. Choose $\lambda_i \in \mathrm{id}(\Omega)$, $g_j \in C_c(\lambda_j)$ so that $f = \Sigma g_j$.

Then we have $a(\Sigma(\lambda_j, g_j)) - a \in I_0(\Omega)$, since for each $i$, $(\omega_i, h_i) - \Sigma(\omega_i\lambda_j, h_i g_j) \in I_0(\Omega)$. Viewing $a, f$ as elements of $C_c(\Omega)$, we see that

$$af = fa = a.$$

Then, since $X$ is a $\sigma$-compact, $C_c(X)$ has a countable increasing approximate identity, $\{f_i\}$, which is also an approximate identity for $C_c(\Omega)$.

The image of $a \in D(\Omega)$ commutes with $C_c(X)$ iff for every $(\lambda, g)$ with $\lambda \in \mathrm{id}(\Omega)$, $(\lambda, g)a - a(\lambda, g) \in I_0(\Omega)$. Choose $f \in C_c(X)$ so that $fa = af = a$. For every $\lambda_1, \ldots, \lambda_n \in \mathrm{id}(\Omega)$ with $\mathrm{supp}(f) \subseteq \bigcup_i \lambda_i$ there are $g_i \in C_c(\lambda_i)$ so that $f = \Sigma g_i^2$.

Hence $\Sigma g_i a g_i - a \in I_0(\Omega)$ for each choice of $\{(\lambda_i, g_i)\}$. Whence $a \in C_c(X)$ (i.e., $a \in C_c(\mathrm{id}(\Omega)) + I_0(\Omega)$). □

3. PROPOSITION. *If $\Omega$, $\Pi$ are equivalent localizations then the associated normed *-algebra are canonically isomorphic.*

*Proof.* Suppose $\Omega \sim \Pi$ on $X$; if $(\omega, f) \in E(\Omega)$ then there exist $\sigma_1, \ldots, \sigma_n \in \Pi$ so that $\mathrm{supp}(f) \subseteq \bigcup_i d(\sigma_i) \subseteq d(\omega)$ and $\sigma_i = \omega d(\sigma_i)$ (i.e.,

$\{\omega, \sigma_1, \ldots, \sigma_n\}$ is coherent). By 2.13 of [44] there are $g_i \in C_c(d(\sigma_i))$ so that $f = \Sigma_i fg_i$. Observe that if $(\gamma_j, h_j) \in E(\Pi)$ with $\operatorname{supp}(f) \subseteq \bigcup_j d(\gamma_j) \subseteq d(\omega)$, $\{\omega, \gamma_1, \ldots, \gamma_m\}$ coherent, and $f = \Sigma fh_j$ then $\Sigma_i(\gamma_j, fh_j) - \Sigma(\sigma_i, fg_i) \in I(\Pi)$; this mapping from $E(\Omega) \to C_c(\Pi)$ extends linearly to $D(\Omega)$ with kernel $I(\Omega)$. It is tedious, but routine, to show that this is an isometric *-isomorphism: $C_c(\Omega) \to C_c(\Pi)$, preserving the masa, $C_c(X)$.

DEFINITION. A *-representation of $C_c(\Omega)$ is a bounded *-homomorphism $\pi$: $C_c(\Omega) \to B(\mathfrak{H})$ where $\mathfrak{H}$ is a separable Hilbert space so that $\pi(C_c(\Omega))\mathfrak{H}$ is dense in $\mathfrak{H}$. For $a \in C_c(\Omega)$ put $|a| = \sup_\pi |\pi(a)|$ where $\pi$ ranges over all *-representations of $C_c(\Omega)$. In fact $|a| \leq |a|_0$.

DEFINITION. Let $C^*(\Omega)$ be the completion of $C_c(\Omega)$ with respect to this norm. Since $C_c(X) \subseteq C_c(\Omega)$ is already equipped with the $C^*$-norm, its closure in $C^*(\Omega)$ is $C_0(X)$. Again, if $\Pi \sim \Omega$, $C^*(\Omega) \cong C^*(\Pi)$. Before proceeding further, we will show that the *-representations of $C_c(\Omega)$ arise from representations of $\Omega$ by partial isometries on a Hilbert space satisfying a mild continuity hypothesis.

4. Suppose $\Omega$ localizes $X$.

DEFINITION. If $\pi$: $\Omega \to B(\mathfrak{H})$ is an ISG morphism (i.e., $\pi(\omega_1\omega_2) = \pi(\omega_1)\pi(\omega_2)$, $\pi(\omega^*) = \pi(\omega)^*$, $\pi(\theta) = 0$) then it is said to be a *-representation of $\Omega$ if $\pi|_{\mathrm{id}(\Omega)}$ extends uniquely to a projection valued measure (PVM) on the Borels in $X$ (call the extension $\pi$) so that $\pi(X) = 1_{\mathfrak{H}}$. Observe that $\pi(\Omega) \subseteq \mathrm{Ist}(\mathfrak{H})$, the collection of partial isometries on $\mathfrak{H}$; if $\lambda \in \mathrm{id}(\Omega)$ then $\pi(\lambda)$ is a projection.

NOTATION. For a bounded Borel function write $\tilde{\pi}(f) = \int f\, d\pi$.

It will be shown that $\pi$ extends uniquely to an ISG morphism $\pi^\alpha$: $\Omega^\alpha \to B(\mathfrak{H})$ so that its restriction to an equivalent localization yields a *-representation. First, the natural correlate of the notion of coherence is defined for elements of $\mathrm{Ist}(\mathfrak{H})$.

DEFINITION. Let $u, v \in \mathrm{Ist}(\mathfrak{H})$. Then $u, v$ are said to cohere if $\{u^*u, v^*v, v^*u, u^*v\}$, $\{uu^*, vv^*, vu^*, uv^*\}$ are two sets of commuting projections.

LEMMA. *If* $u, v \in \mathrm{Ist}(\mathfrak{H})$ *cohere there is a unique* $w \in \mathrm{Ist}(\mathfrak{H})$ *so that* $w^*w = \sup(u^*u, v^*v)$ *and* $u = wu^*u$, $v = wv^*v$.

*Proof.* We have $v^*u = (v^*u)^* = u^*v$ and $uv^* = vu^*$; further,

$$u^*uv^*v = u^*(uv^*)v = u^*(vu^*)v = u^*v = v^*u.$$

If $\xi, \eta \in \mathfrak{H}$ so that $u^*u\xi = \xi$, $v^*v\eta = \eta$, put $w(\xi + \eta) = u\xi + v\eta$. Then $w$ is well defined and a partial isometry with the prescribed properties since

$$\begin{aligned}\langle u\xi + v\eta, u\xi + v\eta\rangle &= \langle u^*u\xi, \xi\rangle + \langle v^*u\xi, \eta\rangle + \langle u^*v\eta, \xi\rangle + \langle v^*v\eta, \eta\rangle \\ &= \langle \xi, \xi\rangle + \langle v^*vu^*u\xi, \eta\rangle + \langle u^*uv^*v\eta, \xi\rangle + \langle \eta, \eta\rangle \\ &= \langle \xi, \xi\rangle + \langle \xi, \eta\rangle + \langle \eta, \xi\rangle + \langle \eta, \eta\rangle = \langle \xi + \eta, \xi + \eta\rangle.\end{aligned}$$

Set $w \equiv 0$ on the orthogonal complement of $u^*u\mathfrak{H} + v^*v\mathfrak{H}$. □

DEFINITION. A sequence $u_1, u_2, \ldots, \in \mathrm{Ist}(\mathfrak{H})$ is said to be coherent if $\{u_i^*u_j: i, j\}$, $\{u_iu_j^*: i, j\}$ are two families of commuting projections.

PROPOSITION. *If $u_1, u_2, \ldots \in \mathrm{Ist}(\mathfrak{H})$ is coherent, then there is a unique $u \in \mathrm{Ist}(\mathfrak{H})$, write $u = \uparrow u_i$ so that $u^*u = \sup\{u_i^*u_i\}$ and $u_i = uu_i^*u_i$ (and consequently $u_i = u_iu_i^*u$).*

*Proof.* Set $p_i = u_i^*u_i$, $q_i = u_iu_i^*$ each $i$, $p = \sup\{p_i\}$, $q = \sup\{q_i\}$; first define $u$ on a dense subspace of $p\mathfrak{H}$ as follows: Let $\xi_1, \ldots, \xi_n \in H$ so that $p_i\xi_i = \xi_i$; then define $u(\Sigma\xi_i) = \Sigma u_i\xi_i$. That this is well defined will follow if it can be shown to be isometric. As in the lemma consider

$$\begin{aligned}\left\langle \sum_i u_i\xi_i, \sum_j u_j\xi_j\right\rangle &= \sum_{i,j}\langle u_j^*u_i\xi_i, \xi_j\rangle = \sum_{i,j}\langle p_jp_i\xi_i, \xi_j\rangle \\ &= \sum_{i,j}\langle p_i\xi_i, p_j\xi_j\rangle = \left\langle \sum_i \xi_i, \sum_j \xi_j\right\rangle.\end{aligned}$$

So $|\Sigma u_i\xi_i| = |\Sigma\xi_i|$ and we are done (since $u \equiv 0$ on the orthogonal complement). □

COROLLARY. *If $\pi\colon \Omega \to B(\mathfrak{H})$ is a *-representation, then it extends uniquely to an ISG morphism $\pi^\alpha\colon \Omega^\alpha \to B(\mathfrak{H})$ (which restricts to a *-representation for any equivalent localization).*

*Proof.* Note that $\pi^\alpha$ is already defined on $\mathrm{id}(\Omega^\alpha)$ because $\pi$ extends uniquely to a projection valued measure. If $\sigma \in \Omega^\alpha$ then there is $\{\omega_1, \omega_2, \ldots\} \in \Omega$, a coherent family with, $d(\sigma) = \bigcup_i d(\omega_i)$ and $\sigma d(\omega_i) = \omega_i$. Since $\pi(\mathrm{id}(\Omega))$ forms a commuting family of projections, $\{\pi(\omega_i)\} \subseteq \mathrm{Ist}(\mathfrak{H})$ is coherent (since $\pi(\omega_i)^*\pi(\omega_u) = \pi(\omega_i^*\omega_j)$ and $\pi(\omega_i)\pi(\omega_j)^* = \pi(\omega_i\omega_j^*)$). Put $\pi^\alpha(\sigma) = \uparrow \pi(\omega_i)$. So $\pi^\alpha(\sigma^*) = \uparrow \pi(\omega_i^*) = \uparrow \pi(\omega_i)^* = \pi^\alpha(\sigma)^*$.

Evidently the construction does not depend on the particular choice of coherent family $\{\omega_i\}$. $\square$

5. Before establishing the correspondence between *-representations of localizations and *-representations of their associated normed *-algebras, a local covariance property for *-representations of localizations is noted. Here again, $\Omega$ localizes $X$.

LEMMA. *If* $\pi\colon \Omega \to B(\mathfrak{H})$ *is a *-representation and* $(\omega, f) \in E(\Omega)$ *(i.e.,* $f \in C_c(d(\omega))$*), then*

$$\tilde{\pi}(f \cdot \omega^*) = \pi(\omega)\tilde{\pi}(f)\pi(\omega^*).$$

*Proof.* It is assumed that $f \geq 0$ and $|f| = 1$. Choose $\varepsilon > 0$ small ($< 1$), and $n = \max\{m\colon m\varepsilon < 1\}$. For $1 \leq i \leq n$ put $\lambda_i = \{x \in X\colon f(x) < i\varepsilon\}$ ($\lambda_i \in \mathrm{id}(\Omega^\alpha)$), $\chi_i$ the characteristic function for $\lambda_i$. Clearly $\pi(\lambda_i) = \tilde{\pi}(\chi_i)$ since $\pi$ is a PVM. Then $\chi_i \cdot \omega^*$ is the characteristic function of $\omega\lambda_i\omega^*$. Then, evidently $\tilde{\pi}(\chi_i\omega^*) = \pi(\omega)\tilde{\pi}(\chi_i)\pi(\omega^*)$. Since $|f - \varepsilon\Sigma\chi_i|_\infty \leq \varepsilon$ and $|f \cdot \omega^* - \varepsilon\Sigma\chi_i \cdot \omega^*| < \varepsilon$, the result follows by linearity and continuity. $\square$

THEOREM. *With* $\pi$ *as above*

$$\tilde{\pi}(\Sigma(\omega_i, f_i)) = \Sigma\pi(\omega_i)\tilde{\pi}(f_i)$$

*defines a *-representation of* $C_c(\Omega)$.

