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STIEFEL'S THEOREM AND TORAL ACTIONS

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The second Stiefel-Whitney class of an orientable manifold admitting an effective codimension three toral action is shown to be Poincaré dual to the fixed point sets of circle subgroups and cyclic subgroups of even order.

0. Introduction. Stiefel [5] proved that the second Stiefel-Whitney class of an orientable 3-manifold is zero. A simple generalization of this theorem is that the second Stiefel-Whitney class of an orientable $(n + 3)$-manifold, M, vanishes if M admits a free action of T^n, the n-torus. If T^n acts almost freely and effectively on an orientable manifold M^{n+3}, that is, if all isotropy subgroups are finite, then $w_2(M)$ need not vanish; in fact $w_2(M)$ is Poincaré dual to the fixed point sets of cyclic subgroups of even order.

In this note, we consider an arbitrary effective action of T^n on an orientable manifold M^{n+3} and show that the Poincaré dual of $w_2(M)$ is represented by the fixed point sets of circle subgroups and cyclic subgroups of even order.

In §1 we establish some notation and give a precise statement of the theorem. In §2 we reduce the proof of the theorem to the case of compact manifolds having only cyclic or circle isotropy subgroups. In §3 we study the cases of only cyclic isotropy subgroups or only circle isotropy subgroups and in §4 we prove the theorem by reducing to the two special cases of §3.

1. Statement of Theorem. If M is a manifold with a smooth action of T^n and H is any subgroup of T^n then $E(M, H) = \{x \in M \mid T^n_x = H\}$ where T^n_x is the isotropy subgroup at x; $E(M, H)$ is open in $F(M, H) = \{x \in M \mid h(x) = x \text{ for all } h \in H\}$, the fixed point set of H [3]. If X is any component of $F(M, H)$ then X is a closed submanifold of M and $\overline{X \cap E(M, H)}$ is either empty or dense in the T^n/H space X by [3]; hence $\overline{E(M, H)}$ is a union of disjoint closed submanifolds of M.

If A^a is a closed submanifold of a manifold B^b, the Poincaré dual of A, $D(A) \in H^{b-a}(B; Z_2) \simeq \operatorname{Hom}(H_{b-a}(B; Z_2); Z_2)$, is defined by $\langle D(A), Z \rangle =$ number of points in $A \pitchfork Z \bmod 2$ where $\pitchfork$ indicates the intersection of A with the cycle Z in general position [4].

If T^n acts smoothly and effectively on an orientable manifold M^{n+3}, then for any cyclic subgroup $H \subset T^n$, $\overline{E(M, H)}$ is a codimension two submanifold since H must act linearly, effectively, and in an orientation preserving manner on the three dimensional slice. If H is isomorphic to S^1, $\overline{E(M, H)}$ will in general have components of codimension two and four; let $\overline{E_2(M, H)}$ denote the components of codimension two. Finally, for T^n acting smoothly and effectively on an orientable manifold M^{n+3}, we define $A(M) \in H^2(M; Z_2)$ by

$$A(M) = \sum_{H \simeq S^1} D(\overline{E_2(M, H)}) + \sum_{H \simeq Z_{2m}} D(\overline{E(M, H)}).$$

THEOREM. *If T^n acts smoothly and effectively on an orientable $(n+3)$-manifold, M^{n+3}, then $A(M) = w_2(M)$, the second Stiefel-Whitney class of M.*

The remainder of the paper is devoted to the proof of this theorem.

