DIRECT LIMITS OF FINITE SPACES OF ORDERINGS

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Spaces of orderings which are direct limits of finite spaces of orderings arise in a natural way. Every space of orderings is canonically a quotient space of such a space. In this paper we examine the internal structure of such spaces. In particular, we examine how the classification theory for finite spaces of orderings carries over to such spaces. We also establish a relationship between spaces of orderings which are direct limits of finite spaces and certain corresponding types of ultrasums of spaces of orderings. This has application to the problem of representing a space of orderings as the space of orderings of a Pythagorean field.

Generally, we use the terminology and notation of [6, 7, 8]. Throughout the paper, X denotes a space of orderings and $G(X)$ denotes the underlying group of X. Thus $G(X)$ has exponent 2, and X is a closed subset of the character group $\chi(G(X))$ satisfying the usual axioms. If f is a (quadratic) form over $G(X)$, then $\dim(f)$ denotes the *dimension* of f, $\dim_{an}(f)$ denotes the *dimension of the anisotropic part* of f, and $\sigma(f)$ denotes the *signature* of f at $\sigma \in X$. We indicate that two forms f, g are *isometric* (resp. *Witt equivalent*) by writing $f \cong g$ or $f \cong_X g$ (resp. $f \sim g$ or $f \sim_X g$). The *value set* of a form f is denoted by $D(f)$ or $D_X(f)$. This consists of all $x \in G(X)$ satisfying $f \cong \langle x, x_2, \ldots, x_n \rangle$ for some $x_2, \ldots, x_n \in G(X)$. The *sum* and *product* of forms f, g is denoted by $f \oplus g$ and $f \otimes g$, and if n is a positive integer, then $n \times f = f \oplus \cdots \oplus f$ (n times). If Y is also a space of orderings, then a *morphism* from X to Y is a map $\varphi^*\colon X \to Y$ induced by a group homomorphism $\varphi\colon G(Y) \to G(X)$ in the sense that $\varphi^*(\sigma) = \sigma \circ \varphi$ for all $\sigma \in X$.

The *Witt ring* of X is denoted by $W(X)$. If $\sigma \in X$, the ring homomorphism from $W(X)$ into $\mathbf{Z}$ induced by σ is also denoted by σ. By [8, Lem. 7.1], $G(X)$ is naturally identified with the group of units of $W(X)$. Also, it is well known (and easily verified, e.g., using [8, Lem. 4.1]) that X is identified with the set of all ring homomorphisms from $W(X)$ into $\mathbf{Z}$. Also, if $\varphi^*\colon X \to Y$ is a morphism of spaces of orderings then the associated group homomorphism $\varphi\colon G(Y) \to G(X)$ induces a ring homomorphism (also denoted by φ) from $W(Y)$ into $W(X)$. Every ring homomorphism from $W(Y)$ into $W(X)$ is of this form for some unique morphism $\varphi^*\colon X \to Y$. In this way the category of Witt rings of spaces of

orderings and the category of spaces of orderings are equivalent. For a ring-theoretic treatment of Witt rings of spaces of orderings, we refer the reader to [3, 10].

1. Direct limits of finite spaces.

As in [7] we will denote the *direct sum* of a finite collection of spaces of orderings $\{X_1, \ldots, X_n\}$ by $X_1 \oplus \cdots \oplus X_n$. The direct sum of an arbitrary collection of spaces of orderings $\{X_i: i \in I\}$ also exists and will be denoted by $\Sigma_{i \in I} X_i$. This is the space of orderings X defined as follows: $G(X)$ is $\Pi_{i \in I} G(X_i)$, the direct product in the category of groups. The projection $G(X) \to G(X_i)$ identifies X_i with a closed subset of the character group $\chi(G(X))$ and $X_i \cap X_j = \varnothing$ if $i \neq j$. X is defined to be the closure of $\cup_{i \in I} X_i$ in $\chi(G(X))$. Note if I is finite then $\cup_{i \in I} X_i$ is already closed. If f is a form over $G(X)$, we denote by f_i the associated form over $G(X_i)$. Since forms over $G(X)$ define continuous functions on X, it is clear that $f \sim g \Leftrightarrow f_i \sim g_i$ for all $i \in I$. The proof that X is a space of orderings and the direct sum of the collection $\{X_i: i \in I\}$ in the category of spaces of orderings is straightforward and will be omitted. $W(X)$ is identified with a subring of the direct product ring $\Pi_{i \in I} W(X_i)$. Observe that $W(X)$ consists of all elements $(f_i)_{i \in I}$ in $\Pi_{i \in I} W(X_i)$ satisfying:

(i) $\dim(f_i) \equiv \dim(f_j) \pmod 2$ for all $i, j \in I$, and
(ii) $\sup\{\dim_{an}(f_i): i \in I\} < \infty$.

If I is finite then (ii) is vacuous.

We consider *direct limits* (of directed systems) of finite spaces of orderings. It turns out that these always exist in the category of spaces of orderings. Let $(X_i: i \in I)$ be a directed system of finite spaces of orderings. Thus I is an ordered set and morphisms $\varphi_{ij}^*: X_i \to X_j$ are specified whenever $i \leq j$. These are assumed to satisfy $\varphi_{jk}^* \circ \varphi_{ij}^* = \varphi_{ik}^*$ whenever $i \leq j \leq k$. It is also assumed that I is directed, i.e., that for all $i, j \in I$, there exists $k \in I$ such that $i, j \leq k$. To define $X := \varinjlim_{i \in I} X_i$ we first use the associated group homomorphism $\varphi_{ij}: G(X_j) \to G(X_i)$ to form $G(X) := \varprojlim_{i \in I} G(X_i)$, the inverse limit in the category of groups. The group homomorphism $\varphi_i: G(X) \to G(X_i)$ induces a group homomorphism $\varphi_i^*: \chi(G(X_i)) \to \chi(G(X))$, and X is defined to be the closure of $\cup_{i \in I} \varphi_i^*(X_i)$ in $\chi(G(X))$.