*Proof.* There are several things to check: non-degeneracy follows from the non-degeneracy of $\pi$ (recall $\pi(X) = 1_{\mathfrak{H}}$). Involution is preserved; for $(\omega, f) \in E(\Omega)$, $(\omega, f)^* = (\omega^*, \bar{f} \cdot \omega^*)$ and

$$\begin{aligned}\tilde{\pi}(\omega^*, \bar{f} \cdot \omega^*) &= \pi(\omega^*)\tilde{\pi}(f\omega^*) = \pi(\omega^*)\pi(\omega)\tilde{\pi}(\bar{f})\pi(\omega^*)\\ &= \pi(\omega^*\omega)\tilde{\pi}(\bar{f})\pi(\omega)^* = (\pi(\omega)\tilde{\pi}(f))^* = \tilde{\pi}(\omega, f)^*\end{aligned}$$

so involution is preserved by linearity. For $(\omega, f), (\sigma, g) \in E(\Omega)$,

$$\begin{aligned}\tilde{\pi}(\omega, f)\tilde{\pi}(\sigma, g) &= \pi(\omega)\tilde{\pi}(f)\pi(\sigma)\tilde{\pi}(g)\\ &= \pi(\omega)\tilde{\pi}(f)\pi(\sigma)\tilde{\pi}(g)\pi(\sigma^*)\pi(\sigma)\\ &= \pi(\omega)\tilde{\pi}(f)\tilde{\pi}(g\sigma^*)\pi(\sigma) = \pi(\omega)\tilde{\pi}(f(g\sigma^*))\pi(\sigma)\\ &= \pi(\omega)\pi(\sigma)\pi(\sigma^*)\tilde{\pi}(f(g\sigma^*))\pi(\sigma) = \pi(\omega\sigma)\tilde{\pi}((f(g\sigma^*))\sigma),\end{aligned}$$

$$\begin{aligned}\tilde{\pi}(\omega, f)\tilde{\pi}(\sigma, g) &= \pi(\omega\sigma)\tilde{\pi}((f\sigma)g) = \tilde{\pi}(\omega\sigma, (f\sigma)g)\\ &= \tilde{\pi}((\omega, f)(\sigma, g)),\end{aligned}$$

whence $\tilde{\pi}$ is an algebra homomorphism (by distributivity). If $\{(\omega_i, f_i)\}$ is coherent, there are open sets $\lambda_i$ with $\operatorname{supp}(f_i) \subseteq \lambda_i \subseteq d(\omega_i)$ and $\{\omega_i\lambda_i\}$ is coherent. Put $u = \uparrow \pi(\omega_i\lambda_i)$; then $\tilde{\pi}(\Sigma(\omega_i, f_i)) = u\tilde{\pi}(\Sigma f_i)$; so $|\tilde{\pi}(\Sigma(\omega_i, f_i))| = |\tilde{\pi}(\Sigma f_i)| \leq |\Sigma f_i|_\infty$. In particular, $I(\Omega) \subseteq \ker(\tilde{\pi})$. If $\{(\omega_i, f_i)\}$ is arbitrary (but finite), let $\{(\sigma_{ij}, g_{ij})\}$ be a coherent partition for $\Sigma(\omega_i, f_i)$. Then

$$\tilde{\pi}(\Sigma(\omega_i, f_i)) = \sum_i \tilde{\pi}\left(\sum_j (\sigma_{ij}, g_{ij})\right)$$

since $I(\Omega) \subseteq \ker(\tilde{\pi})$; so

$$|\tilde{\pi}(\Sigma(\omega_i, f_i))| \leq \sum_i \left(\tilde{\pi}\left(\sum_j (\sigma_{ij}, g_{ij})\right)\right) \leq \sum_i \left|\sum_j g_{ij}\right|_\infty .$$

Taking the minimum over all coherent partitions establishes boundedness. □

For completeness we include:

PROPOSITION. *Every bounded *-representation of $C_c(\Omega)$ arises in this way.*

*Proof.* If $\rho$ is a *-representation of $C_c(\Omega)$ on $\mathfrak{H}$, then a *-representation $\pi$ of $\Omega$ on $\mathfrak{H}$ is to be exhibited so that $\rho = \tilde{\pi}$. For $f \in C_c(X)$ (then certainly $f \in C_c(\Omega)$), take $(\lambda_i, g_i) \in E(\Omega)$ with $\lambda_i \in \operatorname{id}(\Omega)$ and $\Sigma fg_i = f$, so $\rho(f) = \Sigma\rho(\lambda_i, fg_i)$; this defines a non-degenerate *-representation of $C_c(X)$ which implies the existence of a PVM $\pi$ on $X$ so that $\pi(X) = 1$. Consequently $\pi$ is well defined on $\operatorname{id}(\Omega)$. For $\omega \in \Omega$, $d(\omega)$ is paracompact; let $f_i \in C_c(d(\omega))$ be a partition of unity on $d(\omega)$ subordinate to some locally finite covering. Then put

$$\pi(\omega) = \lim_n \sum_{i=1}^n \rho(\omega, f_i)$$

(where limit is taken in weak operator topology in $B(\mathfrak{H})$). Then $\pi(\omega)$ is a partial isometry.

$$\pi(\omega^*)\pi(\omega) = \lim \rho\left(\left(\sum_{i=1}^n f_i\right)^2\right) = \pi(d(\omega))$$

and $\pi$ is the desired *-representation of $\Omega$. □

6. PROPOSITION. *If $\Omega$ simplifies then $C_c(\Omega)$ is algebraically simple.*

*Proof*. Suppose $I \subseteq C_c(\Omega)$ is a non-zero ideal. If $\sigma, \omega \in \Omega$ are incoherent (say $d(\sigma)d(\omega) \neq \theta$), then there is $x \in d(\sigma)d(\omega)$ so that $\sigma(x) \neq \omega(x)$; whence there is $\lambda \in \mathrm{id}(\Omega)$ with $x \in \lambda$ such that $r(\sigma\lambda)r(\omega\lambda) = \theta$. Hence, if $\omega, \sigma$ are incoherent, there is $\lambda \in \mathrm{id}(\Omega)$, $\lambda d(\omega) = \lambda$, such that $\omega(\lambda)\sigma\lambda = \theta$. Now suppose $\Sigma_{i=1}^{n}(\omega_i, f_i) \in I$ $(\neq 0)$. So for each $k$, there is $\lambda \in \mathrm{id}(\Omega)$, $\lambda d(\omega_k) = \lambda$, so that either $\omega_i\lambda$ is coherent with $\omega_k\lambda$ or $r(\omega_i\lambda)r(\omega_k\lambda) = \theta$. Now, choose $\lambda$, $k$ so that there is $x \in \lambda$ with $\Sigma f_i(x) \neq 0$ (where the sum is taken over the indices $i$ for which $\omega_i\lambda$ is coherent with $\omega_k\lambda$). Choose $f \in C_c(\lambda)$ with $f(x) \neq 0$. Then

$$(\omega_k(\lambda), f \cdot \omega_k^*)(\Sigma(\omega_i, f_i))(\lambda, f) = (\omega_k, g) \in I, \qquad g \neq 0.$$

Then $h = |g|^2 \in C_c(X)^+ \cap I$. Now, put $U = \{x: h(x) > 0\}$. Let $K \subseteq X$ be compact; there are $\omega_1, \dots, \omega_n \in \Omega$ such that $d(\omega_i) \subseteq U$ and $K \subseteq \bigcup_i r(\omega_i)$ and $f_i \in C_c(r(\omega_i))^+$ so that $\Sigma f_i(x) = 1$ for each $x \in K$. Set

$$g_i(x) = \begin{cases} ((f_i\omega_i)/h)^{1/2}(x), & x \in d(\omega_i) \text{ and } f_i(\omega_i(x)) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Sigma_i f_i = \Sigma_i(\omega, g_i)h(\omega^*, g_i\omega_i^*) \in I$ so $I = C_c(\Omega)$. □

7. Suppose $G$, a countable group, acts freely on a space $X$; choose a countable basis and let $\Omega$ be the associated localization.

PROPOSITION. $C^*(G, X) \cong C^*(\Omega)$.

*Proof*. It suffices to show that covariant representations of the pair $(G, C_0(X))$ correspond to *-representations of $\Omega$ in such a way that the enveloping algebras are identical. Suppose $\mathfrak{H}$ is a Hilbert space, $u$: $G \to B(\mathfrak{H})$ a unitary representation and $\rho$ a PVM on $X$ evaluated in $B(\mathfrak{H})$ so that $(u, \rho)$ is covariant, i.e.,

$$\rho(gF) = u(g)\rho(F)u(g)^* \quad \text{for } g \in G,\ F \subseteq X \text{ Borel.}$$

If $\omega \in \Omega$ then there is $\lambda \in \mathrm{id}(\Omega)$ so that $\omega = g|_\lambda$. Set $\pi(\omega) = u(g)\rho(\lambda)$ and observe that this yields the desired *-representation. Given $\pi$: $\Omega \to B(\mathfrak{H})$ a *-representation, then let $\rho$ be the unique extension of $\pi\,|\,\mathrm{id}(\Omega)$ to a PVM. For $g \in G$, there are $\omega_i \in \Omega$ so that $\omega_i = g|_{d(\omega_i)}$ and $\bigcup d(\omega_i) = X$. Put $u(g) = \uparrow \pi(\omega_i)$. □

8. Let $\nu = (n_1, \dots, n_k)$ be a multi-index consisting of positive ($\neq 0$) integers, and $S_\nu$ the associated finite localization on $I_{n_1} \vee \cdots \vee I_{n_k}$. It is more or less a routine matter to see that

$$C^*(S_\nu) \cong M_\nu = M_{n_1} \oplus \cdots \oplus M_{n_k}.$$

Refer now to the AF localizations discussed in §3/6; we will show that the associated $C^*$-algebras are unital AF algebras. More explicitly, let $\{m(i, j, k)\}$ be a choice of multiplicity indices; let $\Omega(m)$ be the AF localization on $X(m)$ constructed in §3/6. For $\lambda \in \mathrm{id}(\Omega)$, let $\chi(\lambda)$ denote its characteristic function. Since $\lambda$ is compact and open, $\chi(\lambda) \in C(X(m))$. Further, the collection $\{\chi(\lambda)\colon \lambda \in \mathrm{id}(\Omega)\}$ separates points; hence, by Stone-Weierstrass the algebra generated is dense in $C(X(m))$.

In particular, the collection $\{(\omega, \chi(d(\omega)))\colon \omega \in \Omega(m)\}$ is total in $C^*(\Omega(m))$. Given $l > 0$, the algebra generated by

$$\{(\sigma((u, v), (s, t)), \chi(K(s, t)))\colon (u, v), (s, t) \in E(l), v(l) = t(l)\}$$

is isomorphic to $M_{\nu_l}$. Whence, there is an increasing sequence of finite-dimensional algebras in $C^*(\Omega(m))$; the union is dense because $\{(\omega, \chi(d(\omega)))\colon \omega \in \Omega(m)\}$ is total. Therefore, $C^*(\Omega(m))$ is a unital $AF$ algebra.

Recall that for each $i \geq 0$, $m(i, j, k)$ is an $n(i) \times n(i+1)$ matrix with entries in $\mathbf{Z}^+$ so that no row or column is completely zero. Then $m$ defines a sequence of order morphisms

$$\mathbf{Z}^{n(0)} \xrightarrow{\mu_0} \mathbf{Z}^{n(1)} \xrightarrow{\mu_1} \mathbf{Z}^{n(2)} \xrightarrow{\mu_2} \cdots$$

(where $\mathbf{Z}^{n(i)}$ is viewed as an ordered abelian group) so that $(q_k) = \mu_i(p_j)$ with $0 \leq j < n(i)$ and $0 \leq k < n(i+1)$ is defined by

$$q_k = \sum_j m(i, j, k) p_j.$$

Then $K_0(\Omega(m)) = \varinjlim \mathbf{Z}^{n(i)}$ is the Elliott group of $C^*(\Omega(m))$.

**6. Induced representations and Morita equivalence.**

1. Throughout this section $\Omega$ will be a simplification on $X$. Let $Y$ be an $\Omega$-manifold with atlas $\Sigma = \{\sigma_i\}$. Then $\Omega(\Sigma)$ simplifies $Y$; and the $*$-representations of $\Omega(\Sigma)$ are closely related to those of $\Omega$. It will be seen that $C^*(\Omega(\Sigma))$ is strong Morita equivalent (SME) to $C^*(\Omega)$; in fact the choice of imprimitivity bimodule is prescribed by the equivalence class of the $\Omega$-atlas. Suppose $\pi\colon \Omega \to B(\mathfrak{H})$ is a $*$-representation:

DEFINITION. If $\rho\colon \Omega(\Sigma) \to B(\mathfrak{K})$ is a $*$-representation, then it is said to be induced from $\pi$ if there are partial isometries $T_i\colon \mathfrak{K} \to \mathfrak{H}$ so that $T_i T_j^* = \pi^\alpha(\sigma_i \sigma_j^*)$ and $\rho(\sigma_i^* \omega \sigma_j) = T_i^* \pi^\alpha(\omega) T_j$ each $i, j$ and $\omega \in \Omega^\alpha$.

PROPOSITION. *Induced representations are unique modulo unitary equivalence.*

*Proof*. Suppose $\rho' = \Omega(\Sigma) \to B(\mathfrak{K}')$ is another representation induced from $\pi$; then there are partial isometries $T_i'\colon \mathfrak{K}' \to \mathfrak{H}$ with $T_i'(T_j')^* = \pi^\alpha(\sigma_i\sigma_j^*)$ and $\rho'(\sigma_i^*\omega\sigma_j) = (T_i')^*\pi^\alpha(\omega)T_j'$ all $i$, $j$ and $\omega \in \Omega^\alpha$. Set $R_i = T_i^*T_i' \in \mathrm{Ist}(\mathfrak{K}' \oplus \mathfrak{K})$; then $R_i$ is a coherent sequence of partial isometries:

$$R_iR_j^* = T_i^*T_i'(T_j')^*T_j = T_i^*\pi^\alpha(\sigma_i\sigma_j^*)T_j' = \rho(\sigma_i^*\sigma_i\sigma_j^*\sigma_j)$$

and $\{\rho(\sigma_i^*\sigma_i\sigma_j^*\sigma_j)\}$ is a commuting family of projections on $\mathfrak{K}$, while

$$R_i^*R_j = (T_i')^*T_iT_j^*T_j' = (T_i')^*\pi^\alpha(\sigma_i\sigma_j^*)T_j' = \rho'(\sigma_i^*\sigma_j\sigma_j^*\sigma_j)$$

and $\{\rho'(\sigma_i^*\sigma_i\sigma_j^*\sigma_j)\}$ is a commuting family of projections on $\mathfrak{K}'$. Set $R = \uparrow R_i$. Then $R\colon \mathfrak{K}' \to \mathfrak{K}$ is a unitary.