2. First reduction.

LEMMA 1. *It is suffficient to prove the theorem for compact manifolds without boundary.*

Proof. We wish to prove that $A(M) = w_2(M)$ for an arbitrary orientable $(n+3)$-manifold M with effective smooth T^n action. If $A(M) - w_2(M) \neq 0 \in H^2(M; Z_2)$ then there is a singular cycle $Z \in H_2(M, Z_2)$ with $\langle A(M) - w_2(M), Z\rangle \neq 0$. But any such cycle is contained in a compact, invariant, submanifold with boundary X that is $Z = i_* Z'$ for $Z' \in H_2(X; Z_2)$, $i: X \hookrightarrow M$. Hence

$$\begin{aligned}\langle A(M) - w_2(M), Z\rangle &= \langle A(M) - w_2(M), i_* Z'\rangle \\ &= \langle i^* A(M) - i^* w_2(M), Z'\rangle = \langle A(X) - w_2(X), Z'\rangle\end{aligned}$$

and so it is sufficient to prove the theorem for compact manifolds, possibly with boundary. Finally, let $Y = X \cup_{\partial X} X$ be the double of X and $j: X \hookrightarrow Y$. Then

$$\begin{aligned}\langle w_2(Y) - A(Y), j_* Z'\rangle &= \langle j^* A(Y) - j^* w_2(Y), Z'\rangle \\ &= \langle A(X) - w_2(X), Z'\rangle\end{aligned}$$

so it is sufficient to prove the theorem for Y, that is, for compact manifolds without boundary. □

If T^n acts effectively on the orientable manifold M^{n+3}, then T_x^n, the isotropy subgroup at x, can only be isomorphic to one of $\{e\}$, Z_m, S^1, $S^1 \times Z_m$, T^2 or T^3 because the isotropy subgroup must act linearly, effectively, and in an orientation preserving manner on the slice at x.

LEMMA 2. *It is sufficient to prove the theorem for compact manifold M without boundary, such that $E(M, H) = \phi$ for H isomorphic to T^3, T^2 or $S^1 \times Z_m$ and such that $E(M, H) = E_2(M, H)$ for H isomorphic to S^1.*

Proof. Let $H_1, \ldots, H_r$ be the isotropy subgroups isomorphic to T^3, T^2 or $S^1 \times Z_m$ and suppose that H_1 is isomorphic to T^3. Then H_1 is a maximal isotropy subgroup and hence $E(M, h_1) = F(M, H_1)$ is a closed submanifold of M of dimension $n - 3$ since T^3 must act effectively on fibre of the normal bundle of $F(M, H_1)$ in M. Let U be a closed, invariant tubular neighborhood of $E(M, H_1)$, ∂U the boundary of U. Then we have $H^2(M, M - U; Z_2) \simeq H^2(U, \partial U; Z_2)$ by excision and $H^2(U, \partial U; Z_2) \simeq H_{n+1}(U; Z_2) \simeq H_{n+1}(F(M, H_1); Z_2) = 0$ by Poincaré duality, and hence we have in the exact sequence

$$H^2(M, M - U; Z_2) \to H^2(M; Z_2) \overset{i_*}{\to} H^2(M - U; Z_2)$$

that i_* is injective and hence it is sufficient to prove $i^*(A(M) - w_2(M)) = 0$, that is $A(M - U) = w_2(M - U)$.

By doubling $M - U$ as in Lemma 1 we see that it is sufficient to consider compact manifolds without boundary having isotropy subgroups $H_2, \ldots, H_r$. Repeating this argument a finite number of times removes all isotropy subgroups isomorphic to T^3. If H_s is isomorphic to T^2 and if $E(M, H) = \phi$ for H isomorphic to T^3 then $E(M, H_s) = F(M, H_s)$ is a closed invariant submanifold of codimension four and hence $H_{n+1}(F(M, H_s); Z_2) = 0$ as before and we may repeat the argument. Continuing in this manner, we remove next $F(M, H)$ for H isomorphic to $S^1 \times Z_m$ since codimension $F(M, H)$ is four in this case and finally we remove the components of $F(M, H)$, H isomorphic to S^1, whose codimension is four. □

3. Two special cases.

PROPOSITION 3. *Let T^n act effectively and smoothly on the closed orientable manifold M^{n+3} with all isotropy subgroups finite. Then $w_2(M) = A(M)$.*

LEMMA 4. *With the same hypothesis of Proposition* 3, M^{n+3}/T^n *is a closed orientable* 3-*manifold.*

