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SHALIKA'S GERMS FOR p-ADIC GL(n): THE LEADING TERM

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For an elliptic torus in GL(n) over a p-adic field we calculate the germ associated to the largest unipotent class. It is a constant multiple of $|D|^{-1/2}$, and we compute the constant explicitly.

1. In his landmark paper ([**4**]), J. Shalika proved the existence of germs and established many of their properties (see §3). He conjectured that the germ associated to the trivial unipotent class was a constant, and even predicted its value.

R. Howe ([**2**]) and Harish-Chandra ([**1**]) proved that it is a constant, and J. Rogawski ([**3**]) proved it had the expected value.

In this paper we begin at the other end and calculate the germ associated to the largest unipotent conjugacy class.

Robert Langlands has informed me that he can prove the analogous result for any simply-connected p-adic group. His argument is presented by Vignéras in [**6**].

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2. Let F be a p-adic field, with ring of integers $\mathfrak{o} = \mathfrak{o}_F$, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$, and $q = |\mathfrak{o}/\mathfrak{p}|$. Let $G = \mathrm{GL}(n, F)$, $K = \mathrm{GL}(n, \mathfrak{o})$, and let K_1 be the congruence subgroup (in the obvious notation $K_1 = \{k \in K\colon k \equiv \mathrm{id}, \bmod \mathfrak{p}\}$).

Let

$$u_0 = \begin{bmatrix} 1 & 1 & 0 & & 0 & 0 \\ 0 & 1 & 1 & 0 & & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & & & & \ddots & 1 \\ 0 & 0 & & & & 1 \end{bmatrix}$$

be the element with entries all equal to 1 on the diagonal and superdiagonal and 0 elsewhere.

Let

$$S = \{k \in K\colon k \equiv u_0, \bmod \mathfrak{p}\} = u_0 K_1.$$

PROPOSITION 1. *The only unipotent conjugacy class which meets S is that of u_0.*

Proof. Consider any other unipotent conjugacy class, and let u be an element of the class which is in Jordan canonical form; i.e. u has 1's on the diagonal, 1's and 0's on the superdiagonal (not all 1's), and 0's elsewhere.

Suppose some conjugate of u were in S. Then a conjugate of $u - \mathrm{id}$ would be in $S - \mathrm{id}$. Now every element of $S - \mathrm{id}$ has rank at least $n - 1$ (since even modulo $\mathfrak{p}$ its rank is $n - 1$), but $u - \mathrm{id}$ has rank less than $n - 1$ (since there is at least one zero on the superdiagonal), which is a contradiction. □

3. Let T be an elliptic torus of G, so $T = T_F$ is isomorphic to the multiplicative group $E^\times$ of some extension field E/F of degree n.

Shalika's theoem ([**4**], Theorem 2.1.1) says that for $f \in C_c^\infty(G)$ and $t \in T'$ sufficiently close to the identity (how close depends on f), we have

$$\int_{T\backslash G} f(g^{-1}tg)\, d\dot{g} = \sum_i \Gamma_i(t) \cdot \int_{Z(u_i)\backslash G} f(g^{-1}u_i g)\, d\dot{g} \tag{3.1}$$

where $\{u_i\}$ $(i = 0, 1, \ldots, r)$ is a set of representatives of the unipotent conjugacy classes, and the functions Γ_i do not depend on f (though they do, of course, depend on T). We shall calculate the function Γ_0 corresponding to the element u_0 of §2.

We shall do this by letting f be the characteristic function of the set S defined in §2. By Proposition 1, the integrals on the right-hand side of (3.1) all vanish, except for the one with $i = 0$. Thus to calculate $\Gamma_0(t)$ we need only evaluate the orbital integrals of f over the conjugacy classes of t and u_0.

4. Let $a_0, a_1, \ldots, a_{n-1} \in \mathfrak{p}$, and consider the polynomial $\Phi(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$. We want to find matrices A satisfying: (i) the characteristic polynomial of A is $\Phi(X)$, and (ii) $\mathrm{id} + A \in S$, i.e. $A \equiv u_0 - \mathrm{id}$, mod $\mathfrak{p}$.

In fact, we claim that such a matrix A is uniquely determined by its first $n - 1$ rows. For convenience we write $\chi(A) \in F^n$ for the n-tuple consisting of the coefficients of the characteristic polynomial of $A - \mathrm{id}$, for any matrix A.

PROPOSITION 2. *Suppose the first $n - 1$ rows of A are specified arbitrarily, subject to the congruence condition* (ii). *Then there is a unique A with these first $n - 1$ rows which satisfies* (i) *and* (ii).