We show if $\omega' \in \Omega(\Sigma)$ then $\rho(\omega')R = R\rho'(\omega')$. It is sufficient to demonstrate this for the generating subset $\{\sigma_i^*\omega\sigma_j\colon \omega \in \Omega, i, j\}$. Thus:

$$\begin{aligned}\rho(\sigma_i^*\omega\sigma_j)R &= T_i^*\pi(\omega)T_jR = T_i^*\pi(\omega)T_jR_j = T_i^*\pi(\omega)T_jT_j^*T_j' \\ &= T_i^*\pi(\omega)T_j'\end{aligned}$$

and

$$\begin{aligned}R\rho'(\sigma_i^*\omega\sigma_j) &= R(T_i')^*\pi(\omega)T_j' = R_i(T_i')^*\pi(\omega)T_j' \\ &= T_i^*T_i'(T_i')^*\pi(\omega)T_j' = T_i^*\pi(\omega)T_j'.\end{aligned}$$ □

2. The existence of induced representation is proven via the characterization of an $\Omega$-manifold as the primitive of a reduction of an ampliation.

PROPOSITION. *Induced representation exist.*

*Proof*. There are three cases to consider:

(i) Reductions: If $U \subseteq X$ is open then $\pi(U)$, a projection, is the value of the PVM on $U$. Then the restriction of $\pi$ to the reduced simplification defines a *-representation of $\Omega_U$ on $\pi(U)\mathfrak{H}$.

(ii) Ampliations: Let $\Omega^\infty = \Omega \times S_\infty$ be the ampliation of $\Omega$ ($\Omega^\infty$ simplifies $X \times I_\infty$). Put $\mathfrak{K} = l^2(I_\infty, \mathfrak{H})$; define partial isometries $S_i\colon \mathfrak{K} \to \mathfrak{H}$ by $S_i(\xi) = \xi_i$ each $i \in I_\infty$. Set $\rho\colon \Omega^\infty \to B(\mathfrak{K})$ by $\rho(\omega \times (i, j)) = S_i^*\pi(\omega)S_j$; then $\rho$ is the desired induced representation.

(iii) Primitives: Suppose $\psi\colon X \to Y$ is a local homeomorphism admitted by $\Omega$. An element $\omega \in \Omega$ intertwines $\psi$ if for every $x \in d(\omega)$, $\psi(\omega(x)) = \psi(x)$. Put $\mathfrak{H}_0 = \overline{\mathrm{sp}}\{(\pi(\omega) - \pi(d(\omega)))\xi\colon \xi \in \mathfrak{H}, \omega$ intertwines $\psi\}$, $\mathfrak{K} = \mathfrak{H}_0^\perp$, and let $P$ be the unique projection with range $\mathfrak{K}$. Let

$\Sigma = \{\sigma_i\} \subseteq S_0(Y, X)$ be a complete family of local sections for $\psi$. Then let $S_i$: $\mathfrak{K} \to \mathfrak{H}$ by $S_i = \pi(r(\sigma_i))P$. Then the collection $\{S_i\}$ induces the desired *-representation $\rho$: $\Omega_\psi \to B(\mathfrak{K})$. $\square$

*Note*: Reversing the induction process yields the original *-representation (modulo unitary equivalence).

3. The linking algebra characterization of SME developed in **[10]** may now be invoked. That is, two $C^*$-algebras $A$ and $B$ are SME iff there is a third $C^*$-algebra $C$ (the linking algebra) and two projections $p, q \in M(C)$ (multiplier algebra) with $p + q = 1$ so that $A \cong pCp$, $B \cong qCq$, and neither algebra is contained in a proper ideal (i.e., $A$ and $B$ appear as complementary full corners in the linking algebra). The imprimitivity bimodule is $J \cong pCq$; so if $a \in pCq$ then $a^*a \in B$ and $aa^* \in A$.

THEOREM. *If $\Omega$ simplifies $X$ and $Y$ is a $\Omega$-manifold with atlas $\Sigma = \{\sigma_i\}$, then $C^*(\Omega)$ and $C^*(\Omega(\Sigma))$ are exhibited as complementary full corners in $C^*(\Upsilon(\Sigma))$.*

*Proof*. The linking simplification $\Upsilon(\Sigma)$ was defined to be the simplification on $X \vee Y$ generated by $\Omega$ and $\Sigma$. The reduction of $\Upsilon(\Sigma)$ to $X$, $\Upsilon(\Sigma)_X$, is equivalent to $\Omega$, while the reduction to $Y$, $\Upsilon(\Sigma)_Y \sim \Omega(\Sigma)$. If $\pi$: $\Omega \to B(\mathfrak{H})$ is a *-representation so that $\tilde{\pi}$: $C^*(\Omega) \to B(\mathfrak{H})$ is faithful, then the induced *-representation $\rho$: $\Omega(\Sigma) \to B(\mathfrak{K})$ yields a faithful representation $\tilde{\rho}$: $C^*(\Omega(\Sigma)) \to B(\mathfrak{K})$. Take the induced *-representation $\phi$: $\Upsilon(\Sigma) \to B(\mathfrak{H} \oplus \mathfrak{K})$ so that if $\omega \in \Omega$, $\omega' \in \Omega(\Sigma)$ then $\phi(\omega)|_{\mathfrak{H}} = \pi(\omega)$, $\phi(\omega')|_{\mathfrak{K}} = \rho(\omega')$. Thus $C^*(\Upsilon(\Sigma))$ is the desired linking algebra. The reduced algebras $C^*(\Omega)$, $C^*(\Omega(\Sigma))$ are exhibited as complementary full corners, since the characteristic functions on $X$ and $Y$ are projections in the multiplier algebra.

NOTATION. Write $J(\Sigma) = \{a \in C^*(\Upsilon(\Sigma))$: $a^*a \in C^*(\Omega(\Sigma))$, $aa^* \in C^*(\Omega)\}$; then $J(\Sigma)$ is the canonical imprimitivity bimodule associated to the $\Omega$-manifold $(Y, \Sigma)$.

COROLLARY. *If $Y_1$ and $Y_2$ are $\Omega$-manifolds with $\Omega$-atlases $\Sigma_1$, $\Sigma_2$, then $Y_1 \vee Y_2$ is an $\Omega$-manifold with $\Omega$-atlas $\Sigma_1 \cup \Sigma_2$. Also, $C^*(\Omega(\Sigma_1))$ and $C^*(\Omega(\Sigma))$ embed canonically as complementary full corners in $C^*(\Omega(\Sigma_1 \cup \Sigma_2))$. (Since $Y_1$ is an $\Omega(\Sigma_2)$-manifold and vice versa.)*

NOTATION. Write

$$J(\Sigma_1, \Sigma_2) = \{a \in C^*(\Omega(\Sigma_1 \cup \Sigma_2)) : a^*a \in C^*(\Omega(\Sigma_2)) \text{ and } aa^* \in C^*(\Omega(\Sigma_1))\}.$$

Then $J(\Sigma_1, \Sigma_2)$ is the canonical imprimitivity bimodule associated to the ordered pair of $\Omega$-manifolds $\langle (Y_1, \Sigma_1), (Y_2, \Sigma_2) \rangle$. Viewed as imprimitivity bimodules $J(\Sigma_1, \Sigma_2) \cong J(\Sigma_1)^* \otimes J(\Sigma_2)$ (where the tensor product is $C^*(\Omega)$-balanced).

DEFINITION. Two $\Omega$-manifolds $(Y_1, \Sigma_1)$, $(Y_2, \Sigma_2)$ are said to be weak equivalent (write $(Y_1, \Sigma_1) \simeq (Y_2, \Sigma_2)$) if there is $a \in J(\Sigma_1, \Sigma_2)$ with $a^*a$ strictly positive in $C^*(\Omega(\Sigma_2))$ and $aa^*$ strictly positive in $C^*(\Omega(\Sigma_1))$.

Note that this condition implies $C^*(\Omega(\Sigma_1)) \cong C^*(\Omega(\Sigma_2))$ because the unitary in the polar decomposition of $a$ implements this isomorphism by conjugation. If $Y_1$, $Y_2$ are compact then $a \in J(\Sigma_1, \Sigma_2)$ may be chosen to be unitary. If an $\Omega$-manifold, $(Y, \Sigma)$, is weak equivalent to the disjoint sum of compact $\Omega$-manifolds then $C^*(\Omega(\Sigma))$ has an approximate identity consisting of an increasing sequence of projections. The notion of weak equivalence provides some indication of why Cartan masas (as defined in the Introduction) need not be isomorphic.

## 7. Free localizations.

1. A localization will be said to be free if any element with a fixed point is locally an idempotent. (Renault calls a partial homeomorphism relatively free if its fixed points form a compact open set, he also requires that the domain and range be compact-open cf. 1.2.11 of [**39**].) We will show that each free localization is associated to a unique principal $r$-discrete groupoid with Haar system (as defined by Renault [**39**]), the open $G$-sets of which are identified with the affiliation of the localization.

NOTATION. We use "$R$" for principal groupoid because of its natural kinship with equivalence relations (1.2c) and also because "$G$" is reserved for groups.

DEFINITION. If $\Omega$ localizes $X$ then it is said to be free if for every $\omega \in \Omega$ and $x \in d(\omega)$ with $\omega(x) = x$, there is $\lambda \in \mathrm{id}(\Omega)$, $x \in \lambda$, such that $\lambda\omega = \omega\lambda = \lambda$.

*Fact.* A localization, $\Omega$ on $X$, is free iff for every $\omega \in \Omega$ there are $\lambda_1, \lambda_2, \ldots \in \mathrm{id}(\Omega)$ with $d(\omega) = \bigcup_i \lambda_i$ such that either $\lambda_i\omega\lambda_i = \lambda_i$ or $\lambda_i\omega\lambda_i = \theta$.

*Proof*. Suppose the latter condition is satisfied, and $\omega(x) = x$ for some $\omega \in \Omega$ and $x \in d(\omega)$. Then $x \in \lambda_i$ for some $i$ and $\lambda_i \omega \lambda_i \neq \theta$, so $\lambda_i \omega = \omega \lambda_i = \lambda_i$. Conversely, suppose $\Omega$ is free. Then let $\Lambda = \{\lambda \in \mathrm{id}(\Omega)$: $\lambda \subseteq d(\omega)$ and $\lambda\omega\lambda = \lambda$ or $\theta\}$. Suppose there is an $x \in d(\omega)$ so that $x \notin \lambda$ for each $\lambda \in \Lambda$. Either $\omega(x) = x$ or $\omega(x) \neq x$. In the first case, there is $\lambda' \in \mathrm{id}(\Omega)$, $x \in \lambda'$ so that $\omega\lambda' = \lambda'$; then $\lambda'\omega\lambda' = \lambda'$; this contradicts our assumptions. In the second case, $\omega(x) \neq x$, we can find $\lambda_1, \lambda_2 \in \mathrm{id}(\Omega)$, $\lambda_1\lambda_2 = \theta$ so that $x \in \lambda_1$, $\omega(x) \in \lambda_2$; consequently, there is $\lambda_3 \in \mathrm{id}(\Omega)$ with $x \in \lambda_3$, $\lambda_3\lambda_1 = \lambda_3$, $\omega(\lambda_3)\lambda_2 = \omega(\lambda_3)$; hence $\lambda_3\omega\lambda_3 = \theta$ (write $\omega(\lambda_3)$ for $\omega\lambda_3\omega^*$). □

2. If $X$ is a set and $R \subseteq X \times X$ is an equivalence relation then $R$ is to be viewed as a principal groupoid for our purposes. The set of composable pairs, $R^2 \subseteq R \times R$, is easily seen to be

$$R^2 = \{(x, y)(y, z) \colon x, y, z \in X, (x, y), (y, z) \in R\}.$$

The groupoid inverse map is the flip map $((x, y) \to (y, x))$ restricted to $R$. The range and domain maps ($r$ and $d$) are simply the projections onto the first and second factors.

DEFINITION. If $\Omega$ localizes $X$ freely then let $R(\Omega)$ be the orbital equivalence relation on $X$, i.e., $(x, y) \in R(\Omega)$ iff there is $\omega \in \Omega, y \in d(\omega)$ with $x = \omega(y)$. That this defines an equivalence relation is not difficult to see: $(x, x) \in R(\Omega)$ because $\mathrm{id}(\Omega)$ forms a basis; $(x, y) \in R(\Omega)$ implies $(y, x) \in R(\Omega)$, since if $x = \omega(y)$, then $y = \omega^*(x)$; if $(x, y), (y, z) \in R(\Omega)$. Then $(x, z) \in R(\Omega)$ since there are $\omega, \sigma \in \Omega$ with $x = \omega(y)$, $y = \sigma(z)$ so $x = \omega\sigma(z)$.

THEOREM. *There is a unique principal r-discrete groupoid with Haar system R associated to a free localization $\Omega$ on X, so that $R^0 = X$, and the ISG of open R-sets (qua partial homeomorphisms on $R^0$) is $\Omega^\alpha$. Further each principal r-discrete groupoid with Haar system arises in this way.*

*Proof*. For $\omega \in \Omega$, let $U(\omega) = \{(\omega(x), x) \colon x \in d(\omega)\} \subseteq R(\Omega)$. The collection $\{U(\omega) \colon \omega \in \Omega\}$ defines a topology on $R(\Omega)$ by virtue of being a basis. Equipped with this topology $R(\Omega)$ is a second countable locally compact Hausdorff space since $X$ is. In order that $R(\Omega)$ be a topological groupoid, composition and inversion must be continuous. (Then $R = R(\Omega)$ is the required groupoid.)