Proof. Let π: $M^{n+3} \to M^{n+3}/T^n$ denote the orbit map. To give a chart (U_i, ϕ_i) in M^{n+3}/T^n it is sufficient to give a smooth invariant map $\bar{\phi}_i$: $\pi^{-1}(U_i) \to \mathbf{R}^3$. If $x \in M$ with $T^n_x = e$ then the slice at x, S_x is diffeomorphic to $\mathbf{R}^3$ [1], and we take $U = T^n \times S$, $\bar{\phi}$: $T^n \times S \to S \simeq \mathbf{R}^3$. If $T^n_x = Z_m$ then $S_x \simeq C \times \mathbf{R}$ with $g \in Z_m$ acting linearly on $\mathbf{C} \times \mathbf{R}$ via $g(z, t) = (\xi z, t)$ for ξ an mth root of unity. Define $\bar{\Phi}$: $T^n \times_{Z_m} S_x \to \mathbf{C} \times \mathbf{R}$ via $\bar{\phi}(X, z, t) = (z^m, t)$.

To prove that M/T^n is orientable we note that T^n acts freely on $M - \bigcup_{H \neq e} F(M, H)$ so that $(M - \bigcup_{H \neq e} F(M, H))/T^n$ is orientable and $\bigcup_{H \neq e} F(M, H)/T^n$ consists of isolated curves in M/T^n so M/T^n is orientable also. □

Proof of Proposition 3. Let $x \in H_2(M; Z_2)$ and let x be represented by a submanifold $Q^2 \subset M$ [6]. We must show $\langle w_2(M), Q\rangle = \langle A(M), Q\rangle$. We may assume that Q intersects each fixed point set $F(M, H)$ transversally, that $Q \cap \bigcup_{H \neq e} F(M; H) = \{p_1, \ldots, p_r\}$, and that in a neighborhood of an intersection point $p_i \in Q \cap F(M, H_i)$, Q coincides with a fiber of a tubular neighborhood. More precisely, there is a slice at p_i, $S_{p_i} \simeq C \times \mathbf{R}$ and a neighborhood U_{p_i} of p_i in Q with $U_{p_i} \subset \mathbf{C} \times 0$. To compute $\langle w_2(M), Q\rangle$ we split $T(M)/Q$ as $\theta^{n+1} \oplus \eta^2$ where θ^{n+1} is a trivial bundle and then $\langle w_2(M), Q\rangle = \langle w_2(\eta^2), Q\rangle = \langle \chi(\eta^2), Q\rangle =$ number of zeroes of a generic section of η^2 mod 2, where χ denotes the Euler class. First note that we have a splitting $T(M) = \theta^n \oplus \xi^3$ where θ^n, the bundle of tangents to the orbits, is trivial since all isotropy subgroups are finite. We must now split $\xi^3/Q = \eta^2 \oplus \theta^1$. We have $d\pi$: $T(M) \to T(M/T^n)$ is an epimorphism with kernel θ^n off the fixed point sets; therefore $d\pi$: $\xi^3 \to T(M/T^n)$ is an isomorphism off the fixed point sets, and at a fixed point x of H, $d\pi_x$ has rank 1. Since every orientable 3-manifold is parallelizable by Stiefel's theorem [5], we can find vector fields X_1, X_2, X_3 which are linearly independent at every point of M/T^n. Around each point p_i we can choose a slice $S_{p_i} \simeq \mathbf{C} \times \mathbf{R}$ with coordinate functions (z, t) such that $\pi(S_{p_i})$ is a coordinate chart of M/T^n at $\pi(p_i)$ with coordinate functions (w, t) where $\pi(z, t) = (z^{m_i}, t)$. So we can consider in $\pi(S_{p_i})$ the vector fields $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial t$ where $w = (x, y)$. Since, up to permutation of the indices, the frame $X_1 | \pi(S_{p_i})$, $X_2 | \pi(S_{p_i})$, $X_3 | \pi(S_{p_i})$ is homotopic to the frame $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial t$ we can modify the vector fields X_1, X_2, X_3 to $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ so that in the neighborhood $\pi(S_{p_i})$