THEOREM 1.1. *For any directed system $(X_i: i \in I)$ of finite spaces of orderings, X as defined above is a space of orderings, and X is the direct limit of the directed system $(X_i: i \in I)$ in the category of spaces of orderings.*

Proof. Let $Y = \Sigma_{i \in I} X_i$. Then $G(X) \subseteq G(Y)$ and X is the image of Y under the restriction map $\chi(G(Y)) \to \chi(G(X))$. We prove X is a space of

orderings using [7, Lem. 3.3]. First note, since each $G(X_i)$ is finite, $G(Y)$, with the product topology, is a compact totally disconnected topological group. For $i \leq j$ define $V_{ij} = \{x \in G(Y): \varphi_{ij}(x_j) = x_i\}$. This is closed in $G(Y)$ and $G(X) = \bigcap \{V_{ij}: i, j \in I, i \leq j\}$. Also, if f is a form over $G(Y)$, then the value set $D_Y(f) = \prod_{i \in I} D_{X_i}(f_i)$ is closed in $G(Y)$. Now suppose f, g are forms over $G(X)$ and $D_Y(f) \cap D_Y(g) \neq \varnothing$. Thus the sets $D_Y(f) \cap D_Y(g) \cap V_{ij}$ are closed in $G(Y)$ and using the fact that I is directed and the fact that the maps φ_{ij} preserve value sets, one verifies that the intersection of finitely many such sets is not empty. Thus, by compactness, $D_Y(f) \cap D_Y(g) \cap G(X) \neq \varnothing$. It follows from [7, Lem. 3.3] that X is a space of orderings. The proof that X is the direct limit of the directed system $(X_i: i \in I)$ in the category of spaces of orderings is straightforward and will be omitted.   □

COROLLARY 1.2. *Suppose* $X = \varinjlim_{i \in I} X_i$ *where each* X_i *is finite. Then for any form f over* $G(X)$,

   (i) $f \sim 0 \Leftrightarrow f_i \sim 0$ *for all* $i \in I$,
   (ii) f *is* X-*isotropic* $\Leftrightarrow f_i$ *is* X_i-*isotropic for all* $i \in I$,
   (iii) $D_X(f) = \varprojlim_{i \in I} D_{X_i}(f_i)$.

Proof. Let $Y = \sum_{i \in I} X_i$. The first assertion is clear. By [7, Rem. 3.4] and the proof of Theorem 1.1, f is X-isotropic $\Leftrightarrow f$ is Y-isotropic. On the other hand it is clear that f is Y-isotropic $\Leftrightarrow f_i$ is X_i-isotropic for all $i \in I$. Also, by the proof of [7, Lem. 3.3], $D_X(f) = D_Y(f) \cap G(X)$. Since $D_Y(f) = \prod_{i \in I} D_{X_i}(f_i)$, and $G(X) = \varprojlim_{i \in I} G(X_i)$, this implies $D_X(f) = \varprojlim_{i \in I} D_{X_i}(f_i)$.   □

THEOREM 1.3. *Suppose* $X = \varinjlim_{i \in I} X_i$ *where each* X_i *is finite. Then* $W(X)$ *is the subring of the ring-theoretic inverse limit* $\varprojlim_{i \in I} W(X_i)$ *consisting of all* $(f_i)_{i \in I}$ *satisfying* $\sup\{\dim_{an}(f_i): i \in I\} < \infty$.

Proof. We use the notation of Theorem 1.1. It is clear that $W(X) \subseteq \varprojlim W(X_i)$ and every element $(f_i)_{i \in I}$ of $W(X)$ satisfies $\sup\{\dim_{an}(f_i): i \in I\} < \infty$. Since I is directed and $\varphi_{ij}(f_j) \sim f_i \Rightarrow \dim(f_i) \equiv \dim(f_j)$ (mod 2), we know that every element of $\varprojlim_{i \in I} W(X_i)$ satisfying this latter condition is represented by a form over $G(Y)$ so we are reduced to proving the following *Claim*: Suppose f is a form over $G(Y)$ satisfying $\varphi_{ij}(f_j) \cong f_i$ for all $i, j \in I$ such that $i \leq j$. Then there exists a form g over $G(X)$ satisfying $f \cong g$. For by hypothesis, any intersection of finitely many of the closed sets $D_Y(f) \cap V_{ij}$ is not empty. Thus, by compactness

of $G(Y)$, $D_Y(f) \cap G(X) \neq \emptyset$, say $x_1 \in D_Y(f) \cap G(X)$. Thus $f \cong \langle x_1 \rangle \oplus g$ where g is a form over $G(Y)$ of dimension $n - 1$, where $n = \dim(f)$. Since g satisfies the same hypothesis as f, by induction on n, there exist $x_2, \ldots, x_n \in G(X)$ such that $g \cong \langle x_2, \ldots, x_n \rangle$, so $f \cong \langle x_1, \ldots, x_n \rangle$. $\qquad\square$

A space of orderings which is expressible as a direct limit of finite spaces of orderings will be referred to as a *DLO finite space* for short. Note, if X is a DLO finite space of orderings then, with the inverse limit topology, $G(X)$ is a compact totally disconnected topological group and $W(X)$ is a totally disconnected topological ring.

REMARK 1.4. Suppose X is any space of orderings. The subspace of X generated by a finite set $\{\sigma_1, \ldots, \sigma_n\} \subseteq X$ consists of all $\sigma \in X$ expressible as $\sigma = \sigma_1^{e_1} \cdots \sigma_n^{e_n}$ where $e_1, \ldots, e_n \in \{0, 1\}$. In particular, this subspace is finite. Thus, the finite subspaces of X form a directed system when ordered by inclusion. Suppose we denote the direct limit of this system by $\hat{X}$. Let $\varphi^*: \hat{X} \to X$ be the natural morphism which exists by the universal property of $\hat{X}$. This is surjective and hence identifies X with a quotient space of $\hat{X}$. The corresponding inclusions $\varphi: G(X) \to G(\hat{X})$ (resp. $\varphi: W(X) \to W(\hat{X})$) identify $G(X)$ (resp. $W(X)$) with a dense subgroup of $G(\hat{X})$ (resp. with a dense subring of $W(\hat{X})$). Since the results in [2] are a bit confusing on this point, we emphasize that φ is *not* surjective except in the trivial case where X is finite. Note also, by [9, Th. 1.4] and Cor. 1.2 (ii), if f is a form over $G(X)$, then $\varphi(f)$ is $\hat{X}$-isotropic iff f is X-isotropic.

Suppose X is DLO finite, say $X = \varinjlim_{i \in I} X_i$. Let $X^c := \bigcup_{i \in I} \varphi_i^*(X_i)$ where $\varphi_i^*: X_i \to X$ is the induced morphism. Thus X^c is dense in X. Note that the kernel of $\varphi_i: G(X) \to G(X_i)$ (resp. $\varphi_i: W(X) \to W(X_i)$) is the intersection of the kernels of $\sigma \circ \varphi_i: G(X) \to \{\pm 1\}$ (resp. $\sigma \circ \varphi_i: W(X) \to \mathbf{Z}$), $\sigma \in X_i$. Thus the inverse limit topology on $G(X)$ (resp. $W(X)$) is the weakest topology such that the map $\sigma: G(X) \to \{\pm 1\}$ (resp. $\sigma: W(X) \to \mathbf{Z}$) is continuous for each $\sigma \in X^c$. The following theorem shows, in particular, that X^c depends only on the topology on $G(X)$ (resp. $W(X)$), i.e., is independent of the particular presentation of X as a direct limit of finite spaces.

THEOREM 1.5. *Suppose X is DLO finite and $\sigma \in X$. Then the following are equivalent*:

    (i) $\sigma \in X^c$,

    (ii) $\sigma: W(X) \to \mathbf{Z}$ *is continuous*,

    (iii) $\sigma: G(X) \to \{\pm 1\}$ *is continuous*.