*Composition.* Suppose $(x, y), (y, z) \in R(\Omega)$ and $U$ is an open neighborhood of $(x, z)$. Then it is necessary to show that there is an open subset $V \subseteq R^2$, with $((x, y), (y, z)) \in V$, that maps into $U$ via the composition map. Let $\omega, \sigma \in \Omega$ so that $y = \sigma(z)$ and $x = \omega(y)$. Then $x = \omega(\sigma(z))$. There is $\beta \in \Omega$ so that $(x, z) \in U(\beta) \subseteq U$; since $\Omega$ is free and $\beta^*\omega\sigma$ leaves $z$ fixed, there is $\lambda \in \mathrm{id}(\Omega)$, $z \in \lambda$, $\lambda d(\beta) = \lambda d(\omega\sigma) = \lambda$ so that $\beta\lambda = \omega\sigma\lambda$. Then $U(\sigma\lambda)$ is an open neighborhood of $(y, z)$ and $U(\omega r(\sigma\lambda))$ is an open neighborhood of $(x, y)$. Setting

$$V = R^2 \cap (U(\omega r(\sigma\lambda)) \times U(\sigma\lambda))$$

yields the desired open set. So composition is continuous.

*Inversion.* To show that the inversion map on $R(\Omega)$ $((x, y) \to (y, x))$ is a homeomorphism, it clearly suffices to show that it is continuous pointwise (since it is a bijection and its own inverse). If $(x, y) \in R(\Omega)$ then there is $\omega \in \Omega$ with $y \in d(\omega)$ so that $y = \omega(x)$ and consequently $U(\omega)$ is an open neighborhood of $(x, y)$ and $U(\omega^*)$ is an open neighborhood of $(y, x)$. Observe that for each $(v, w) \in U(\omega)$, $(w, v) = (v, w)^{-1} = (\omega^*(v), \omega(w))$, so inversion is continuous.

By Proposition 1.2.8 of **[37]** it remains to show that $r\colon R \to X$ is a local homeomorphism, but this is evident from the construction and the basis chosen. (If $\omega \in \Omega$, then $r|_{U(\omega)}\colon U(\omega) \cong r(\omega)$.) Open $R$-sets in $R(\Omega)$ are identified with elements in $S_0(X)$ in a straightforward manner: an open $R$-set $T$ is homeomorphic to its images under the range and domain mappings; so let $\sigma_T \in S_0(X)$ be the unique partial homeomorphism for which $T = \{(\sigma_T x, x)\colon x \in d(\sigma_T)\}$. Choose $\omega_i \in \Omega$ so that $T = \bigcup_i U(\omega_i)$. The collection $\{\omega_i\}$ is coherent and for each $i$, $\sigma_T d(\omega_i) = \omega_i$. Since $d(\sigma_T) = \bigcup_i d(\omega_i)$, $\sigma_T \in \Omega^\alpha$. Further, if $\sigma \in \Omega^\alpha$ then $T(\sigma) = \{(\sigma(x), x)\colon x \in d(\sigma)\}$ is an open $R$-set. It remains to show that if $R$ is a principal $r$-discrete groupoid admitting a Haar system then $R = R(\Omega)$ for some localization on the unit space $X$ $(= R^0)$. This is established already, in essence, by Proposition 1.2.8 in **[39]**, where it is shown (among other things) that $R$ must have a basis, $\mathcal{U}$, of open $R$-sets. It is assumed here that all spaces are second countable so $\mathcal{U}$ may be chosen to be countable. Identifying $\mathcal{U}$ with elements in $S_0(X)$, $\Omega$ is defined to be the smallest ISG containing it. Then $\Omega$ localizes $X$ and $R = R(\Omega)$ (if $(x, y) \in R$, there is $U \in \mathcal{U}$ with $(x, y) \in U$, and consequently $y \in d(\sigma_U)$, $\sigma_U(y) = x$). $\square$

It seems to be true that $C^*(\Omega)$ is isomorphic to $C^*(R(\Omega), \sigma)$ (as defined by Renault) at least if the continuous cocycle $\sigma$ is trivial. In any case, it is clear that $C_c(\Omega) = C_c(R(\Omega))$ if the Haar system is the counting measure.

3. For this class of groupoids, Renault constructs a reduced $C^*$-algebra, $C^*_{\mathrm{red}}(R, \sigma)$, (in addition to the usual $C^*$-algebra associated to a groupoid) wherein the norm is derived from representations induced from measures on the unit space. Some of his results should be noted:

PROPOSITION. 2.4.6 [**39**]. *There is an order preserving bijection between the lattice of ideals in the reduced $C^*$-algebra and the lattice of invariant open sets in the unit space. (So in particular $C^*_{\mathrm{red}}(R(\Omega), \sigma)$ is simple iff $\Omega$ simplifies.)*

He defines a conditional expectation $P$.

PROPOSITION. 2.4.8 [**39**]. *The conditional expectation $P$: $C^*_{\mathrm{red}}(R, \sigma) \to C_0(R^0)$ is the unique such with range $C_0(R^0)$; furthermore it is faithful.*

The conditional expectation is defined first on $C_c(R, \sigma)$ as restriction to $R^0$ and then extended. Uniqueness is proven by a partition-of-unity argument.

4. Suppose $\Omega$ is a free localization on $X$.

THEOREM. *There is a conditional expectation $P$: $C^*(\Omega) \to C_0(X)$ and an ideal $I \subseteq C^*(\Omega)$ so that $I^+ = \{a \geq 0\colon P(a) = 0\}$ and*
(i) *$C_0(X)$ is maximal abelian,*
(ii) *$\{a \in C^*(\Omega)\colon a^*P(b)a = P(a^*ba)$ each $b \in C^*(\Omega)\}$ is total.*

*Proof.* Consider the following directed set:

$$\Delta = \left\{(K, \{\lambda_i\})\colon K \subseteq X \text{ compact}, \lambda_1, \ldots, \lambda_n \in \mathrm{id}(\Omega) \text{ and } K \subseteq \bigcup \lambda_i\right\}.$$

For $\delta = (K, \{\lambda_1, \ldots, \lambda_n\})$ and $\gamma = (L, \{\mu_1, \ldots, \mu_m\}) \in \Delta$ put $\delta \leq \gamma$ if $K \subseteq L$ and for each $j$, $1 \leq j \leq m$, with $\mu_j \cap K \neq \varnothing$ there is $i$, $1 \leq i \leq n$, so that $\lambda_i \mu_j = \mu_j$. For each $\delta \in \Delta$, $\delta = (K, \{\lambda_i\})$, choose $f_i^\delta \in C_c(\lambda_i)^+$ so that $\Sigma_i (f_i^\delta)^2(x) = 1$ for each $x \in K$, and $|\Sigma (f_i^\delta)^2|_\infty = 1$. Define $P_\delta(a) = \Sigma f_i^\delta a f_i^\delta$ for $a \in C^*(\Omega)$. Then $P_\delta$ is completely positive and:

*Claim.* $|P_\delta| = 1$. Let $\mathfrak{H}$ be a faithful $C^*(\Omega)$ module (i.e., a representation but this notation is suppressed for clarity of presentation). Let $\mathfrak{H}^n = \mathfrak{H} \oplus \cdots \oplus \mathfrak{H}$ ($n$ times), put $\rho(a)(\xi_1, \ldots, \xi_n) = (a\xi_1, \ldots, a\xi_n)$ for $a \in C^*(\Omega)$. Let $V_\delta$: $\mathfrak{H} \to \mathfrak{H}^n$ be defined by $V_\delta(\xi) = (f_1^\delta \xi, \ldots, f_n^\delta \xi)$. Then $V_\delta^* V_\delta = \Sigma (f_i^\delta)^2$ so $|V_\delta^* V_\delta| = 1$; then $P_\delta(a) = V_\delta^* \rho(a) V_\delta$ and $|P_\delta| = 1$.

*Claim*. For each $a \in C_c(\Omega)$ there is $\delta \in \Delta$ so that if $\gamma \geq \delta$ then $P_\gamma(a) = P_\delta(a) \in C_c(X)$. First, observe that if $\delta = (K, \{\lambda_i\})$ and $f \in C_c(X)$ with $\operatorname{supp} f \subseteq K$ then $P_\delta(f) = f$. Next consider $(\omega, f) \in E(\Omega)$; since $K = \operatorname{supp} f$ is compact and $\Omega$ is free there are $\lambda_i \in \mathrm{id}(\Omega)$ so that $K \subseteq \bigcup \lambda_i$ and $\lambda_i \omega \lambda_i = \lambda_i$ or $\theta$, let $\delta = \{K, \{\lambda_i\}\}$. Put $n(\delta) = \{i\colon \lambda_i \omega \lambda_i = \lambda_i\}$. Suppose $\gamma = (L, \{\mu_j\}) \geq \delta$. Then $\mu_j \omega \mu_j = \mu_j$ or $\theta$, set $n(\gamma) = \{j\colon \mu_j \omega \mu_j = \mu_j\}$; observe that if $\lambda_i \mu_j \neq \theta$ then $i \in n(\delta)$ iff $j \in n(\gamma)$. We have

$$\begin{aligned} P_\delta((\omega, f)) &= \sum_i (\lambda_i, f_i^\delta)(\omega, f)(\lambda_i, f_i^\delta) \\ &= \sum_{i \in n(\delta)} \left(\lambda_i \omega \lambda_i, f\left(f_i^\delta\right)^2\right) = \sum_{i \in n(\delta)} \left(\lambda_i, f\left(f_i^\delta\right)^2\right) \in C_c(X), \end{aligned}$$

and since $(\omega, f) = \Sigma_i(\omega\lambda_i, f(f_i^\delta)^2)$,

$$\begin{aligned} P_\gamma((\omega, f)) &= P_\gamma\Big(\sum_i \left(\omega\lambda_i, f\left(f_i^\delta\right)^2\right)\Big) \\ &= \sum_i \sum_j (\mu_j, f_j^\gamma)\left(\omega\lambda_i, f\left(f_i^\delta\right)^2\right)(\mu_j, f_j^\gamma) \\ &= \sum_{i \in n(\delta)} \sum_j (\mu_j, f_j^\gamma)\left(\omega\lambda_i, f\left(f_i^\delta\right)^2\right)(\mu_j, f^\gamma) \\ &\qquad \left(\text{if } i \notin n(\delta) \text{ then } \mu_j \omega \lambda_i \mu_j = \theta \text{ for each } j\right) \\ &= P_\gamma\left(P_\delta((\omega, f))\right) = P_\delta((\omega, f)) \quad \text{since } P_\delta((\omega, f)) \in C_c(X). \end{aligned}$$

For $a = \Sigma_i(\omega_i, f_i)$, choose $\delta_i \in \Delta$ for each $i$ so that if $\gamma_i \geq \delta_i$ then $P_{\gamma_i}((\omega_i, f_i)) = P_{\delta_i}((\omega_i, f_i))$. Choose $\delta \in \Delta$ so that $\delta \geq \delta_i$ for each $i$. Then if $\gamma \geq \delta$, $P_\gamma(a) = P_\delta(a) \in C_c(X)$. Define $P\colon C_c(\Omega) \to C_c(X)$ by $P(a) = \lim_{\delta \in \Delta} P_\delta(a)$.

*Claim*. If $f \in C_c(X)$ then $P(f) = f$ and if $a \in C_c(\Omega)$, $P(af) = P(fa) = fP(a)$. For $f \in C_c(X)$, take $\delta = (K, \{\lambda_i\})$ so that $\operatorname{supp} f \subseteq K$. Then evidently $P_\delta(f) = f$. For $a \in C_c(\Omega)$ let $\gamma \in \Delta$ be such that $P_\gamma(a) = P(a)$. Then

$$P_\gamma(fa) = \sum_j f_j^\gamma f a f_j^\gamma = f \Sigma f_j^\gamma a f_j^\gamma = f P_\gamma(a) = f P(a)$$

(and if $\gamma' \geq \gamma$, $P_{\gamma'}(fa) = fP(a)$) so $P(fa) = fP(a)$; the fact that $P(af) = fP(a)$ is proven similarly.

*Claim*. $P$ extends uniquely to a conditional expectation on $C^*(\Omega)$. For $a \in C^*(\Omega)$ set

$$P(a) = \lim_{\delta \in \Delta} P_\delta(a).$$

Then $P(a) \in C_0(X)$. Choose $a_n \in C_c(\Omega)$ so that $a_n \to a$. Since $|P| = 1$, $P(a_n)$ converges to some $f \in C_0(X)$. For $\varepsilon > 0$, there is $n$ so that $|P(a_n) - f| < \varepsilon$ and $|a - a_n| < \varepsilon$; take $\delta_n \in \Delta$ so that $P_\delta(a_n) = P(a_n)$ for $\delta \geq \delta_n$. Hence for $\delta \geq \delta_n$,

$$|P_\delta(a) - f| \leq |P_\delta(a) - P_\delta(a_n)| + |P(a_n) - f| < 2\varepsilon.$$

Thus $P(a) = f$.