we have $\tilde{X}_i = \partial/\partial x$, $\tilde{X}_2 = \partial/\partial y$, $\tilde{X}_3 = \partial/\partial t$. Note that the vector field Y_1 on S_{p_i} given by $Y_1 = \partial/\partial t$ satisfies $d\pi(y_1) = \tilde{X}_3$. Then on $Q - P_i$ we have a well defined nonzero vector field Y_2 that corresponds via $d\pi$ to the vector field $\tilde{X}_3$, since $d\pi$ is an isomorphism off the fixed point set. Restricting the vector field Y_1 to U_i we have that $Y_1 = Y_2$ in $U_i - \{P_i\}$. So Y_2 extends to a nonzero vector field Y in Q. Let us use this vector field to split $\xi^3|_Q \simeq \eta^2 \oplus \theta$. Now $Z = d\pi^{-1}(\tilde{X}_1)$ is a section of $\eta^2 | Q - \{P_i\}$. Denoting by χ_i the index of Z at P_i we have that $\chi(\eta^2) \bmod 2 = \sum_{i=1}^r \chi_i \bmod 2$. The map π restricted to U_i is given by $\pi(z) = z^{m_i}$ and $d\pi(Z) = \partial/\partial x$ for $Z | U_i - p_i$. Writing this in a matrix form we have

$$m_i r^{m_i} \begin{pmatrix} \cos(m_i - 1)\theta & -\sin(m_i - 1)\theta \\ \sin(m_i - 1)\theta & \cos(m_i - 1)\theta \end{pmatrix} \begin{pmatrix} z_1^i \\ z_2^i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $z = re^{i\theta}$ and $Z = (z_1^i, z_2^i)$ so in U_i the vector Z can be written in the form $Z = (1/m_i r^{m_i})(\cos(m_i - 1)\theta, -\sin(m_i - 1)\theta)$ which shows that the degree of Z at p_i is $1 - m_i$. Then $\chi(\eta^2) \bmod 2$ is the number of p_i's mod 2 such that m_i is even. □

LEMMA 5. *Let π: $E^{n+3} \to B^{n+1}$ be a smooth T^n vector bundle ν such that*

(i) *$F(E^{n+3}, H) = E(E^{n+3}, H) = B^{n+1}$ for some $H \simeq S^1$,*

(ii) *E is an orientable manifold,*

(iii) *T^n acts effectively on E^{n+1}.*

Then $w_2(E^{n+3}) = A(E^{n+3}) = D(B^{n+1})$.

Proof. Since H acts effectively on the fibers of E by (iii), $T_x^n = e$ for $x \notin B$ and hence $A(E^{n+3}) = D(B^{n+1})$. We have $T(E) = \pi^*(T(E) | B) = \pi^*(T(B) \oplus \nu)$. Since $H \simeq S^1$ acts effectively on E, E has a complex structure and is hence orientable and thus B is orientable and $w_2(E) = \pi^*(w_2(B) + w_2(\nu))$. Since $T^n/H \simeq T^{n-1}$ acts freely on B, B/T^{n-1} is an orientable 2-manifold and thus $w_2(B/T^{n-1}) = 0$. From the fibration p: $B \to B/T^{n-1}$ we have that $T(B) = p^*T(B | T^{n-1}) \oplus T_F$ where T_F, the tangent bundle along the fibers is a trivial bundle [**1**] and hence $w_2(B) = p^*w_2(B | T^{n-1}) = 0$. Thus $w_2(E) = \pi^* w_2(\nu)$. Now any $z \in h_2(E; Z_2)$ can be written as $z = i_* z'$, $z' \in H_2(B; Z_2)$ and z' can be represented by a 2-manifold $Q \subset B$. Then