Proof. We use the notation of the proof of Theorem 1.1. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial. To prove (iii) $\Rightarrow$ (i) suppose σ: $G(X) \to \{\pm 1\}$ is continuous. Since $G(X) = \varprojlim_{i \in I} G(X_i)$, there exists $i \in I$ and a character τ on $G(X_i)$ such that $\sigma = \tau \circ \varphi_i$. For $j \geq i$ define the character σ_j on $G(X_j)$ by $\sigma_j = \tau \circ \varphi_{ij}$. If $\sigma_j \in X_j$ for some $j \geq i$, we are done. Assume this is not the case. Then by [**8**, Lem. 4.1], there exist $a_j, b_j \in G(X_j)$ such that $b_j \in D_{X_j}\langle 1, a_j \rangle$, $\sigma_j(a_j) = 1$, $\sigma_j(b_j) = -1$. For each $j \geq i$ let W_j denote the set of $(a, b) \in G(Y) \times G(Y)$ such that $b_j \in D_{X_j}\langle 1, a_j \rangle$, $\sigma_j(a_j) = 1$, but $\sigma_j(b_j) = -1$. This is closed in $G(Y) \times G(Y)$ so for each $j \geq i$ and each $k \leq j$, the sets $W_j \cap (V_{jk} \times V_{jk})$ are closed in $G(Y) \times G(Y)$. One checks that the intersection of finitely many of these sets is not empty, so by compactness of $G(Y) \times G(Y)$, there exist $a, b \in G(X)$ such that $b \in D_X\langle 1, a \rangle$, $\sigma(a) = 1$, but $\sigma(b) = -1$. This contradicts $\sigma \in X$. $\qquad\square$

COROLLARY 1.6. *Suppose X, Y are DLO finite and φ^*: $X \to Y$ is a morphism. Then the following are equivalent*:

    (i) $\varphi^*(X^c) \subseteq Y^c$,

    (ii) φ: $W(Y) \to W(X)$ *is continuous*,

    (iii) φ: $G(Y) \to G(X)$ *is continuous*.

Proof. The topology on $W(X)$ is the weakest such that each $\sigma \in X^c$ defines a continuous function on $W(X)$, so (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (iii) is trivial. Finally, (iii) $\Rightarrow$ (i) follows from Theorem 1.5. $\qquad\square$

Although a DLO finite space X may have several presentations as a direct limit of finite spaces, there is a standard presentation. To obtain this we denote by $\mathfrak{U}(X)$ the set of all finite subspaces of X which lie in X^c.

COROLLARY 1.7. *Suppose X is DLO finite. Then $\mathfrak{U}(X)$ is a directed system of finite spaces of orderings and X is the direct limit of this system.*

Proof. Since a finite product of continuous characters on $G(X)$ is again a continuous character on $G(X)$, it follows using Theorem 1.5 and the argument in Remark 1.4 that the finite subspace of X generated by $\sigma_1, \ldots, \sigma_n \in X^c$ lies in X^c. This proves that $\mathfrak{U}(X)$ is directed. The image of $G(X)$ in $\varprojlim_{Y \in \mathfrak{U}(X)} G(Y)$ via the natural map is dense and since the map is continuous and $G(X)$ is compact, the image is closed. Thus the map is surjective. It is injective since X^c is dense in X. It should be clear now that X is the direct limit of the directed system $\mathfrak{U}(X)$. $\qquad\square$

We are now in a position to characterize DLO finite spaces internally.

COROLLARY 1.8. *Suppose X is a space of orderings and $G(X)$ has the structure of a compact topological group such that $X^c := \{\sigma \in X: \sigma$ is continuous on $G(X)\}$ is dense in X. Then X is DLO finite and the topology on $G(X)$ is the induced inverse limit topology.*

Proof. Immediate from the proof of Corollary 1.7.   □

The content of Corollary 1.8 may also be phrased as follows:

COROLLARY 1.9. *Let X be a space of orderings. Then X is DLO finite iff there exists a subset $B \subseteq X$ satisfying:*
   (i) *$[B] \cap X$ is dense in X. (Here, $[B]$ denotes the linear span of B in $\chi(G(X))$.)*
   (ii) *The natural map $\varphi: G(X) \to \{\pm 1\}^B$ is onto. (Note, φ, is already 1-1 by (i).)*

Proof. Suppose X has a subset B satisfying (i) and (ii). Then the isomorphism φ induces a topology on $G(X)$ giving it the structure of a compact totally disconnected topological group. Also, $X^c = [B] \cap X$ is dense in X, so X is DLO finite by Corollary 1.8. Observe that B is a maximal linearly independent subset of X^c. Conversely, if X is DLO finite and B is any maximal linearly independent subset in X^c, then $[B] \cap X = X^c$ is dense in X and φ is continuous and has dense image in $\{\pm 1\}^B$ so by compactness of $G(X)$, φ is onto.   □

2. **Decomposition of DLO finite spaces.** In this section we show how the structure theory of finite spaces of orderings given in [6] carries over to DLO finite spaces of orderings which satisfy certain finiteness conditions. Throughout, we work in the category consisting of DLO finite spaces together with morphisms which are continuous in the sense of Corollary 1.6.

If Δ is any compact totally disconnected topological group of exponent 2, we denote by $\chi^c(\Delta)$ the group of continuous characters on Δ. Note, for any such Δ, $\Delta \cong \{\pm 1\}^\alpha$, where α denotes the $\mathbf{Z}/2\mathbf{Z}$ dimension of $\chi^c(\Delta)$.

THEOREM 2.1. *Suppose X is the group extension of Y by the group Δ in the sense of [7] where Y is DLO finite and Δ is compact totally disconnected with exponent 2. Then X has, in a natural way, the structure of a DLO finite space.*

Proof. By definition, $G(X) = G(Y) \times \Delta$ and $X = Y \times \chi(\Delta)$. Giving $G(X)$ the product topology, $X^c = Y^c \times \chi^c(\Delta)$, and X is DLO finite using Corollary 1.8. $\square$

For any space of orderings X, the *translation group* $\mathrm{gr}(X)$ is defined as in [9]. For X DLO finite, the group to consider is $\mathrm{gr}(X^c) := \{\gamma \in \chi^c(G(X)) : \gamma X^c = X^c\}$. Using the density of X^c in X and the modification of [6, Lem. 4.2] mentioned in [9, Rem. 2.1], it follows that $\mathrm{gr}(X^c) = \mathrm{gr}(X) \cap \chi^c(G(X))$. Note if X is the group extension of Y by Δ as in Theorem 2.1, then $\mathrm{gr}(X^c) = \mathrm{gr}(Y^c) \times \chi^c(\Delta)$ and Y is the residue space of X corresponding to $\chi^c(\Delta) \subseteq \mathrm{gr}(X^c)$. Conversely, we have the following:

THEOREM 2.2. *Suppose X is DLO finite and H is any subgroup of $\mathrm{gr}(X^c)$. Then the residue space Y of X corresponding to H is DLO finite and X is the group extension of Y by a compact totally disconnected group $\Delta \cong \chi(H)$ as in Theorem 2.1. In particular, if $H = \mathrm{gr}(X^c)$, then $\mathrm{gr}(Y^c) = 1$.*

Proof. By definition Y is the quotient space of X obtained by restricting elements of X to $G(Y) := \{a \in G(X) : \gamma(a) = 1 \text{ for all } \gamma \in H\}$. This is closed in $G(X)$ since each $\gamma \in H$ is continuous. Y is DLO finite by an easy application of Corollary 1.8. Also, since $G(Y)$ is a closed subgroup of $G(X)$, there exists a closed subgroup $\Delta \subseteq G(X)$ such that $G(X) = G(Y) \times \Delta$. The rest of the assertions are clear. $\square$

THEOREM 2.3. *Suppose X_i is DLO finite for all $i \in I$. Then $X = \Sigma_{i \in I} X_i$ is DLO finite and $X^c = \dot{\bigcup}_{i \in I} X_i^c$.*

Proof. $G(X) = \prod_{i \in I} G(X_i)$ is compact totally disconnected with the product topology. Since $\dot{\bigcup}_{i \in I} X_i$ is dense in X and X_i^c is dense in X_i, $\dot{\bigcup}_{i \in I} X_i^c$ is dense in X. $\dot{\bigcup}_{i \in I} X_i^c \subseteq X^c$, so X^c is dense in X. Thus X is DLO finite by Corollary 1.8. Suppose $\sigma \in X^c$. By continuity, σ factors through $\prod_{i \in J} G(X_i)$ for some finite set $J \subseteq I$. Let $Y = \Sigma_{i \in J} X_i$ and $Z = \Sigma_{i \in K} X_i$ where $K = I \setminus J$. Then $X = Y \oplus Z$ so by choice of J, $\sigma \in Y$. Since J is finite there exists $i \in J$ such that $\sigma \in X_i$. Using the continuity of σ once more, this implies $\sigma \in X_i^c$. $\square$

For X DLO finite, we define a relation $\sim_c$ on X^c by declaring $\sigma \sim_c \tau$ to mean that either $\sigma = \tau$ or there exists a 4-element fan $V \subseteq X^c$ with $\sigma, \tau \in V$. Thus if $\mathfrak{U}(X)$ is defined as in Corollary 1.7, then for $\sigma, \tau \in X^c$, $\sigma \sim_c \tau$ holds iff σ, τ lie in the same connected component of Y for some $Y \in \mathfrak{U}(X)$. Thus, by results in [6, 9], $\sim_c$ is an equivalence relation on X^c.

The equivalence classes of X^c with respect to $\sim_c$ are called the *connected components* of X^c. We say X^c is *connected* if it has only one connected component.

THEOREM 2.4. *Every DLO finite space X decomposes uniquely as $X \cong \Sigma_{i \in I} X_i$ where X_i^c is connected for each $i \in I$.*

Proof. If $X = \Sigma_{i \in I} X_i$ is such a decomposition then, by Theorem 2.3, $X^c = \dot{\bigcup}_{i \in I} X_i^c$, so by [**8**, Cor. 7.5(i)], the sets X_i^c, $i \in I$, are the connected components of X^c and X_i is the subspace of X obtained by closing X_i^c topologically. This proves uniqueness.

To prove existence, let $\{C_i: i \in I\}$ be the connected components of X^c. For each $Y \in \mathfrak{U}(X)$, let $I(Y) = \{i \in I: Y \cap C_i \neq \varnothing\}$. Thus $Y = \Sigma_{i \in I(Y)} Y_i$ where $Y_i = Y \cap C_i$. This follows from the decomposition theory in [**6**, **9**]. For $i \in I$, let $\mathfrak{U}(X)_i = \{Y \in \mathfrak{U}(X): Y \subseteq C_i\}$. Note $\mathfrak{U}(X)_i$, ordered by inclusion, is a directed system of finite spaces of orderings. Denote by X_i the direct limit of this system. Observe that the inclusions $Y \subseteq X$ for $Y \in \mathfrak{U}(X)_i$ together with the universal property of X_i yields a morphism $\psi_i^*: X_i \to X$ for each $i \in I$ and hence a morphism $\psi^*: \Sigma_{i \in I} X_i \to X$. On the other hand, for each $Y \in \mathfrak{U}(X)$, $Y = \Sigma_{i \in I(Y)} Y_i$ as above, so the inclusions $Y_i \subseteq X_i$ induce a morphism $Y \to \Sigma_{i \in I(Y)} X_i \to \Sigma_{i \in I} X_i$. By the universal property of X this lifts to a morphism $\varphi^*: X \to \Sigma_{i \in I} X_i$. It is clear that $\psi^* \circ \varphi^* = 1$ and $\varphi^* \circ \psi^* = 1$ so $X \cong \Sigma_{i \in I} X_i$. $\square$

We denote the *stability index* of a space of orderings X by $\mathrm{st}(X)$. Various characterizations of $\mathrm{st}(X)$ are given in [**8**, **11**]. Suppose X is DLO finite and $\mathfrak{U}(X)$ is defined as in Corollary 1.7. If $Y \in \mathfrak{U}(X)$, then $\mathrm{st}(Y) \leq \mathrm{st}(X)$. On the other hand, if p is an anisotropic Pfister form over $G(X)$ and if, for each $Y \in \mathfrak{U}(X)$, there exists a form q_Y over $G(X)$ such that $p \cong_Y 2 \times q_Y$, then, by Theorem 1.3, there exists a form q over $G(X)$ such that $p \cong_X 2 \times q$. It follows from [**8**, Th. 6.2] that $\mathrm{st}(X) = \sup\{\mathrm{st}(Y): Y \in \mathfrak{U}(X)\}$.

THEOREM 2.5. *For X DLO finite and $n \geq 0$, the following are equivalent:*

  (i) $\mathrm{st}(X) \leq n$,

  (ii) $X \in \mathcal{P}_n$ *where $\mathcal{P}_0$ denotes the class of singleton spaces and, inductively, $\mathcal{P}_{k+1}$ consists of all DLO finite spaces Y expressible in the form $Y \cong (\Sigma_{i \in I} Y_i^{(1)}) \oplus (\Sigma_{j \in J} \{\sigma_j\})$ where $Y_i \in \mathcal{P}_k$ for all $i \in I$. (Here, $Y_i^{(1)}$ denotes the group extension of Y_i by $\{\pm 1\}$ and $\{\sigma_j\}$ denotes a singleton space.)*

Proof. (ii) ⇒ (i): If Y is as in (ii) and $k \geq 0$, then $\text{st}(Y) \leq k + 1 \Leftrightarrow$ $\text{st}(Y_i) \leq k$ for all $i \in I$. Thus, by induction, if $X \in \mathcal{P}_n$ then $\text{st}(X) \leq n$.