*Claim*. $C_0(X)$ is maximal abelian. Let $a \in C^*(\Omega)$ commute with $C_0(X)$; since $X$ is $\sigma$-compact there is an increasing sequence of compact subsets $\{K_n\}$ which exhausts $X$. Choose $\delta_n \in \Delta$, $\delta_n = (K_n, \{\lambda_i^n\})$:

$$P_{\delta_n}(a) = \sum_i f_i^{\delta_n} a f_i^{\delta_n} = \sum_i \left(f_i^{\delta_n}\right)^2 a \to a.$$

This is because $\{\Sigma_i (f_i^{\delta_n})^2\}$ forms an approximate identity for $C_c(\Omega)$ and consequently for $C^*(\Omega)$. Hence $P(a) = a \in C_0(X)$. Finally if $a \in E(\Omega)$ $(a = (\omega, f))$ then $a^* P(b) a = P(a^* b a)$ for all $b$. This is checked easily for $b \in E(\Omega)$ extends linearly to $C_c(\Omega)$ and obtains generally by continuity. Evidently $E(\Omega)$ is total in $C^*(\Omega)$. One may now infer that there is an ideal $I \subseteq C^*(\Omega)$ so that $\{a \geq 0: P(a) = 0\} = I^+$, since if $a \geq 0$ and $P(a) = 0$ then $P(b^* a b) = 0$ for any $b \in C^*(\Omega)$. □

COROLLARY. *If $\Omega$ simplifies then $C^*(\Omega)/I$ is simple.*

*Proof*. Suppose $C^*(\Omega)/I$ is not simple. Then there is a proper ideal $J \subseteq C^*(\Omega)$ so $I \subseteq J$ but $I \neq J$. Let $a \geq 0$ with $a \in J$ but $a \notin I$, $\delta \in \Delta$; then $P_\delta(a) \in J$ and since $P_\delta(a) \to P(a)$, $P(a) \in J$; $P(a) \neq 0$ since $a \geq 0$ and $a \notin I$. Since $C_c(\Omega)$ is algebraically simple (§5/6) and dense in $C^*(\Omega)$ it follows that $J = C^*(\Omega)$. So $C^*(\Omega)/I$ is simple. □

COROLLARY. *If $\Omega$ simplifies and $C^*(\Omega)$ has a faithful finite trace $\tau$ then $C^*(\Omega)$ is simple.*

*Proof*. In view of the above corollary it suffices to show that $I = 0$. Suppose $a \in I^+$, $a \neq 0$; then $\tau(a) > 0$. But since $\tau$ is finite it must be norm continuous. For each $b \in C_c(\Omega)$, $\delta \geq \delta_0$, $\tau(b) = \tau(P_\delta(b))$; so by continuity $\tau(a) = \tau(P(a))$, but this is a contradiction since $P(a) = 0$. So $I = 0$ and $C^*(\Omega)$ is simple. □

5. Most of the examples considered heretofore are of free localizations. Each AF localization is certainly free since finite localizations are free. In this case the conditional expectation $P$ was constructed by Stratila-Voiculescu in [**46**], p. 17, and shown to be faithful (I.2.7, p. 24).

If $G$ is discrete and acts freely on a space $X$, then amenability for $G$ will guarantee that $P$ is faithful. Related discussions may be found in [**20, 31, 38**].

([**39**] 2.3.6) Renault's notion of measure-wise amenability guarantees in the case of principal $r$-discrete groupoids that the conditional expectation is faithful. It is interesting to observe that if $P$ is faithful, $C^*(\Omega)$ embeds in the multiplier algebra of a continuous trace algebra SME to $C_0(X)$. (This is because in line with Rieffels treatment of conditional expectations in [**41**] a faithful representation of $C^*(\Omega)$ is induced from $C_0(X)$ via $P$.)

**8. Compact primitives.**

1. Rieffel's discovery [**44**] of a countable family of inequivalent projections in an irrational rotation algebra suggests the following result, a condition of sufficiency for the existence of a projection. Suitably modified, the condition yields an increasing sequence of projections which forms an approximate identity. Throughout this section $\Omega$ will denote a simplification on $X$.

DEFINITION. A space $Y$ is called a compact primitive of $\Omega$ if $Y$ is compact and a primitive of a reduction (i.e., there is an open set $U \subseteq X$ and a local homeomorphism $\psi$: $U \to Y$ admitted by $\Omega_U$). Evidently a compact primitive is an $\Omega$-manifold.

2. THEOREM. *Let $Y$ be a compact primitive of $\Omega$, so there is an open set $U \subseteq X$ and a local homeomorphism $\psi$: $U \to Y$ admitted by $\Omega_U$; let $\Sigma = \{\sigma_i\} \subseteq S_0(Y, X)$ be a complete set of local sections for $\psi$. Then there is a partial isometry $u \in J(\Sigma)$ so that $u^*u = 1$ $(\in C^*(\Omega(\Sigma)))$.* (§6/3).

*Proof.* For each $i$ choose $f_i \in C_c(d\sigma_i))$ so that for each $y \in Y$ there is $i$ for which $f_i(y) \neq 0$. Since $Y$ is compact there is $m > 0$ so that $Y = \bigcup_{i=1}^m \{y\colon f_i(y) \neq 0\}$. In addition, for each $y \in Y$ set $n(y) = \{(i, j)\colon y \in \sigma_i^*\sigma_j\}$ (Note: $\sigma_i^*\sigma_j \in \mathrm{id}(\Omega(\Sigma))$ and put $f(y) = \Sigma_{i,j \in n(y)}(\bar{f}_j f_i)(y)$. Then we require $f(y) > 0$ for all $y$. (This requirement is satisfied if, for instance, each $f_i \geq 0$.) Now set $g_i(y) = f_i(y)/(f(y))^{1/2}$. Then $g_i \in C_c(d(\sigma_i))$ and $u = \Sigma(\sigma_i, g_i) \in J(\Sigma)$ (since $u \in C^*(\Upsilon(\Sigma))$; cf. §4/3). Then

$$u^* = \sum_{j=1}^m (\sigma_j^*, \bar{g}_j \sigma_j^*)$$

and

$$u^*u = \sum_{i,j}\left(\sigma_j^*\sigma_i, \left(\bar{g}_j\sigma_j^*\sigma_i\right)g_i\right) = h \in C(Y)$$

since $\{\sigma_i\}$ is left coherent, $\sigma_j^*\sigma_i \in \mathrm{id}(\Omega(\Sigma))$. Then

$$h(y) = \sum_{i,j\in n(y)}\left(\sigma_j^*\sigma_i, \bar{g}_j g_i\right) = \frac{1}{f(y)}\sum_{i,j\in n(y)}\left(\sigma_j^*\sigma_i, \bar{f}_j f_i\right) = 1. \qquad \square$$

COROLLARY. *Set $p = uu^* \in C(\Omega)$, $p$ is a projection and $pC^*(\Omega)p = uC^*(\Omega(\Sigma))u^* \cong C^*(\Omega(\Sigma))$. Also, if $v \in J(\Sigma)$ is another such ($v^*v = 1$) then $q = vv^* \sim p$.*

*Proof.* Let $w = vu^* \in C^*(\Omega)$. Then $ww^* = vu^*uv^* = q$ and $w^*w = uv^*vu^* = p$. $\square$

DEFINITION. If $X$ is compact then two compact primitives $(Y_1, \Sigma_1)$ and $(Y_2, \Sigma_2)$ are said to be complementary if there is $u_1 \in J(\Sigma)$ with $u_1^*u_1 = 1 \in C^*(\Omega(\Sigma_1))$, and $u_2 \in J(\Sigma_2)$ with $u_2^*u_2 = 1 \in C^*(\Omega(\Sigma_2))$ so that $u_1u_1^* + u_2u_2^* = 1 \in C^*(\Omega)$. In this case $X \simeq (Y_1 \vee Y_2, \Sigma_1 \cup \Sigma_2)$. An example of this phenomenon appears in the next section.

DEFINITION. A sequence of compact primitives $\{(Y_i, \Sigma_i)\}$ is said to be complementary if there are $u_i \in J(\Sigma_i)$ with $u_i^*u_i = 1 \in C^*(\Omega(\Sigma_i))$ and the collection $\{u_iu_i^*\colon i\}$ are orthogonal projections in $C^*(\Omega)$ so that $p_n = \sum_{i=1}^n u_iu_i^*$ is an approximate identity. In this case $X \simeq (\bigvee_i Y_i, \bigcup_i \Sigma_i)$ ($X$ is certainly an $\Omega$-manifold).

3. A condition for the existence of an approximate identity consisting of an increasing sequence of projections is offered which seems to be weaker than the existence of a complementary sequence of compact primitives.

THEOREM. *Suppose there is a sequence of compact sets $K_n \subseteq X$ so that $X = \bigcup_n K_n$ and a sequence of compact primitives $\psi_n\colon U_n \to Z_n$ such that:*

(i) *$K_n \subseteq U_n \subseteq K_{n+1}$ each $n$.*

(ii) *If $\psi_n(x) = \psi_n(y)$, $x \in U_n$, $y \in K_n$ then $x = y$ (each $n$).*

*Then there is an increasing sequence of projections, $\{p_n\} \subseteq C^*(\Omega)$, which form an approximate identity.*

*Proof*. We require:

LEMMA. *Suppose $\varnothing \neq V_1 \subseteq K \subseteq V_2 \subseteq X$ with $K$ compact $V_1$, $V_2$ open and there is a compact primitive defined by $\psi$: $V_2 \to Z$ so that if $x \in V_2$, $y \in K$ with $\psi(x) = \psi(y)$ then $x = y$. Then there is a projection $p \in C^*(\Omega_{V_2})$ so that if $a \in C^*(\Omega_{V_1})$, $pa = ap = a$.*

*Proof*. Let $\Sigma = \{\sigma_i\}$ be a family of local sections for $\psi$ and put $Y = \psi(K)$. Then if $y \in Y \cap d(\sigma_i)d(\sigma_j)$, it follows that $\sigma_i(y) = \sigma_j(y) \in K$. Let $u = \Sigma_i(\sigma_i, g_i)$ be as in 2. Then

$$p = uu^* = \sum_{i,j} (\sigma_i, g_i)(\sigma_j^*, \bar{g}_j\sigma_j^*) = \sum_{i,j} \left(\sigma_i\sigma_j^*, (g_i\bar{g}_j)\sigma_j^*\right).$$

Observe that if $\lambda \in \mathrm{id}(\Omega)$, $\lambda \subseteq K$ then $\sigma_i\sigma_j^*\lambda$ is an idempotent; so if $f \in C_c(\lambda)$ then

$$uu^*(\lambda, f) = \sum_{i,j} \left(\sigma_i\sigma_j^*, \left((g_i\bar{g}_j)\sigma_j^*\lambda\right)f\right) = (\lambda, f)$$

because

$$\sum_{i,j\in n(\psi(y))} \left((g_i\bar{g}_j) \cdot \sigma_j^*\right)(y) = 1 \quad \text{each } y \in K.$$

Similarly if $(\omega, f) \in E(\Omega)$ with $r(\omega) \subseteq V_1$, then

$$uu^*(\omega, f) = (\omega, f),$$

and if $(\omega, f) \in E(\Omega)$ with $d(\omega) \subseteq V_1$, then

$$(\omega, f)uu^* = (\omega, f),$$

so if $(\omega, f) \in E(\Omega)$ with $r(\omega), d(\omega) \subseteq V_1$,

$$(\omega, f)p = p(\omega, f) = (\omega, f).$$

The proof is completed by linearity and the observation that $C_c(\Omega_U)$ is dense in $C^*(\Omega_U)$. □

*Proof of Theorem*. Choose $p_n$ as in the lemma. Then $p_{n+1} \geq p_n$ each $n$. Also $p_n \in C^*(\Omega_{U_n})$. By Urysohn's lemma there is $f_n \in C_c(U_n)$ so that $f_n(x) = 1$ for each $x \in K_n$. Then $\{f_n\}$ forms an approximate identity for $C^*(\Omega)$ since $X = \bigcup K_n$ (and consequently if $K$ is compact there is some $n$ for which $K \subseteq K_n$). For $\varepsilon > 0$, $a \in C^*(\Omega)$, there is $n \geq 0$ so that if $m > n$, then $|f_m a - a| < \varepsilon$; for $m > n + 1$:

$$\begin{aligned} |p_m a - a| &\le |p_m a - p_m f_{m-1} a| + |p_m f_{m-1} a - f_{m-1} a| + |f_{m-1} a - a| \\ &\le |p_m|\,|a - f_{m-1} a| + |f_{m-1} a - a| < 2\varepsilon \end{aligned}$$

since $p_m f_{m-1} = f_{m-1}$. So $\{p_m\}$ forms the desired approximate identity.

REMARK. So $C^*(\Omega)$ may be realized as the inductive limit of the primitive algebras $C^*((\Omega_{U_n})_{\psi_n})$.

EXAMPLE. If $\Omega$ is the simplification associated to translations of the reals by a countable dense subgroup then $C^*(\Omega)$ has an approximate identity consisting of an increasing sequence of projections. In the next section this example will be dealt with in somewhat greater detail.