$$\langle A(E), z \rangle = \langle D(B), z \rangle = Q \pitchfork B \bmod 2$$

and

$$\begin{aligned}\langle w_2(E), z\rangle &= \langle \pi^* w_2(\nu), i_* z'\rangle = \langle w_2(\nu), \pi_* i_* z'\rangle \\ &= \langle w_2(\nu), z'\rangle = \langle w_2(\nu \mid Q), Q\rangle = \langle \chi(\nu \mid Q), Q\rangle \\ &= \text{number of zeroes mod 2 of a generic section } s,\end{aligned}$$

where χ denotes the Euler class. But this generic section can be used to put Q in general position with respect to B and the number of points in the intersection $s(Q) \cap B$ is just the number of zeroes of the section. □

4. Proof of Theorem.

DEFINITION. A T^n manifold M^{n+3} is said to be a nice T^n manifold if

(i) M is closed and orientable,

(ii) every isotropy subgroup is cyclic or isomorphic to S^1,

(iii) for H isomorphic to S^1, such that $E(M, H) \neq \phi$, $E(M, H)$ has codimension two,

(iv) for every component F of $E(M, H)$, H isomorphic to S^1, $w_2(\nu(F, M)) \neq 0$ where $\nu(F, M)$ is the normal bundle of F in M.

LEMMA 6. *The theorem is true for nice T^n manifolds.*

Proof. Let E be a tubular neighborhood of $\cup_{H \simeq S^1} E(M, H)$ with boundary ∂E. Consider the cohomology Mayer-Vietoris sequence for $(M, \bar{E}, M - E)$ with Z_2 coefficients

$$\to H^1(\bar{E}) \oplus H^1(M - E) \xrightarrow{i_1^* + i_2^*} H^1(\partial E) \xrightarrow{\delta} H^2(M)$$

$$\xrightarrow{j_1^* \oplus j_2^*} H^2(E) \oplus H^2(M - E).$$

By Proposition 3, $j^* w_2(M) = w_2(M - E) = A(M - E) = j_2^* A(M)$, and by Lemma 5 $j_1^* w_2(M) = w_2(\bar{E}) = A(\bar{E}) = j_1^* A(M)$, hence $w_2(M) - A(M) \in$ image δ. To prove the lemma it suffices to show i_1^*: $H^1(E) \to H^1(\partial E)$ is onto and hence that $\delta = 0$. To prove that i_1^* is onto, we consider the Gysin sequence for one component π: $E_1 \to F$ of the vector bundle $E \to \cup_{H \simeq S^1} E(M, H)$

$$\to H^1(E_1) \xrightarrow{i_1^*} H^1(\partial E) \xrightarrow{\beta} H^0(E_1) \xrightarrow{\alpha} H^2(E_1)$$

and note that $H^0(E_1) \simeq Z_2$ since E_1 is connected and that $\alpha(1) = \pi^* w_2(\nu(F, E_1))$ [2], and $\pi^* w_2(\nu(F, E_1)) \neq 0$ by condition (iv). Hence α is

1-1, β is zero and $i_i^* \mid H^1(E_1)$ is onto. Taking direct sums over the components of E yields $i_1^*\colon H^1(E) \to H^1(\partial E)$ is onto. □

To complete the theorem we need one final reduction lemma.

LEMMA 7. *It is sufficient to prove the theorem for nice actions of T^n.*

Proof. We shall show that for any smooth effective action of T^n on an orientable manifold M^{n+3} and any $x \in H_2(M; Z_2)$ there exists a nice action of T^n on a manifold $\tilde{M}^{n+3}$ and an $\tilde{x} \in H_2(\tilde{M}, Z_2)$ such that $\langle w_2(M), x\rangle = \langle w_2(\tilde{M}), \tilde{x}\rangle$ and $\langle A(M), x\rangle = \langle A(\tilde{M}), \tilde{x}\rangle$. Clearly that will prove Lemma 7.