Proof. The familiar argument about characteristic polynomials of matrices with ones on the superdiagonal and zeroes elsewhere except for the last row can be adapted to this situation. It shows that the matrix relating the coefficients $\chi(A)$ to the entries in the last row of A is congruent, mod $\mathfrak{p}$, to a diagonal matrix with alternate 1's and -1's on the diagonal, so the system has a unique solution. □

Now suppose $t \in T'$; write $t = \mathrm{id} + x$, and assume t is such that $\chi(t) \in \mathfrak{p}^n$ (here and in what follows, superscripts on $\mathfrak{p}$ refer to Cartesian products of copies of $\mathfrak{p}$, and not to powers of the ideal). There is a map C_t: $T\backslash G \to G$ given by C_t: $g \mapsto g^{-1}tg = t^g$. Let $\overline{G}_1(t) = C_t^{-1}(S)$. The measure of $\overline{G}_1(t)$ is the orbital integral of f over the conjugacy class of t.

For each matrix $s \in S$, define $P(s)$ to be the $(n-1) \times n$ matrix obtained by deleting the last row of $s - u_0$. This gives a mapping P: $S \to M_{n-1,n}(\mathfrak{p}) \cong \mathfrak{p}^{(n-1)n}$. Composing these two maps gives a map

$$P \circ C_t\colon \overline{G}_1(t) \to S \to \mathfrak{p}^{(n-1)n}. \tag{4.1}$$

Proposition 2 shows that this composite mapping is bijective for each fixed $t \in T'$ as above. We need to calculate its Jacobian as a function of t.

Let $T_1 \subset T \cap K_1$ and $\overline{G}_2 \subset T\backslash G$ be open sets so that $T_1^{\overline{G}_2} \subseteq S$ (in particular $\overline{G}_2 \subseteq \overline{G}_1(t)$, for all $t \in T_1$). Consider the diagram in Figure 1.

$$\begin{tikzcd} T_1 \times \overline{G}_2 \arrow[r] \arrow[d, "\chi\times\mathrm{id}"'] & S \arrow[d, "\chi\times P"'] \arrow[dr] & \\ \mathfrak{p}^n \times \overline{G}_2 \arrow[r, dashed] & \mathfrak{p}^n \times \mathfrak{p}^{(n-1)n} \arrow[r, leftrightarrow] & \mathfrak{p}^n \times \mathfrak{p}^{(n-1)n} \end{tikzcd}$$

FIGURE 1

The top horizontal map takes (t, g) to t^g; the diagonal map takes $s \in S$ to ((last row of $s - u_0$), $P(s)$). The map at the bottom right is the identity on the second factor and on the first factor is a linear transformation congruent mod $\mathfrak{p}$ to a diagonal matrix whose diagonal entries are ± 1 (it is the map which arose in the proof of Proposition 2). The horizontal arrow on the bottom left is the identity on the first factor, and, for fixed $\chi(t)$ in the first factor, acts on the second factor as the restriction of the map $P \circ C_t$ of (4.1).

In the next paragraph we discuss the Jacobians of these maps.

5. The Jacobian of the map $T \times T\backslash G \to G$ given by $(t, g) \to t^g$ is $D(t) = \det(\mathrm{id} - \mathrm{Ad}(t))_{\mathfrak{g}/\mathfrak{t}}$, where $\mathfrak{g}$ and $\mathfrak{t}$ are the Lie algebras of G and T respectively (see [5], pp. 196–198).

The diagonal map in Figure 1 is the identity map relative to the obvious coordinates and the Jacobian of the map at the bottom right has modulus 1, so the Jacobian of their composition $\chi \times P$ has modulus 1 also.

It remains to calculate the Jacobian of the map χ. By [7], Theorem 1, Chapter II, it is possible to pick a basis $\alpha_1 = 1, \alpha_2, \ldots, \alpha_n$ for E/F, so that the ring of integers $\mathfrak{o}_E$ is $\sum \mathfrak{o}_F \alpha_i$. Let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n$ be the various F-embeddings of E into $\bar{E}$, and let L be a finite extension of F containing all the E^{σ_i}.

Now suppose $t \in T_1$, $t = \mathrm{id} + x$. Regarding x as an element of E, write $x = \sum x_i \alpha_i$. The mapping which takes the column vector $(x_i) \in F^n$ to the column $(x^{\sigma_i}) \in L^n$ is a linear map, with matrix $[\alpha_j^{\sigma_i}]$. We regard it as a linear map $\mathfrak{L}\colon L^n \to L^n$.

Let $\mathfrak{S}\colon L^n \to L^n$ be the map which takes an n-tuple of elements of L to the n-tuple of all their elementary symmetric functions.