4. By relaxing the requirement that $Y$ be compact in 2, a result of interest may still be salvaged.

PROPOSITION. *Let $U$ be open and let $\psi$: $U \to Y$ be a local homeomorphism admitted by $\Omega_U$; if $\Sigma = \{\sigma_i\} \subseteq S_0(Y, X)$ is a complete family of local sections for $\psi$, then there is $a \in J(\Sigma)$ so that $a^*a$ is strictly positive in $C^*(\Omega(\Sigma))$.*

*Proof*. Since $Y$ is paracompact it may be assumed that $\{d(\sigma_i)\}$ forms a locally finite cover for $Y$. Choose $f_i \in C_c(d(\sigma_i))$ with $\Sigma_i |f_i|_\infty < \infty$, $Y = \bigcup_i \{y: f_i(y) \neq 0\}$ and, further, for each $y \in Y$, there is $\sigma_i \in \Sigma$, $y \in d(\sigma_i)$ so that $\Sigma_{y \in \sigma_i^* \sigma_j} f_j(y) \neq 0$ (this is possible if for instance each $f_i \geq 0$).
Then

$$a = \sum_{i=1}^{\infty} (\sigma_i, f_i) \in J(\Sigma)$$

(since if $a_n = \Sigma_{i=1}^n (\sigma_i, f_i)$, then $a_n \in C^*(\Upsilon(\Sigma))$ and for $m > n$,

$$|a_m - a_n|_0 = \left| \sum_{i=n+1}^{m} (\sigma_i, f_i) \right|_0 \leq \sum_{i=n}^{\infty} |f_n|_\infty )$$

and $f = a^*a \in C_0(Y)$, and for each $y \in Y$, $f(y) > 0$. So $a^*a$ is strictly positive (since $(f/|f|_\infty)^{1/n}$ forms an approximate identity for $C_0(Y)$ and hence for $C^*(\Omega(\Sigma))$). □

COROLLARY. *With $(Y, \Sigma)$ and a as above, the hereditary subalgebra in $C^*(\Omega)$ determined by $aa^*$ is isomorphic to $C^*(\Omega(\Sigma))$.*

*Proof*. Let $a = u|a|$ be the polar decomposition in $C^*(\Upsilon(\Sigma))^{**}$. Then $u^*u = 1 \in C^*(\Omega(\Sigma))$ and $p = uu^*$ is the open projection associated to the hereditary subalgebra of $C^*(\Omega)$ determined by $aa^*$.

**9. Applications.**

1. Let $\Omega$ denote the simplification associated to the action by translation of a countable dense subgroup $G \subseteq \mathbf{R}$. Then, evidently, $\Omega$ is free since $G$ acts freely and $C^*(\Omega) \cong C^*(G, \mathbf{R})$ is simple since it has a faithful trace ($\tau(a) = \int P(a)\, d\mu$ where $\mu$ is Lebesgue measure on $\mathbf{R}$). A special case of interest is when $G = G_\alpha = \{n + m\alpha\colon n, m \in \mathbf{Z}\}$ where $\alpha$ is an irrational number. In this case the compact primitive defined by the quotient map $\psi\colon \mathbf{R} \to \mathbf{R}/\mathbf{Z}$ (which is certainly a local homeomorphism and is admitted by $\Omega$ because $\mathbf{Z} \subseteq G_\alpha$) gives rise to the irrational rotation algebra, i.e., $C^*(\Omega_\psi) \cong C^*(\mathbf{Z}_\alpha, \mathbf{R}/\mathbf{Z})$. Putting $\phi\colon \mathbf{R} \to \mathbf{R}/\mathbf{Z}_\alpha$, we have $C^*(\Omega_\phi) \cong C^*(\mathbf{Z}, \mathbf{R}/\mathbf{Z}_\alpha)$ (which is also an irrational rotation algebra for $\alpha' = 1/\alpha$). The fact that $C^*(\Omega_\psi)$ and $C^*(\Omega_\phi)$ are strong Morita equivalent follows from [**43**], and a classification of unital $C^*$-algebras in this strong Morita equivalence class appears in [**44**].

If $A$ is a simple unital AF algebra with comparability of projections, then Elliott shows that $K_0(A)$ is a subgroup of $\mathbf{R}$ [**23**] (and dense unless $A$ is finite dimensional). The existence of a projection $p \in C^*(K_0(A), \mathbf{R})$ and a unital embedding $A \to pC^*(K_0(A), R)p$ is demonstrated in the last part of this section. A preliminary classification of compact $\Omega$-manifolds modulo weak equivalence (§6/3) is shown to be indexed by $G^+ = \{g \in G\colon g > 0\}$ and this in turn provides some insight into the structure of the diagonally defined hereditary subalgebras of $C^*(\Omega)$ (i.e., $C^*(\Omega_U)$ for $U \subseteq \mathbf{R}$ open).

2. Notation. For ease of presentation, the following notational conventions will be used in this section. For $t \in G^+$, the quotient map $\psi\colon \mathbf{R} \to \mathbf{R}/\mathbf{Z}t$ defines a compact primitive $X(t)$ (understood to be an $\Omega$-manifold with explicit reference to the equivalence class of $\Omega$-atlases suppressed since any collection of local sections of $\psi$ whose domains exhaust $X(t)$ will do) and the induced simplification is denoted $\Omega(t)$. For $s, t \in G^+$ write $X(s \vee t)$ for $X(s) \vee X(t)$ when convenient and let $\Omega(s \vee t)$ be the induced simplification. Let $J(s, t) \subseteq C^*(\Omega(s \vee t))$ denote the canonically specified imprimitivity bimodule for $C^*(\Omega(s))$ and $C^*(\Omega(t))$ (§6/3), so that for $a \in J(s, t)$, $a^*a \in C^*(\Omega(t))$ and $aa^* \in C^*(\Omega(s))$. (for $s, t, r \in G^+$, $J(s \vee t, r) \cong J(s, r) \oplus J(t, r)$ is similarly defined.) Set $\Lambda_0 = \{\lambda\colon G^+ \to \mathbf{Z}^+ : \lambda(t) = 0$ except for a finite number of $t$'s$\}$; writing $nX_t$ for the disjoint sum of $n$ copies of $X_t$, put $X(\lambda) = \bigvee_t \lambda(t) X_t$ where the disjoint sum is taken over the indices $t$, for which $\lambda(t) \neq 0$. Let $\Omega(\lambda)$ denote the

induced simplification. Under coordinate-wise addition, $\Lambda_0$ is an abelian semigroup so that $X(\lambda_1 + \lambda_2) \cong X(\lambda_1) \vee X(\lambda_2)$. Again for $\lambda_1, \lambda_2 \in \Lambda_0$, let $J(\lambda_1, \lambda_2) \subseteq C^*(\Omega(\lambda_1 + \lambda_2))$ denote the canonical imprimitivity bimodule so that if $a \in J(\lambda_1, \lambda_2)$ then $a^*a \in C^*(\Omega(\lambda_2))$, $aa^* \in C^*(\Omega(\lambda_1))$. Define $v\colon \Lambda_0 \to G^+$ by $v(\lambda) = \Sigma_t t\lambda(t)$. The key result is that if

$$v(\lambda_1) = v(\lambda_2) \quad \text{then} \quad X(\lambda_1) \simeq X(\lambda_2).$$

3. The following lemma establishes that $X(s)$ and $X(t)$ appear as complementary primitives in $X(s+t)$.

LEMMA. *Suppose $s, t \in G^+$ and $r = s + t$. Then there are $u \in J(s, r)$, $v \in J(t, r)$ so that $uu^* = 1$ (in $C^*(\Omega(s))$) and, $vv^* = 1$ (in $C^*(\Omega(t))$) while $u^*u + v^*v = 1$ (in $C^*(\Omega(r))$).*

*Proof.* Both $X(s)$ and $X(t)$ are to be viewed as compact primitives of $X(r)$ so the methods of the last section (§8/2) may be invoked. Identify $x \in X(s)$ ($X(t)$, $X(r)$ respectively) with $x \in \mathbf{R}$ (mod $s$) ((mod $t$), (mod $r$), respectively). Choose $\delta$ with $0 < \delta < \frac{1}{3}\min(s, t)$. Define $\omega_1, \omega_2 \in S_0(X(r), X(s))$ by

$$d(\omega_1) = (-\delta, s/2 + 2\delta), \qquad \omega_1(x) = x \quad (\text{in } X(s)),$$
$$d(\omega_2) = (s/2 - \delta, s + 2\delta), \quad \omega_2(x) = x \quad (\text{in } X(s)).$$

Observe that $\omega_1^*, \omega_2^*$ are left coherent and $X(s) = r(\omega_1) \cup r(\omega_2)$ so $X(s)$ is in fact a compact primitive of $X(r)$. Define $\sigma_1, \sigma_2 \in S_0(X(r), X(t))$ by

$$d(\sigma_1) = (s - \delta, s + t/2 + 2\delta), \quad \sigma_1(x) = x - s \quad (\text{in } X(t)),$$
$$d(\sigma_2) = (s + t/2 - \delta, r + 2\delta), \quad \sigma_2(x) = x - s \quad (\text{in } X(t)).$$

Again $\sigma_1^*, \sigma_2^*$ are left coherent while $X(t) = r(\sigma_1) \cup r(\sigma_2)$. Now choose $\lambda$, $\mu\colon [0, \delta] \to [0, 1]$ continuous with $\lambda(0) = 0$, $\lambda(\delta) = 1$ and $\lambda^2(x) + \mu^2(x) = 1$ for each $x \in [0, \delta]$. Choose $\nu \in C[s + \delta, r]$ with $|\nu(x)| = 1$ and $\nu(s + \delta) = i$, $\nu(r) = -i$. Define $f_1, f_2 \in C_c(\mathbf{R})$:

$$f_1(x) = \begin{cases} \lambda(x), & x \in [0, \delta), \\ 1, & x \in [\delta, s/2), \\ \mu^2(x - s/2), & x \in [s/2, s/2 + \delta), \\ 0, & \text{otherwise}, \end{cases}$$

$$f_2(x) = \begin{cases} \lambda^2(x - s/2), & x \in [s/2, s/2 + \delta), \\ 1, & x \in [s/2 + \delta, s), \\ \mu(x - s), & x \in [s, s + \delta), \\ 0, & \text{otherwise}. \end{cases}$$

Set $u = (\omega_1, f_1) + (\omega_2, f_2)$ and note $u \in J(s, r)$. Define $g_1, g_2 \in C(X_r)$ (here $i^2 = -1$):

$$g_1(x) = \begin{cases} i\lambda(x-s), & x \in [s, s+\delta), \\ \nu(x), & x \in [s+\delta, s+t/2), \\ \nu(x)\mu^2(x-s-t/2), & x \in [s+t/2, s+t/2+\delta), \\ 0, & \text{elsewhere}, \end{cases}$$

$$g_2(x) = \begin{cases} \nu(x)\lambda^2(x-s-t/2), & x \in [s+t/2, (s+t/2)+\delta), \\ \nu(x), & x \in [(s+t/2)+\delta, r), \\ -i\mu(x), & x \in [r, r+\delta), \\ 0, & \text{elsewhere}. \end{cases}$$

Set $v = (\sigma_1, g_1) + (\sigma_2, g_2)$, and note $v \in J(t, r)$. Then $u, v$ are the desired partial isometries.

*Claim.* $uu^* = 1$ ($\in C^*(\Omega(s))$); ($\omega_2\omega_1^*, \omega_1\omega_2^* \in \operatorname{id}(\Omega(s))$),

$$(\omega_1, f_1)(\omega_1^*, \bar{f}_1\omega_1^*) = (f_1)^2\omega_1^* = (f_1)^2 \in C(X(s)),$$

$$(\omega_1, f_1)(\omega_2^*, \bar{f}_2\omega_2^*) = (\omega_1\omega_2^*, (f_1\bar{f}_2)\omega_2^*) = \lambda^2\mu^2(x-s/2) \in C(X(s)),$$

$$(\omega_2, f_2)(\omega_1^*, \bar{f}_1\omega_1^*) = (\omega_2\omega_1^*, (f_2\bar{f}_1)\omega_1^*) = \lambda^2\mu^2(x-s/2) \in C(X(s)),$$

$$(\omega_2, f_2)(\omega_2^*, \bar{f}_2\omega_2^*) = (f_2)^2\omega_2^* = (f_2)^2 \in C(X(s)),$$

whence $uu^* = h \in C(X(s))$ and

$$h(x) = (f_1(x))^2 + (f_2(x))^2 + 2(\lambda\mu(x-s/2))^2.$$

Since $\lambda^2 + \mu^2 = 1$ and $\lambda^4 + 2\lambda^2\mu^2 + \mu^4 = 1$, we have $uu^* = 1$. A similar calculation shows that $vv^* = 1$ ($\in C^*(\Omega(t))$). It remains to show that $u^*u + v^*v = 1$.

$$u^*u = ((\omega_1^*, \bar{f}_1\omega_1^*) + (\omega_2^*, \bar{f}_2\omega_2^*))((\omega_1, f_1) + (\omega_2, f_2)).$$

Note that

$$d(\omega_1^*\omega_2) = (s/2-\delta, s/2+2\delta) \cup (s-\delta, s+2\delta);$$

if $x \in (s/2-\delta, s/2+2\delta)$ then $\omega_1^*\omega_2(x) = x$, while if $x \in (s-\delta, s+2\delta)$ then $\omega_1^*\omega_2(x) = x - s$. Thus:

$$(\omega_1^*, \bar{f}_1\omega_1^*)(\omega_1, f_1) = f_1^2,$$

$$\begin{aligned}(\omega_1^*, \bar{f}_1\omega_1^*)(\omega_2, f_2) &= (\omega_1^*\omega_2, (\bar{f}_1(\omega_1^*\omega_2))f_2) \\ &= \lambda^2\mu^2(x-s/2) + (\omega_1^*\omega_2, \lambda\mu(x-s)),\end{aligned}$$

$$\begin{aligned}\left(\omega_2^*, \bar{f}_2\omega_2^*\right)(\omega_1, f_1) &= \left(\omega_2^*\omega_1, \left(\bar{f}_2(\omega_2^*\omega_1)\right)f_1\right)\\ &= \lambda^2\mu^2(x - s/2) + \left(\omega_2^*\omega_1, \lambda\mu(s)\right),\\ \left(\omega_2^*, \bar{f}_2\omega_2^*\right)(\omega_2, f_2) &= f_2^2.\end{aligned}$$