(i) ⇒ (ii): Suppose X is DLO finite, $\text{st}(X) \leq n$. By Theorems 2.2, 2.4, and induction on n, it is enough to show that if X^c is connected and $|X^c| \neq 1$, then $\text{gr}(X^c) \neq 1$. Since $\text{st}(Y) \leq \text{st}(X) \leq n$ for all $Y \in \mathcal{U}(X)$, it follows from the proof of [**9**, Th. 2.6] that there is a uniform bound on the length of chains

$$(*) \qquad\qquad Y_{\gamma_0} \underset{\neq}{\subseteq} Y_{\gamma_1} \underset{\neq}{\subseteq} \cdots \underset{\neq}{\subseteq} Y_{\gamma_l},$$

where $Y \in \mathcal{U}(X)$, $\gamma_i \neq 1$, $\gamma_i \in \chi^c(G(X))$, and Y_{γ_i} is a non-fan for $i = 0, \dots, l$. Suppose $(*)$ is a chain of this type of maximal length. We *claim* $\gamma_l \in \text{gr}(X^c)$. For let $\sigma \in X^c$. Since X^c is connected, there is a finite subspace $Z \subseteq X^c$ with $Y \subseteq Z$, $\sigma \in Z$, and with σ and Y_{γ_l} lying in the same connected component of Z. Since Y is a subspace, $Y_{\gamma_i} = Z_{\gamma_i} \cap Y$. Thus Z_{γ_i} is a non-fan for $i = 0, \dots, l$ and these sets are linearly ordered by inclusion by [**11**, Th. 1.3]. From this and $(*)$ it follows that $Z_{\gamma_{i-1}} \underset{\neq}{\subseteq} Z_{\gamma_i}$ for $i = 1, \dots, l$. By the maximal choice of l and the structure theory in the finite case, Z_{γ_l} is a connected component in Z. Thus $\sigma \in Z_{\gamma_l}$. In particular, this implies $\sigma \gamma_l \in X^c$. Since $\sigma \in X^c$ is arbitrary, this proves $\gamma_l \in \text{gr}(X^c)$. Thus we have $\text{gr}(X^c) \neq 1$ and the proof is complete. We remark the proof given does not cover the trivial case where Y is a fan for all $Y \in \mathcal{U}(X)$. In this case, if $\sigma, \tau \in X^c$ are arbitrary then $\gamma = \sigma\tau \in \text{gr}(X^c)$ so the result is valid in this case also. $\qquad\square$

THEOREM 2.6. *Suppose X is DLO finite and $n \geq 0$. Then the following are equivalent*:

(i) $\sup\{|E(Y, \sigma)| : Y \in \mathcal{U}(X), \sigma \in Y\} \leq n$. (*Here $E(Y, \sigma)$ is defined as in* [**11**].)

(ii) $X \in \mathcal{Q}_n$ *where $\mathcal{Q}_0$ denotes the class of singleton spaces and, inductively, $\mathcal{Q}_{k+1}$ is the class of all DLO finite spaces Y expressible as $Y \cong \sum_{i \in I} Y_i^{(\alpha_i)}$ where α_i is a cardinal number and $Y_i \in \mathcal{Q}_k$ for all $i \in I$.* (*Here, $Y^{(\alpha)}$ denotes the group extension of Y by the group $\{\pm 1\}^{\alpha}$.*)

Proof. This follows by induction using results in [**11**], but the same sort of proof as in Theorem 2.5. $\qquad\square$

EXAMPLE 2.7. The following illustrate why some finiteness conditions are necessary to get structure results like Theorems 2.5 and 2.6:

(i) $X = \{\sigma_0\} \oplus (\{\sigma_1\} \oplus (\{\sigma_2\} \oplus \cdots)^{(1)})^{(1)}$. This satisfies $X \cong \{\sigma_0\} \oplus X^{(1)}$, so if we attempt to decompose X using Theorems 2.4 and 2.2 successively, we have "infinite descent".

(ii) $X = \cdots ((\{\sigma_0\}^{(1)} \oplus \{\sigma_1\})^{(1)} \oplus \{\sigma_2\})^{(1)} \cdots$. This satisfies $\text{gr}(X^c) = 1$ but at the same time X^c is connected, so both types of decomposition are "blocked" right from the start.

REMARK 2.8. It is not known if DLO finite spaces exist with X^c connected, $|X| \neq 1$, and $\text{gr}(X) = 1$. The following shows that if X is such a space then X^c is uncountable. For suppose X^c is countable. Then there exists a countable sequence

$$(*) \qquad\qquad X_0 \subseteq X_1 \subseteq \cdots$$

where X_i is a finite subspace of X and $X^c = \bigcup_{i \geq 0} X_i$. Thus $X = \varinjlim_{i \geq 0} X_i$. Since X^c is connected we can choose $(*)$ so that $\text{gr}(X_i) \neq 1$ for all $i \geq 0$. Denote by $[X_i]$ the linear span of X_i in $\chi(G(X))$. First assume there exists $n \geq 1$ such that $\text{gr}(X_i) \subseteq [X_{i-1}]$ for all $i \geq n$. Since X_{i-1} is a subspace of X_i this implies $\text{gr}(X_i) \subseteq \text{gr}(X_{i-1})$ for $i \geq n$. Since these groups are finite and $\neq 1$, there exists $\gamma \in \bigcap_{i \geq n} \text{gr}(X_i)$, $\gamma \neq 1$. Thus $\gamma \in \text{gr}(X^c) \subseteq \text{gr}(X)$, so $\text{gr}(X) \neq 1$. Now assume no such n exists. Then, replacing $(*)$ by a suitable subsequence, we can assume $\text{gr}(X_i) \nsubseteq [X_{i-1}]$ for all $i \geq 1$. Thus there exists $a_i \in G(X)$ such that $a_i \in X_{i-1}^{\perp}$ but $a_i \notin \text{gr}(X_i)^{\perp}$. Thus, if $b_{i-1} \in G(X)$ is arbitrary, there exists $b_i \in G(X)$ such that $b_i b_{i-1} \in X_{i-1}^{\perp}$ and $b_i \notin \text{gr}(X_i)^{\perp}$. (Take $b_i = b_{i-1}$ if $b_{i-1} \notin \text{gr}(X_i)^{\perp}$, and $b_i = a_i b_{i-1}$ if $b_{i-1} \in \text{gr}(X_i)^{\perp}$.) Since $G(X) = \varprojlim_{i \geq 0} G(X_i)$, this yields an element $x \in G(X)$ satisfying $x_i \notin \text{gr}(X_i)^{\perp}$ for all $i \geq 0$. Thus, by [10, Th. 6.6], and Corollary 1.2, x and $-x$ are both rigid and $\text{gr}(X) \neq 1$.

3. Topological structure of $X^c \subseteq X$.

For a space of orderings X and $\sigma \in X$ the *local stability index* $\text{st}(X, \sigma)$ is defined as in [11]. Thus, $\text{st}(X) = \sup\{\text{st}(X, \sigma): \sigma \in X\}$.