We may assume, by Lemma 2, that the action of T^n on M^{n+3} satisfies the niceness conditions (i), (ii) and (iii). Let x be represented by a closed submanifold $Q^2 \subset M^{n+3}$ which is transverse to $F(M, H)$ for every isotropy subgroup H isomorphic to S^1. To construct the manifold $\tilde{M}$ we will first choose an invariant neighborhood U of Q^2 and then construct $\tilde{M}$ so that U is contained in $\tilde{M}$. The class $\tilde{x}$ will then be represented by $Q \subset U \subset \tilde{M}$. Clearly then

$$\langle A(M), i_*Q\rangle = \langle i^*A(M), Q\rangle = \langle A(U), Q\rangle = \langle j^*A(\tilde{M}), Q\rangle$$

and

$$\begin{aligned}\langle w_2(M), i_*Q\rangle &= \langle i^*w_2(M), Q\rangle = \langle w_2(U), Q\rangle \\ &= \langle j^*w_2(\tilde{M}), Q\rangle = \langle w_2(\tilde{M}), j_*Q\rangle\end{aligned}$$

where $i\colon U \hookrightarrow M, j\colon U \hookrightarrow \tilde{M}$ are inclusions.

Since Q is transverse to $\bigcup_{H\simeq S^1} F(M, H)$ the intersection is finite, say $\{P_1,\ldots,P_r\}$. Also for any isotropy subgroup $H \simeq S^1$, $F(M, H)$ is an $(n+1)$-dimensional oriented closed manifold with free action of $T^n/H \simeq T^{n-1}$, hence $F(M, H)/T^n$ is a 2-manifold and $\pi\colon \bigcup_{H\simeq S^1} F(M, H) \to \bigcup_{H\simeq S^1} F(M, H)/T^n$ is a union of principal T^{n-1} bundles. Choose neighborhoods V_i of $\pi(P_i)$ in $F(M, H)/T^n$ with V_i diffeomorphic to the open disc $\mathring{D}^2$. Then $\pi^{-1}(V_i) \simeq T^n/H \times V_i$. Let $C = \bigcup_{H\simeq S^1} F(M, H) - \bigcup_{i=1}^r \pi^{-1}(V_i)$. Note that C is compact and invariant and hence $M - C$ is an open invariant neighborhood of Q. Let U be an open invariant neighborhood of Q, $Q \subset U \subset M - C$ such that $\overline{U} \subset M - C$ and $\overline{U}$ is a manifold with boundary and let M_1^{n+3} = double of $\overline{U} = \overline{U} \cup_{\partial\overline{U}} \overline{U} = \partial(\overline{U} \times I)$. Note that M_1 is not a nice T^n manifold, in fact, the normal bundle of every fixed point set $F(M_1, H)$ in M_1 is trivial. But M_1 does have the property that $H^2(F; Z_2) \neq 0$ for any component F of a fixed point set $F(M_1, H)$, $H \simeq S^1$. (One could have a T^2 action on M^5 for example, with $F(M^5, H) \simeq S^3$). To prove the last statement we note that

$$\bigcup_{H \simeq S^1} F(\bar{U}, H) \subset \bigcup_{H \simeq S^1} F(M - C, H) = \bigcup_{i=1}^{r} T^n/H_i \times V_i$$

and hence

$$\bigcup_{H \simeq S^1} F(\bar{U}, H) = \bigcup_{i=1}^{r} T^n/H_i \times W_i$$

for W_i a compact 2-manifold with boundary $\subset V_i \simeq \mathring{D}^2$. Thus

$$\bigcup_{H \simeq S^1} F(M_1, H) = \bigcup_{i=1}^{r} T^n/H_i \times \left(W_i \cup_{\partial W_i} W_i\right)$$

and

$$H^2\left(W_i \cup_{\partial W_i} W_i; Z_2\right) \neq 0.$$

To construct the manifold $\tilde{M}$ we shall modify the manifold M_1 by twisting the normal bundle at each component of each fixed point set $F(M_i, H)$, $H \simeq S^1$ to satisfy condition (iv) of niceness. Let $F_1, \ldots, F_r$ be the components of $\cup_{H \simeq S^1} F(M_1, H)$ and let $\{p_i, \ldots, p_r\} = Q \cap \cup_{H \simeq S^1} F(M_1, H)$ with $p_i \in F_i$, F_i a component of $F(M_i, H_i)$. Choose $q_i \in F_i$ so that p_i and q_i are on different orbits. Choose a tubular neighborhood C_i of the orbit q_i, diffeomorphic to $T^n \times_{H_i} \mathbf{C} \times D^2$.