Put $h_1(x) = f_1^2(x) + f_2^2(x) + 2\lambda^2\mu^2(x - s)$; then

$$h_1(x) = \begin{cases} \lambda^2(x), & x \in [0, \delta), \\ 1, & x \in [\delta, s), \\ \mu^2(x - s), & x \in [s, s + \delta), \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$u^*u = h_1 + \left(\omega_2^*\omega_1, \lambda\mu\right) + \left(\omega_1^*\omega_2, (\lambda\mu)\omega_1^*\omega_2\right).$$

(Note that $0 \in d(\omega_2^*\omega_1)$ and $\omega_2^*\omega_1(0) = s$.) A similar calculation yields (note that an appropriate selection of $v$ above is essential to obtain this):

$$v^*v = h_2 - \left(\sigma_1^*\sigma_2, \lambda\mu\right) - \left(\sigma_2^*\sigma_1, (\lambda\mu)\sigma_2^*\sigma_1\right)$$

where

$$h_2(x) = \begin{cases} \lambda^2(x - s), & x \in [s, s + \delta), \\ 1, & x \in [s + \delta, r), \\ \mu^2(x - r), & x \in [r, r + \delta), \\ 0, & \text{elsewhere}. \end{cases}$$

Observe that $h_1 + h_2 = 1$ (in $C^*(\Omega(r))$) and $(-\delta, 2\delta) \subseteq d(\sigma_1^*\sigma_2), d(\omega_2^*\omega_1)$ and if $x \in (-\delta, 2\delta)$ then $\sigma_1^*\sigma_2(x) = \omega_2^*\omega_1(x) = x + s$; whence $(\sigma_1^*\sigma_2, \lambda\mu) = (\omega_2^*\omega_1, \lambda\mu)$. Finally,

$$\begin{aligned} u^*u + v^*v = h_1 &+ \left(\omega_2^*\omega_1, \lambda\mu\right) + \left(\omega_1^*\omega_2, (\lambda\mu)\cdot\omega_1^*\omega_2\right)\\ &+ h_2 - \left(\sigma_1^*\sigma_2, \lambda\mu\right) - \left(\sigma_2^*\sigma_1, (\lambda\mu)\cdot\sigma_2^*\sigma_1\right) = 1. \qquad \square\end{aligned}$$

COROLLARY. $X(s + t) \simeq X(s \vee t) = X(s) \vee X(t)$.

4. The following technical lemma is needed:

LEMMA. *Suppose* $\Pi$ *is a simplification and that* $X_1, X_2, Y_1, Y_2$ *are compact* $\Pi$*-manifolds with* $X_1 \simeq Y_1$ *and* $X_2 \simeq Y_2$. *Then* $X_1 \vee X_2 \simeq Y_1 \vee Y_2$.

*Proof.* By weak equivalence choose $u_1 \in J(X_1, Y_1)$ and $u_2 \in J(X_2, Y_2)$ (where explicit reference to $\Omega$-atlases has been suppressed for clarity) so that

$$u_1^* u_1 = 1 \quad (\text{in } C^*(\Pi(Y_1))), \qquad u_1 u_1^* = 1 \quad (\text{in } C^*(\Pi(X_1))),$$
$$u_2^* u_2 = 1 \quad (\text{in } C^*(\Pi(Y_2))), \qquad u_2 u_2^* = 1 \quad (\text{in } C^*(\Pi(X_2))).$$

Then $u = u_1 + u_2$ is the desired isometry in $J(X_1 \vee X_2, Y_1 \vee Y_2)$. □

PROPOSITION. *Suppose* $\lambda \in \Lambda_0$. *Then* $X(\lambda) \simeq X(v(\lambda))$.

*Proof.* For $\lambda \in \Lambda_0$, put $n(\lambda) = \Sigma_t \lambda(t)$ (this is a finite integer). Proceed by induction on $n(\lambda)$. Suppose $X(\lambda) \simeq X(v(\lambda))$ when $n(\lambda) < n_0$. Now suppose $n(\lambda) = n_0$; then there are $\lambda_1, \lambda_2 \in \Lambda_0$ with $\lambda = \lambda_1 + \lambda_2$; hence $n(\lambda) = n(\lambda_1) + n(\lambda_2)$, and $0 < n(\lambda_1), n(\lambda_2) < n_0$.

$$X(\lambda) = X(\lambda_1 + \lambda_2) = X(\lambda_1) \vee X(\lambda_2) \simeq X(v(\lambda_1)) \vee X(v(\lambda_2)).$$

The weak equivalence follows from the lemma and the induction hypothesis. Note that $v(\lambda) = v(\lambda_1 + \lambda_2) = v(\lambda_1) + v(\lambda_2)$. The result now follows from the result in 3:

$$X(\lambda) \simeq X(v(\lambda_1)) \vee X(v(\lambda_2)) \simeq X(v(\lambda_1) + v(\lambda_2)) = X(v(\lambda)). \quad \square$$

5. Again a technical lemma is required:

LEMMA. *Suppose* $\Pi$ *is a simplification and* $\{X_i\}$ *and* $\{Y_i\}$ *are two sequences of compact* $\Pi$*-manifolds so* $X_i \simeq Y_i$ *each* $i$. *Then* $\bigvee_i X_i \simeq \bigvee_j Y_j$.

*Proof.* Put $X = \bigvee_i X_i$ and $Y = \bigvee_j Y_j$; then for each $i$ there is $u_i \in J(X_i, Y_i) \subseteq J(X, Y)$ so that $u_i^* u_i = 1$ (in $C^*(\Pi(Y_i))$) and $u_i u_i^* = 1$ (in $C^*(\Pi(X_i))$). If $C_i > 0$ is a sequence of scalars for which $\Sigma_i C_i < \infty$, then put $a = \Sigma_i C_i u_i \in J(X, Y)$. Then $a^*a$ and $aa^*$ are both strictly positive, hence the result. □

DEFINITION. Let $\Lambda_1 = \{\lambda\colon G^+ \to \mathbf{Z}^+ : \Sigma_t t\lambda(t) < \infty\}$, and for $\lambda \in \Lambda_1$, set $X(\lambda) = \bigvee_t \lambda(t) X(t)$ and $v(\lambda) = \Sigma_t t\lambda(t)$.

PROPOSITION. *Suppose* $\lambda, \mu \in \Lambda_1$. *If* $v(\lambda) = v(\mu)$ *then* $X(\lambda) \simeq X(v)$.

*Proof.* There are $\lambda_i, \mu_j \in \Lambda_0$ satisfying:

$$\text{(i)} \qquad r = \sum_i v(\lambda_i) = \sum_i v(\mu_j),$$

$$\text{(ii)} \quad \sum_1^n \lambda_i(t) \le \lambda(t) \quad \text{each } n > 0,\ t \in G^+,$$
$$\sum_1^n \mu_j(t) \le \mu(t) \quad \text{each } n > 0,\ t \in G^+,$$
$$\text{(iii)} \quad \sum_1^n v(\lambda_i) < \sum_1^n v(\mu_j) < \sum_1^{n+1} v(\lambda_i) \quad \text{each } n.$$

Put $s_1 = v(\lambda_1)$ and $t_1 = v(\mu_1) - s_1$, and define $s_i$, $t_i$ recursively by $s_i = v(\lambda_i) - t_{i-1}$ and $t_i = v(\mu_i) - s_i$. Note that $s_i, t_i \in G^+$; put $Z = \bigvee_i (X(s_i) \vee X(t_i))$. By the proposition in 4 $X(\mu_i) \simeq X(s_i) \vee X(t_i)$. And so $X(\mu) = \bigvee_i X(\mu_i) \simeq Z$ where weak equivalence follows from the above lemma. Also $X(\lambda_1) \simeq X(s_1)$, and for $i > 1$ $X(\lambda_i) \simeq X(s_i) \vee X(t_{i-1})$; so

$$X(\lambda) \simeq \bigvee_i X(\lambda_i) \simeq X(s_1) \vee \bigvee_{i>1} (X(s_i) \vee X(t_{i-1})) = Z. \qquad \square$$

REMARK. The choice of $\lambda_i \in \Lambda_0$ required that for each $t \in G^+$ there is $n$ so that $\lambda(t) = \sum_{i=1}^n \lambda_i(t)$ (because $v(\lambda) = \sum_i^n v(\lambda_i)$).

DEFINITION. Let $\Lambda^\infty = \{\lambda\colon\ G^+ \to \mathbf{Z}^+ \cup \{\infty\}\colon\ \sum_t t\lambda(t) = \infty\}$ (use usual extended arithmetic so $t \cdot \infty = \infty$ for $t > 0$). For $\lambda \in \Lambda^\infty$ set $X(\lambda) = \bigvee_t \lambda(t) X(t)$, where $\infty \cdot X(t)$ is understood to be the countable disjoint sum of an infinite number of copies of $X(t)$.

PROPOSITION. *For $\lambda \in \Lambda^\infty$, there are $\lambda_i \in \Lambda_0$ so that for each $t \in G^+$, $\lambda(t) = \sum_i \lambda_i(t)$ (and consequently $\sum_i v(\lambda_i) = \infty$).*

*Proof.* Let $T_i \subseteq G^+$ be an ascending sequence of finite subsets so that $G^+ = \bigcup_i T_i$. Set

$$\mu_i(t) = \begin{cases} \min(i, \lambda(t)) & \text{for } t \in T_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu_i \in \Lambda_0$, and for each $t$, $\mu_i(t)$ is non-decreasing. Moreover $\mu_i(t) \to \lambda(t)$. Set $\lambda_1 = \mu_1$, and $\lambda_i(t) = \mu_i(t) - \mu_{i-1}(t)$. $\square$

REMARK. $X(\lambda) = \bigvee_i X(\lambda_i)$.

COROLLARY. *If $\lambda, \mu \in \Lambda^\infty$ then $X(\lambda) \simeq X(\mu)$; in fact:*

$$C^*(\Omega(X(t))) \otimes \mathcal{K} \cong C^*(\Omega(X(\lambda))) \quad \textit{for } t \in G^+,\ \lambda \in \Lambda^\infty.$$

*Proof.* The proof proceeds in exactly the same manner as the proof of the first proposition above.

6. Let $m$ be Lebesgue measure, and $U \subseteq \mathbf{R}$ open (non-empty) with $m(U) < \infty$. Viewing $U$ as an $\Omega$-manifold (via reduction), it is to be proven that $U \simeq X(\lambda)$ for $\lambda \in \Lambda_1$ with $v(\lambda) = m(U)$. Further, if $U$ is open with $m(U) = \infty$ then $U \simeq X(\lambda)$ for $\lambda \in \Lambda^\infty$.

LEMMA. *Let $U = (c, d)$ with $(c < d)$; there are $t_i \in G^+$ with $d - c = \Sigma_i t_i$ so that $U \simeq \vee_i X(t_i)$ (so $C^*(\Omega_U)$ has an approximate identity consisting of projections).*

*Proof.* Since $G$ is dense, there is a strictly decreasing sequence $c_i \in G$, and a strictly increasing sequence $d_i \in G$ such that:

(i) $c_1 < d_1$,

(ii) $c_i - c_{i+1} < c_{i-1} - c_i$; $d_{i+1} - d_i < d_i - d_{i-1}$;

(iii) $c_i \to c$; $d_i \to d$.

Set $s_i = d_i - c_i$, $s_0 = 0$, and $t_i = s_i - s_{i-1}$, and choose $\delta_i > 0$ so that $\delta_i < \frac{1}{4}\min\{d_{i+1} - d_i, c_i - c_{i+1}\}$. We will define a sequence of projections $\{p_i\}$ in accordance with (§8/3) and 3 in this section.

Take $\omega_i \in \Omega_U$ with $[c_i, c_i + \delta] \subseteq d(\omega_i)$ and $\omega_i(x) = x + s_i$. Define $f_i, g_i \in C_c(U)$ by

$$f_i(x) = \begin{cases} \lambda^2(x - c_i), & x \in [c_i, c_i + \delta_i), \\ 1, & x \in [c_i + \delta_i, d_i), \\ \mu^2(x - d_i), & x \in [d_i, d_i + \delta_i), \\ 0 & \text{elsewhere}; \end{cases}$$

$$g_i(x) = \begin{cases} \lambda\mu(x - c_i), & x \in [c_i, c_i + \delta_i), \\ 0 & \text{elsewhere}. \end{cases}$$

Now put $p_i = f_i + (\omega_i, g_i) + (\omega_i, g_i)^*$. Then by §8/3 $p_{i+1} \geq p_i$ and $\{p_i\}$ is an approximate identity. Further, there is $u_i \in J(s_i)$ so that $p_i = u_i u_i^*$ (and $u_i^* u_i = 1$). By 3 (this section), there is $v_i \in J(s_{i+1}, t_{i+1})$ so that $v_i^* v_i = 1$ and $v_i v_i^* + u_{i+1}^* p_i u_{i+1} = 1$. Hence for each $i$, there is $w_i \in J(t_i)$, $w_i^* w_i = 1$ and $p_{i-1} + w_i w_i^* = p_i$. Since $s_i \to d - c$, $\Sigma_i t_i = d - c$. □

THEOREM. *Let $U \subseteq \mathbf{R}$ be open (non-empty) and of finite measure. Then there is $\lambda \in \Lambda_1$ with $v(\lambda) = m(U)$ so that $U \simeq X(\lambda)$.*

*Proof.* The open set $U$ must be the disjoint sum of no more than a countable number of open intervals $U_i$. Evidently, $m(U) = \Sigma_i m(U_i)$. For each $i$, there is $\lambda_i \in \Lambda_1$ with $v(\lambda_i) = m(U_i)$ and $U_i \simeq X(\lambda_i)$. Set $\lambda = \Sigma_i \lambda_i$. (Observe that $v(\lambda) = \Sigma_i v(\lambda_i) = m(U)$, so $X(\lambda) = \bigvee_i X(\lambda_i)$.) To establish that $U \simeq X(\lambda)$, it remains to show that if $\{p_j^i\}$ forms an approximate identity for $C^*(\Omega_{U_i})$ (each $i$) then

$$p_k = \sum_{i+j=k} p_j^i$$

forms an approximate identity for $C^*(\Omega_U)$; but this is clear. □

COROLLARY. *If $U, V \subseteq X$ are open so that $m(U) = m(V) < \infty$, then $C^*(\Omega_U) \cong C^*(\Omega_V)$.*

*Proof.* There are $\lambda_1, \lambda_2 \in \Lambda_1$, $v(\lambda_1) = v(\lambda_2) = m(U) = m(V)$, and $U \simeq X(\lambda_1)$, $V \simeq X(\lambda_2)$. But $X(\lambda_1) \simeq X(\lambda_2)$. Hence $U \simeq V$. □

REMARK. The restriction $m(U) < \infty$ in the corollary may actually be removed. There is no difficulty involved in constructing the appropriate approximate identity for $U = \mathbf{R}$, $(c, \infty)$, or $(-\infty, d)$ and then establishing weak equivalence with $X(\lambda)$ for $\lambda \in \Lambda^\infty$. In particular, $C^*(\Omega)$ is stable and so is $C^*(\Omega_U)$ when $m(U) = \infty$.