THEOREM 3.1. *Suppose X is DLO finite and $\sigma \in X^c$. Then the following are equivalent*:
  (i) $\{\sigma\}$ *is open in X^c*,
  (ii) $\{\sigma\}$ *is open in X*,
  (iii) $\text{st}(X, \sigma) < \infty$.
Further, $\text{st}(X, \sigma) = \sup\{\text{st}(Y, \sigma): Y \in \mathfrak{U}(X), \sigma \in Y\}$.

Proof. The equivalence of (i) and (ii) follows from the density of X^c in X. Also, if $\{\sigma\}$ is open in X, then by definition of the topology on X, there exist $a_1, \ldots, a_n \in G(X)$ such that $X(a_1, \ldots, a_n) = \{\sigma\}$. (Recall $X(a_1, \ldots, a_n) := \{\tau \in X: \tau(a_i) = 1 \text{ for all } i = 1, \ldots, n\}$.) Thus, by [11, Th. 3.4], (ii) $\Rightarrow$ (iii). Let $l = \sup\{\text{st}(Y, \sigma): Y \in \mathfrak{U}(X), \sigma \in Y\}$. Clearly $\text{st}(X, \sigma) \geq l$. Now suppose $l < \infty$. Suppose $\sigma \in Y$ where $Y \in \mathfrak{U}(X)$.

Since Y is finite, $\{\sigma\}$ is open in Y so by [**11**, Th. 3.4] there exist elements $a_{1Y}, \ldots, a_{lY} \in G(X)$ such that $Y(a_{1Y}, \ldots, a_{lY}) = \{\sigma\}$. Consider the Pfister form $p_Y = \langle 1, a_{1Y} \rangle \otimes \cdots \otimes \langle 1, a_{lY} \rangle$. By the proof of Theorem 1.3 there exists a form $p = \langle b_1, \ldots, b_s \rangle$ over $G(X)$ (where $s = 2^l$) such that $p \cong_Y p_Y$ for all $Y \in \mathcal{U}(X)$ such that $\sigma \in Y$. Note that $\sigma(p) = 2^l$ whereas $\tau(p) = 0$ for all $\tau \in X^c$, $\tau \neq \sigma$. It follows, in particular, that $2^{s-l} \times p$ and $\langle 1, b_1 \rangle \otimes \cdots \otimes \langle 1, b_s \rangle$ have the same signature at all $\tau \in X^c$ and hence are isometric. Thus, applying [**8**, Lem. 6.3] repeatedly, p is a Pfister form, say $p \cong \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_l \rangle$. Then it is clear that $X(a_1, \ldots, a_l) \cap X^c = \{\sigma\}$ and hence that $X(a_1, \ldots, a_l) = \{\sigma\}$. This proves $\mathrm{st}(X, \sigma) \leq l$ and completes the proof. $\square$

THEOREM 3.2. *Suppose X is DLO finite. Then the following are equivalent:*

(i) $\mathrm{st}(X) < \infty$,

(ii) X^c *with the induced topology is discrete, and X is the Stone-Čech compactification of X^c.*

The proof of Theorem 3.2 uses another result which is of some independent interest:

THEOREM 3.3. *Suppose X is any space of orderings, $|X| > 1$, and f is a form over $G(X)$ satisfying the inequality $\dim(f) > |\sigma(f)| \cdot 2^{\mathrm{st}(X, \sigma) - 1}$ for all $\sigma \in X$. (Here, we observe the convention $0 \cdot \infty = 0$.) Then f is isotropic.*

Note, in particular, Theorem 3.3 asserts that, if $\mathrm{st}(X) = 1$, then every totally indefinite form over $G(X)$ is isotropic. Theorem 3.3 should be viewed as a generalization of this well-known property of SAP spaces.

Proof of Theorem 3.3. By [**9**, Th. 1.4], we can assume X is finite. The proof is by induction on the global stability index $\mathrm{st}(X)$. If $X = X_1 \oplus \cdots \oplus X_s$ and if the result is true for each X_i with $|X_i| > 1$, then it is true for X. Thus, by [**9**, Th. 1.3] we can assume $X = Y^{(1)}$, notation as in Theorem 2.5, and $|Y| > 1$. Thus $f \cong f_1 \oplus f_2 t$ where t is an element of $G(X) \backslash G(Y)$ and f_1, f_2 are forms over $G(Y)$. Scaling f by t if necessary, we can assume $\dim(f_1) \geq \dim(f_2)$. Let $\sigma \in Y$. Note if $e \in \{\pm 1\}$ is suitably chosen, then $|\sigma(f_1) + e\sigma(f_2)| \geq |\sigma(f_1)|$. Extend σ to an element $\tau \in X$ in such a way that $\tau(t) = e$. Thus $|\tau(f)| \geq |\sigma(f_1)|$. Also $\mathrm{st}(X, \tau) = \mathrm{st}(Y, \sigma) + 1$ so

$$2 \cdot \dim(f_1) \geq \dim(f) > |\tau(f)| \cdot 2^{\mathrm{st}(X, \tau) - 1} \geq |\sigma(f_1)| \cdot 2^{\mathrm{st}(Y, \sigma)}$$

so

$$\dim(f_1) > |\sigma(f_1)| \cdot 2^{\mathrm{st}(Y,\sigma)-1}$$

for all $\sigma \in Y$. Finally, st $(Y) = \mathrm{st}(X) - 1$ so by induction on the stability index, f_1 is Y-isotropic, so f is X-isotropic. □

Proof of Theorem 3.2. Assume (i), say $\mathrm{st}(X) = 2^k < \infty$. Thus, by Theorem 3.1, X^c is discrete in X. To show X is the Stone-Čech compactification of X^c, we show that every function $\varphi \colon X^c \to \{\pm 1\}$ extends to a continuous function $\psi \colon X \to \{\pm 1\}$. Consider $2^k \varphi \colon X^c \to \mathbf{Z}$. By [8, Ths. 5.5 and 6.4], for all $Y \in \mathfrak{A}(X)$, there exists an anisotropic form f_Y over $G(Y)$ representing $2^k \varphi$. Thus, by Theorem 3.3, there exists $\sigma \in Y$ such that

$$\dim(f_Y) \le |\sigma(f_Y)| \cdot 2^{\mathrm{st}(Y)-1} \le 2^k 2^{k-1} = 2^{2k-1},$$

so by Theorem 1.3, there exists a form f over $G(X)$ such that $\sigma(f) = 2^k \varphi(\sigma)$ for all $\sigma \in X^c$. Define $\psi \colon X \to \{\pm 1\}$ by $\psi(\sigma) = 2^{-k}\sigma(f)$ for all $\sigma \in X$. This is continuous and extends φ.