Let

$$M_2 = \left(M_1 - \bigcup_{i=1}^{r} C_i'\right) \cup_f \bigcup_{i=1}^{r} C_i''$$

where

$$C_i' = T^n \times_{H_i} \mathbf{C}(2) \times \mathring{D}^2, \qquad C_i'' = T^n \times_{H_i} \mathbf{C}(1) \times D^2,$$

where $\mathbf{C}(r)$ denotes the closed disk of radius r in $\mathbf{C}$, and f: $T^n \times_{H_i} \mathbf{C}(1) \times \partial D^2 \to T^n \times_{H_i} \mathbf{C}(2) \times \partial D^2$ is given by $f(t, z, \theta) = (t, e^{i\theta}z, \theta)$. Note that $F(M_2, H_i) = F(M_1, H_i)$. See Figure 1.

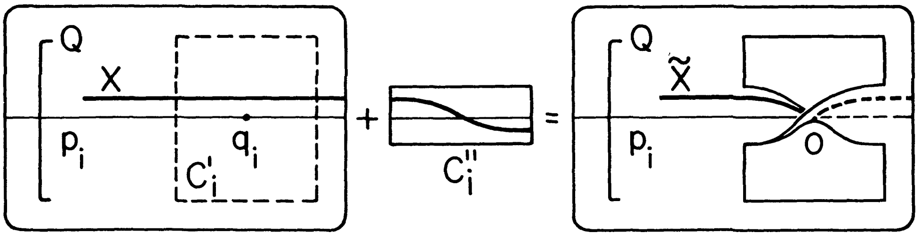

FIGURE 1

Finally $\tilde{M}$ is constructed by choosing $M_3 \subset M_2$ so that M_3 is a manifold with boundary and $\cup_{H \simeq S^1} F(M_2, H) \subset \text{int } M_3$, $U \subset M_3$ and setting $\tilde{M} =$ double of M_3. $\tilde{M}$ clearly satisfies conditions (i), (ii) and (iii) of niceness and $Q \subset U \subset \tilde{M}$. So we need only show that condition (iv) is satisfied. To that end we note that $\nu(F(M_1, H_i))$ is a trivial bundle and therefore has a nonvanishing section X. Then, in M_2, we have a nonvanishing section of $\nu(F(M_2, H_i), M_2) \mid F(M_2, H_i) - T^n/H_i \times \mathring{D}^2$. Recall that $F(M_2, H_i) = T^n/H \times (W_i \cup_{\partial W_i} W_i) = T^n/H \times B^2$ where B^2 is a closed 2-manifold.

We shall show that $\langle w_2(\nu(F(M_2, H_i), H_2), B^2 \rangle \neq 0$. The section X restricts to a section of $\nu(F(M_2, H_i), M_2) \mid B^2 - \mathring{D}^2$ and we want to look at this section in local coordinates on D^2. Then $\tilde{X}$: $\partial D^2 \to \mathbf{C}$ has the form $\tilde{X}(\theta) = e^{i\theta}X(\theta)$ where $\deg X = 0$, X: $\partial D^2 \to \mathbf{C} - \{0\}$, since X extends to $D^2 \subset M_1$ and therefore $\deg \tilde{X}(\theta) = 1$ and hence extending $\tilde{X}$ generically gives $\langle \chi(\nu(F(M_2, h_i)), B^2 \rangle \neq 0$ where χ is the Euler class. □

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