7. Some facts concerning the order structure of $G$ are collected.

DEFINITION. For $\lambda \in \Lambda_0$, put $d(\lambda) = \{t \in G^+ : \lambda(t) > 0\}$. For $n \in \mathbf{Z}^+$ set $(n\lambda)(t) = n(\lambda(t))$ so $n\lambda \in \Lambda_0$. For $t_i, \ldots, t_n \in G^+$, put $\mathrm{Sp}(t_i) = \{\Sigma_i k_i t_i : k_i \in \mathbf{Z}^+\}$.

LEMMA. *If $t_0, t_1, \ldots, t_n \in G^+$ are distinct and non-zero, then there are $\lambda_i \in \Lambda_0$, $0 \le i \le n$ with $v(\lambda_i) = t_i$ and $d(\lambda_0) \subseteq \bigcup_{i>0} d(\lambda_i)$.*

*Proof.* If $t_0 \in \mathrm{Sp}(t_1, \ldots, t_n)$ then there is nothing to prove. By reindexing if necessary we may assume $t_1 < t_i$ for $i > 1$. Suppose $t_0 < t_1$. Set $\lambda_0(t_0) = 1$, and for $i \ge 1$ put $\lambda_i(t_0) = \max\{n: nt_0 \le t_i\}$, $\lambda_i(t_i - t_0\lambda_i(t_0)) = 1$ unless $\lambda_i(t_0)t_0 = t_i$; so the prescribed conditions are met in this case. If $t_1 < t_0$, let $n_0 = \max\{n: nt_1 < t_0\}$ and let $s = t_0 - nt_1$. For $i \ge 1$ set $\lambda_i(s) = \max\{n: ns \le t_i\}$ and $\lambda_i(t_i - s\lambda_i(s)) = 1$ unless $t_i = s\lambda_i(s)$. Set $\lambda_0(s) = n_0\lambda_1(s) + 1$; for $t \ne s$ set $\lambda_0(s) = n_0\lambda_1(t)$. □

DEFINITION. If $\lambda, \mu \in \Lambda_0$ with $v(\lambda) = v(\mu)$ then write $\mu \ll \lambda$ if for each $t \in d(\mu)$ there is $\lambda_t \in \Lambda_0$, $d(\lambda_t) \subseteq d(\lambda)$, $v(\lambda_t) = t$ so that $\lambda = \Sigma_t \mu(t)\lambda_t$.

PROPOSITION. *If $t \in G^+$, there is a sequence $\lambda_i \in \Lambda_0$ with $\lambda_1 \ll \lambda_2 \ll \lambda_3 \ll \cdots$ so $v(\lambda_i) = t$ and $\bigcup_i \operatorname{Sp}(d(\lambda_i)) = G^+$.*

*Proof.* This follows from the above lemma and the fact that $G^+$ is countable.

8. REMARK. If $d(\lambda) = \{t\}$ and $\lambda(t) = n$, then it is not hard to see that $C^*(\Omega(\lambda)) \cong M_n(C^*(\Omega(t)))$. In fact there is a canonical unital embedding $M_n \to C^*(\Omega(\lambda))$.

DEFINITION. Suppose $\lambda \in \Lambda_0$, and $d(\lambda) = \{t_1,\ldots,t_n\}$. Then $X(\lambda) = \bigvee_{i,j} Y_{ij}$ where $1 \leq i \leq n$, $1 \leq j \leq \lambda(t_i)$ and $\sigma_{ij}$: $X(t_i) \cong Y_{ij}$. Put

$$S(\lambda) = \{\sigma_{ij}\sigma_{ik}^*\colon 1 \leq i \leq n,\, 1 \leq j, k \leq \lambda(t_i)\} \cup \{\theta\}.$$

*Facts.* Then $S(\lambda) \subseteq \Omega(\lambda)^\alpha$; if $\nu = (\lambda(t_1),\ldots,\lambda(t_n))$ then $S(\lambda) \cong S_\nu$ (§2/4) as ISG's.

DEFINITION. Put $u(\sigma) = (\sigma, \chi(d(\sigma)))$ for $\sigma \in S(\lambda)$, where $\chi(d(\sigma))$ is the characteristic function on $d(\sigma)$ (which is continuous since $d(\sigma)$ is compact open).

*Facts.* For $\sigma \in S(\lambda)$, $u(\sigma)$ is an isometry in $C^*(\Omega(\lambda))$. Put $M(\lambda) = \operatorname{sp}\{u(\sigma)\colon \sigma \in S(\lambda)\}$. Then $1 \in M(\lambda) \subseteq C^*(\Omega(\lambda))$ and $M(\lambda) \cong \bigoplus M_{\lambda(t_i)}$.

PROPOSITION. *Suppose $\mu \ll \lambda$ with $v(\mu) = v(\lambda) = t(\lambda, \mu \in \Lambda_0)$. Then there is $u \in J(\lambda, \mu)$ (unitary) so that $uM(\mu)u^* \subseteq M(\lambda)$ is a unital embedding of finite dimensional subalgebras.*

*Proof.* Since $\mu \ll \lambda$, there are $\lambda_i \in \Lambda_0$, $1 \leq i \leq n$, so that $t_i = v(\lambda_i)$, $\{t_i\} = d(\mu)$ and $\lambda = \sum_i \mu(t_i)\lambda_i$. Say $d(\lambda) = \{s_1,\ldots,s_m\}$; then $\lambda(s_j) = \sum_i \mu(t_i)\lambda_i(s_j)$. As in (§3/6) there is a unital embedding of ISG's $\rho$: $S^\alpha(\mu) \to S^\alpha(\lambda)$ given by the multiplicity matrix $[\lambda_i(s_j)]$. Then there is a partition of $X(\lambda) = \bigvee_i \mu(t_i)X(\lambda_i)$ so that elements in $\rho(S^\alpha(\mu))$ transpose the disjoint summands (e.g., sending one copy of $X(\lambda_i)$ to another). Then since $X(\mu) = \bigvee_i \mu(t_i)X(t_i)$ and $X(t_i) \simeq X(\lambda_i)$, there is $u \in J(\lambda, \mu)$ with $uS(\mu)u^* = \rho(S(\mu)) \subseteq S(\lambda)^\alpha$. Then the matrix algebras embed as required. □

9. THEOREM. *Let $A$ be the unique $AF$ algebra with Elliott group, $K_0(A) = G$ (as ordered group) and dimension range, $D(A) = \{s \in G^+ : s \leq t\}$. Then $A$ unitally embeds in $C^*(\Omega(t))$.*

*Proof.* By the proposition in 7, there are $\lambda_i \in \Lambda_0$ with $v(\lambda_i) = t$ each $i$ and $\lambda_1 \ll \lambda_2 \ll \lambda_3 \ll \cdots$, $\bigcup \mathrm{Sp}(d(\lambda_i)) = G^+$. And, by repeated application of the proposition in 8, $M(\lambda_i) \subseteq C^*(\Omega(t))$, unitally and $M(\lambda_1) \subseteq M(\lambda_2) \subseteq \cdots$ is an increasing sequence of finite dimensional algebras. Hence its closure is an AF algebra $A$ unitally contained in $C^*(\Omega(t))$. Suppose $d(\lambda_i) = \{t_1^i, \ldots, t_{n_i}^i\}$. Then

$$\mathbf{Z}^{n_1} \to \mathbf{Z}^{n_2} \to \cdots$$

are the associated order group morphisms (the multiplicity matrices are given as in 8). And if $\nu = (k_1, \ldots, k_{n_i}) \in \mathbf{Z}^{n_i}$ then $\phi_i$: $\mathbf{Z}^{n_i} \to G$ by $\phi_i(\nu) = \sum_j k_j t_j^i$ while $G^+ = \bigcup_i \phi_i((\mathbf{Z}^{n_i})^+)$, and $D(A) = \{s \in G^+ : s \leq t\}$. □

**10. Appendices.**

1. *Cuntz/Krieger simplifications.* In **[13]** the authors present a class of traceless simple $C^*$-algebras related to the field of topological Markov chains generalizing an earlier construction of Cuntz in **[12]**.

If $A$ is an $n \times n$ with entries in $\{0, 1\}$, no row or column is completely 0, and $A$ is irreducible (for each $i, j$ there is an $m$ so that $(A^m)_{ij} > 0$) but not a permutation matrix, then $\mathcal{O}_A$ is generated by isometries $\{S_i\}$ on a Hilbert space $\mathfrak{H}$ satisfying

$$(\dagger) \qquad \begin{aligned} &(S_i S_i^*)(S_j S_j^*) = 0 \quad \text{if } i \neq j, \\ &S_i^* S_i = \sum_j A_{ij} S_j S_j^*. \end{aligned}$$

Consider the one-sided shift for the associated topological Markov chain:

Let $X_A = \{(j_k)\ k \geq 0\colon 1 \leq j_k \leq n \text{ each } k,\ A(j_k, j_{k+1}) = 1\}$. Then $X_A$ is a zero-dimensional compactum, i.e., is compact and has a basis consisting of compact open sets. This basis is indexed by multi-indices $\mu = (j_1, \ldots, j_n)$ with $A(j_k, j_{k+1}) = 1$; and $K_\mu = \{(i_k) \in X_A\colon i_k = j_k \text{ for } k = 1, \ldots, n\}$. Define the one-sided shift $\sigma_A\colon X_A \to X_A$ by

$$(\sigma_A(j))_k = j_{k+1}.$$

Then $\sigma_A$ is a local homeomorphism and in fact the localization generated by $\sigma_A$ and $\{K_\mu\colon \mu\}$ simplifies. Denote this simplification $\Omega(A)$; then $C^*(\Omega(A)) \cong \mathcal{O}_A$ (observe that $\Omega(A)$ is not free). Let $\sigma_i$ be local sections for $\sigma_A$ so that $\sigma_i \sigma_A = K_i$ (simple multi-index). Then under any *-representation on $\pi$ of $\Omega(A)$, $\pi(\sigma_i)$ satisfy the relations (†).

2. This example is closely related to 7d of [35]; in particular, the argument given there guarantees the simplicity of the algebra associated to the simplification considered here.

Let $\mathbf{T}$ denote the unit circle viewed as complex numbers of unit modulus ($\mathbf{T} = \{e^{i\theta}: 0 \leq \theta < 2\pi\}$). Define $\beta: \mathbf{T} \to \mathbf{T}$ by $\beta(z) = z^2$. Then $\beta$ is a local homeomorphism. Let $\Sigma$ be a complete collection of local sections for $\beta$ so that $\{d(\sigma): \sigma \in \Sigma\}$ forms a countable basis. Let $\Omega$ be the localization generated by $\Sigma$ (observe that $\Sigma \subseteq S_0(\mathbf{T})$).

*Claim.* $\Omega$ simplifies. Set $\Pi_0 = \mathrm{id}(\Omega)$ and define $\Pi_n$ inductively as follows:

$$\Pi_{n+1} = \{\sigma_1 \omega \sigma_2^*: \omega \in \Pi_n, \sigma_1, \sigma_2 \in \Sigma\}.$$

Then $\Pi_n$ is a dynamical localization (§3/7) whose ample group $\Gamma(\Pi_n)$ consists precisely of rotations by roots of unity of order $2^n$. Since $\Pi_n \subseteq \Omega$ for each $n$, $\Omega$ simplifies. However, $\Omega$ is not free because $\beta(1) = 1$.

Let $\psi: \mathbf{R} \to \mathbf{T}$ be the standard quotient map. Then the imprimitive $\Omega^\psi$ (simplifies $\mathbf{R}$) is dynamical. The ample group $\Gamma(\Omega^\psi)$ is the semidirect product of the integers (realized as multiplication by powers of 2) and the dyadic rationals (by translation). Then $C^*(\Omega^\psi)$ is simple by the argument in [35]; thus $C^*(\Omega)$ is simple.

Viewing $\mathbf{T}$ as an $\Omega$-manifold ($\Omega$-atlas given by any open covering), it is not difficult to see that it is weak equivalent to its two-fold ampliation. Hence the algebra $\mathcal{O}_2$ [12] is unitally contained in $C^*(\Omega)$.

## INDEX OF TERMS AND NOTATIONS

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