Suppose (ii) holds but (i) fails. Thus

$$\mathrm{st}(X) = \sup\{\mathrm{st}(Y) \colon Y \in \mathfrak{A}(X)\}$$
$$= \sup\{\mathrm{st}(Y,\sigma) \colon Y \in \mathfrak{A}(X), \sigma \in Y\} = \infty$$

so there are arbitrarily large finite fans in X^c, but, by Theorem 3.1, $\mathrm{st}(X,\sigma) < \infty$ holds for all $\sigma \in X^c$. These facts can be used to construct a sequence of fans $V_1, V_2, \ldots$ in X^c of increasing size and mutually disjoint. (For if $V_1, \ldots, V_{k-1}$ are already constructed just let $m = \sup\{\mathrm{st}(X,\sigma) \colon \sigma \in V_1 \cup \cdots \cup V_{k-1}\}$ and pick V_k to be any fan in X^c with $|V_k| > 2^m$. then $V_k \cap V_i = \varnothing$ for $i = 1, \ldots, k-1$.) Now pick $\sigma_i \in V_i$ for each $i \ge 1$, and define $\varphi \colon X^c \to \{0,1\}$ by $\varphi(\sigma_i) = 1$ for $i = 1, 2, \ldots$ and $\varphi(\sigma) = 0$ if $\sigma \notin \{\sigma_1, \sigma_2, \ldots\}$. This extends to a continuous $\psi \colon X \to \{0,1\}$ so by [8, Lem. 5.4] there exists $k \ge 0$ and a form f over $G(X)$ such that $\sigma(f) = 2^k \cdot \psi(\sigma)$ for all $\sigma \in X$. This contradicts [8, Th. 5.5] since if i is sufficiently large, $|V_i| > 2^k$ so $\sum_{\sigma \in V_i} \sigma(f) = 2^k \not\equiv 0 \bmod |V_i|$. □

COROLLARY 3.4. *Suppose X is DLO finite, $\mathrm{st}(X) < \infty$, and $\varphi \colon X^c \to \mathbf{Z}$ is any function. Then the following are equivalent*:

(i) *There exists a form f over $G(X)$ such that $\sigma(f) = \varphi(\sigma)$ for all $\sigma \in X^c$.*

(ii) *$\varphi(X^c)$ is a finite set and $\sum_{\sigma \in V} \varphi(\sigma) \equiv 0 \bmod |V|$ holds for all finite fans $V \subseteq X^c$.*

Proof. (i) $\Rightarrow$ (ii) is true by [**8**, Th. 5.5]. (ii) $\Rightarrow$ (i) follows by an argument similar to the first half of the proof of Theorem 3.2. $\square$

EXAMPLE 3.5. We conclude this section with an example of how Corollary 3.4 can fail if $\mathrm{st}(X) = \infty$. Define $X_1 = \{\sigma\}$ and for each $i \geq 1$ define $X_{i+1} = (X_i \oplus X_i)^{(1)}$, notation as in Theorem 2.5. Thus $X = \sum_{i=1}^{\infty} X_i$ is DLO finite by Theorem 2.3. We *claim* there exists an anisotropic form f_i over $G(X_i)$ such that $\dim(f_i) = 2^i$ and $\sigma(f_i) = \pm 2$ for all $\sigma \in X_i$. Define $f_1 = \langle 1, 1 \rangle$. Suppose f_i is defined. Let h_i be the hyperbolic form over $G(X_i)$ with dimension equal to $2^i = \dim(f_i)$. Define forms f_i', f_i'' over $G(X_i \oplus X_i)$ by $f_i' = (f_i, h_i)$ and $f_i'' = (h_i, f_i)$. Finally, define $f_{i+1} = f_i' \oplus f_i'' t_i$, where t_i is an element of $G(X_{i+1}) \setminus G(X_i \oplus X_i)$. This proves the claim. Since $X^c = \bigcup_{i=1}^{\infty} X_i$ by Theorem 2.3, the function $\varphi \colon X^c \to \mathbf{Z}$, defined by $\varphi(\sigma_i) = \sigma_i(f_i)$ if $\sigma_i \in X_i$, is well defined. Since $\varphi(\sigma) = \pm 2$ and since each non-trivial fan in X^c lies in some X_i, φ satisfies condition (ii) of Corollary 3.4. However, since f_i is anisotropic of dimension 2^i, there cannot be a form f over $G(X)$ satisfying (i) of Corollary 3.4. Note, by Theorem 3.1, X^c is discrete in X, so φ is even continuous.

4. Direct limits and ultrasums. Recall that an *ultrafilter* $\mathfrak{U}$ on a set I is a collection of subsets of I satisfying: (i) $\varnothing \notin \mathfrak{U}$ (ii) if $J, K \in \mathfrak{U}$ then $J \cap K \in \mathfrak{U}$, and (iii) if $J \notin \mathfrak{U}$, then $I \setminus J \in \mathfrak{U}$.

Let $\{X_i \colon i \in I\}$ be any collection of spaces of orderings, let $X = \sum_{i \in I} X_i$, and if $J \subseteq I$, let $X_J = \sum_{i \in J} X_i$. Thus, if $K = I \setminus J$, then $X = X_J \oplus X_K$ so, in particular, X_J is a subspace of X. Thus, if $\mathfrak{U}$ is any ultrafilter on I, then $X_{\mathfrak{U}} := \bigcap_{J \in \mathfrak{U}} X_J$ is a subspace of X. We refer to $X_{\mathfrak{U}}$ as the *ultrasum* of the collection $\{X_i \colon i \in I\}$ determined by $\mathfrak{U}$. Note that $G(X_{\mathfrak{U}}) = G(X)/H_{\mathfrak{U}}$, where $H_{\mathfrak{U}}$ is the subgroup consisting of all $x \in G(X)$ satisfying $\{i \in I \colon x_i = 1\} \in \mathfrak{U}$, and $W(X_{\mathfrak{U}}) = W(X)/J_{\mathfrak{U}}$, where $J_{\mathfrak{U}}$ is the ideal consisting of all $f \in W(X)$ satisfying $\{i \in I \colon f_i \sim 0\} \in \mathfrak{U}$.

REMARK 4.1. Suppose we denote by $\beta(I)$ the Stone-Čech compactification of the discrete set I. Thus the clopen subsets of $\beta(I)$ are in 1-1 correspondence with the subsets of I and the points of $\beta(I)$ are in 1-1 correspondence with the ultrafilters on I. One can verify that the Witt rings $W(X_J)$, $J \subseteq I$, form a *sheaf of Witt rings* over $\beta(I)$ in the terminology of [**10**, Ch. 8]. The *stalks* of this sheaf are the Witt rings $W(X_{\mathfrak{U}})$, $\mathfrak{U} \in \beta(I)$.

LEMMA 4.2. *Suppose K is the ultraproduct of the Pythagorean fields $\{K_i \colon i \in I\}$ determined by some ultrafilter $\mathfrak{U}$. Then K is a Pythagorean field and*

X_K is naturally identified with the ultrasum $(\Sigma_{i \in I} X_{K_i})_{\mathfrak{A}}$. (Here, X_K denotes the space of orderings of the field K).

Proof. By definition, K is the ring $(\prod_{i \in I} K_i)/M_{\mathfrak{A}}$ where $M_{\mathfrak{A}}$ is the ideal consisting of all $a \in \prod_{i \in I} K_i$ such that $\{i \in I: a_i = 0\} \in \mathfrak{A}$. The check that K is a Pythagorean field is straightforward. For the second assertion, one simply verifies that the natural map from $\dot{K}/\dot{K}^2$ into $(\prod_{i \in I} \dot{K}_i/\dot{K}_i^2)/H_{\mathfrak{A}}$ is a group isomorphism which preserves value sets of binary forms. The result then follows from [**8**, Lem. 4.1]. More explicitly, it follows from

LEMMA 4.3. *Suppose* X, Y *are spaces of orderings. Then a group homomorphism* $\alpha: G(Y) \rightarrow G(X)$ *induces a morphism* $\alpha^*: X \rightarrow Y$ *iff* $\alpha(-1) = -1$ *and* $\alpha(D_Y\langle 1, a \rangle) \subseteq D_X\langle 1, \alpha(a) \rangle$ *for all* $a \in G(Y)$.

Proof. One implication is clear. The other follows from [**8**, Lem. 4.1]. □

In case I is a directed set, ultrafilters *compatible with the ordering* on I are of special interest. These are just ultrafilters $\mathfrak{A}$ on I with the property that for all $i \in I$, the set $I(i) := \{j \in I: j \geq i\}$ belongs to $\mathfrak{A}$. Such ultrafilters exist by an easy application of Zorn's Lemma.

LEMMA 4.4. *Suppose* $(X_i: i \in I)$ *is a directed system of finite spaces of orderings and* $\mathfrak{A}$ *is an ultrafilter on* I *compatible with the ordering. Then there are natural morphisms* $\alpha^*: (\Sigma_{i \in I} X_i)_{\mathfrak{A}} \rightarrow \varinjlim_{i \in I} X_i$ *and* $\beta^*: \varinjlim_{i \in I} X_i \rightarrow (\Sigma_{i \in I} X_i)_{\mathfrak{A}}$ *such that* $\alpha^* \circ \beta^* = $ *the identity.*

Proof. Let $X = \Sigma_{i \in I} X_i$. α^* is just the composite of the inclusion $X_{\mathfrak{A}} \subseteq X$ followed by the projection $X \rightarrow \varinjlim_{i \in I} X_i$. The definition of β^* is a bit more interesting. First note that, by the universal property of the direct limit, it is enough to define morphisms $\beta_i^*: X_i \rightarrow X_{\mathfrak{A}}$ such that, if $i \leq j$, then $\beta_i^* = \beta_j^* \circ \varphi_{ij}^*$ where the $\varphi_{ij}^*: X_i \rightarrow X_j$ are the morphisms defining the directed system. Let $x \in G(X)$. For fixed $y \in G(X_i)$ define $I(i, y) = \{j \in I: j \geq i$ and $\varphi_{ij}(x_j) = y\}$. Since $G(X_i)$ is finite, $I(i) \in \mathfrak{A}$, and $I(i)$ is the disjoint union of the sets $I(i, y), y \in G(X_i)$, it follows that there exists a unique $\beta_i(x) \in G(X_i)$ such that $I(i, \beta_i(x)) \in \mathfrak{A}$. Observe that $\beta_i: G(X) \rightarrow G(X_i)$ is a group homomorphism, $\beta_i(-1) = -1$, and if $y \in D\langle 1, x \rangle$, then $\beta_i(y) \in D\langle 1, \beta_i(x) \rangle$. Thus β_i induces a morphism $\beta_i^*: X_i \rightarrow X$ by Lemma 4.3. Note that the kernel of β_i contains all x such that $\{j \in I: x_j = 1\} \in \mathfrak{A}$ so $\beta_i^*: X_i \rightarrow X_{\mathfrak{A}}$. Finally, if $j \geq i$, then $\beta_i = \varphi_{ij} \circ \beta_j$

so $\beta_i^* = \beta_j^* \circ \varphi_{ij}^*$. This completes the definition of β^*. Observe that if $x \in \varprojlim_{i \in I} G(X_i)$, then $\beta_j(x) = x_j$. Thus $\alpha^* \circ \beta^* = 1$. $\square$

THEOREM 4.5. *Suppose X is any DLO finite space. Then there exists a Pythagorean field K and morphisms α^*: $X_K \to X$, β^*: $X \to X_K$ such that $\alpha^* \circ \beta^* = 1$.*

Proof. Suppose $X = \varinjlim_{i \in I} X_i$ where each X_i is finite. By [1] or [4] there exists a Pythagorean field K_i with $X_{K_i} \cong X_i$. The result follows by applying Lemmas 4.2 and 4.4. $\square$

To analyze further the conclusion of Theorem 4.5 we note the following:

LEMMA 4.6 *Suppose X, Y are spaces of orderings and α^*: $X \to Y$ and β^*: $Y \to X$ are morphisms satisfying $\alpha^* \circ \beta^* = 1$. Then*

(i) *α^* identifies Y with a quotient space of X, and for any form f over $G(Y)$, if $\alpha(f)$ is isotropic then f is isotropic, and*

(ii) *β^* identifies Y with a subspace of X.*

Proof. The hypothesis implies α^* is onto and β^* is 1-1. It also implies $\beta \circ \alpha = 1$ so α is 1-1 and β is onto. The assertion that Y is a quotient space of X is now clear. Also, if $\alpha(f)$ is isotropic, then so is $\beta(\alpha(f)) = f$, so (i) is clear. To complete the proof of (ii), it is necessary to check that $\beta^*(Y)$ is a subspace of X. This amounts to showing that if $\sigma \in X$ satisfies $\sigma(a) = 1$ for all $a \in \ker(\beta)$, then $\sigma \in \beta^*(Y)$. Let $b \in G(X)$. Then $\beta(\alpha(\beta(b))) = \beta(b)$ so $\alpha(\beta(b)) \cdot b \in \ker(\beta)$. Thus $\sigma(\alpha(\beta(b)) \cdot b) = 1$, i.e.,

$$\sigma(b) = \sigma(\alpha(\beta(b))) = (\beta^*(\alpha^*(\sigma)))(b).$$

Since $b \in G(X)$ is arbitrary, this implies $\sigma = \beta^*(\alpha^*(\sigma))$, so $\sigma \in \beta^*(Y)$. $\square$

COROLLARY 4.7. *Every DLO finite space of orderings is isomorphic to a subspace of the space of orderings of a suitably chosen Pythagorean field.*

Proof. Immediate from Theorem 4.5 and Lemma 4.6(ii). $\square$

COROLLARY 4.8. *Every space of orderings X is isomorphic to a quotient space of the space of orderings of a suitably chosen Pythagorean field K. Further, this representation of X can be achieved in such a way that for a form f over $G(X)$, f is X-isotropic iff f is X_K-isotropic.*

Proof. Immediate from Remark 1.4, Theorem 4.5, and Lemma 4.6(i). $\square$

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