Put this all together, as in Figure 2.

$$
\begin{array}{ccccc}
L^n & \xrightarrow{\mathfrak{L}} & L^n & \xrightarrow{\mathfrak{S}} & L^n \\
\Uparrow & & & & \Uparrow \\
F^n & & & & F^n \\
\uparrow & & & & \Uparrow \\
T_1 & & \dashrightarrow & & \mathfrak{p}^n
\end{array}
$$

FIGURE 2

The vertical arrow at lower left is the map described above which takes $t \in T_1$ to the coordinates (x_i) of $x = t - \mathrm{id}$. The other vertical arrows are the natural inclusions. So the bottom horizontal map takes $t \in T_1$ to the n-tuple consisting of the elementary symmetric functions of the eigenvalues of $x = t - \mathrm{id}$, which, apart from signs, is $\chi(x)$.

The map $\mathfrak{S} \circ \mathfrak{L}\colon L^n \to L^n$ is a polynomial map taking F^n to F^n, so it is defined over F. The Jacobian matrix of $\mathfrak{S}$ is easily calculated, and a straightforward manipulation with row operations shows that its determinant equals the van der Monde determinant

$$
\det \begin{vmatrix} 1 & 1 & & \cdots \\ -r_1 & -r_2 & -r_3 & \cdots \\ r_1^2 & r_2^2 & r_3^2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} = \pm \prod_{i>j} (r_i - r_j).
$$

From this we can calculate the Jacobian of $\mathcal{S} \circ \mathcal{L}$ and find that at a point in L^n which is the image of $t = \mathrm{id} + x \in T_1$, the Jacobian has modulus

$$\left|\det\left(\alpha_j^{\sigma_i}\right)\right| \left|\prod_{i>j} \left(x^{\sigma_i} - x^{\sigma_j}\right)\right| = \left|D_{E/F}\right|^{1/2} \cdot \left|D(t)\right|^{1/2}.$$

Here $D_{E/F}$ is the discriminant of E over F and the absolute value is the absolute value on F.

By Figure 2, this enables us to calculate the modulus of the Jacobian of χ. Relative to the coordinates on T_1 provided by the basis $\{\alpha_i\}$, the Jacobian of χ has the same modulus as $\mathcal{S} \circ \mathcal{L}$, namely

$$\left|D_{E/F}\right|^{1/2} \cdot \left|D(t)\right|^{1/2}.$$

Referring again to Figure 1, we now know that the top map has Jacobian $D(t)$ and the vertical map on the right has Jacobian of modulus 1. Going the other way, the modulus of the Jacobian of the vertical map on the left is $|D_{E/F}|^{1/2}|D(t)|^{1/2}$, so the modulus of the Jacobian of the bottom left horizontal map is $|D_{E/F}|^{-1/2} \cdot |D(t)|^{1/2}$. Since the bottom map is $\mathrm{id} \times (P \circ C_t)$, we see that the modulus of the Jacobian of $P \circ C_t$ is $|D_{E/F}|^{-1/2} \cdot |D(t)|^{1/2}$.

6. To finish the calculation of orbital integrals we must specify the normalizations of the measures involved. The natural additive measure dx on E gives $\mathfrak{o}_E$ a total mass of 1. We note this measure can be realized as the natural product measure when E is identified with F^n via the basis $\{\alpha_i\}$ and the measure on F is the natural one for which $\mathfrak{o}_F$ has measure 1. On $E^\times$ we take the corresponding measure $d^\times x = |x|_E^{-1}\, dx$, and this will give the measure on $T \cong E^\times$. If $t = \mathrm{id} + x \in T_1$, then $|t|_E = 1$ so dt is just the product $dx_1 \cdots dx_n$.

For the Haar measure on G we make a slightly unusual choice, noting that since it occurs on both sides of equation (3.1) this normalization does not affect the germ Γ_0. We choose the measure on G whose restriction to K equals the restriction to K of the natural product measure on $M_n(F) \cong F^{n^2}$.

Choices of measures on G and T imply a choice of measure on $T\backslash G$ (and on K and S, where the natural measure is just the restriction of Haar measure on G). Referring again to Figure 1, we see that these measures are consistent with the standard measures on $\mathfrak{p}^n$ and $\mathfrak{p}^{(n-1)n}$ (restrictions of the standard measures on F^n and $F^{(n-1)n}$).

PROPOSITION 3. *Suppose $t \in T$ is such that $\chi(t) \in \mathfrak{p}^n$. Then*

$$\int_{T\backslash G} f(g^{-1}tg)\, d\dot{g} = q^{-(n-1)n}|D_{E/F}|^{1/2} \cdot |D(t)|^{-1/2}.$$

Proof. Recall from (4.1) the bijection $P \circ C_t$: $\overline{G}_1(t) \to \mathfrak{p}^{(n-1)n}$. The orbital integral of f is just the measure of $\overline{G}_1(t)$. Since the measure of $\mathfrak{p}^{(n-1)n}$ is $q^{-(n-1)n}$ and from §5 the Jacobian of $P \circ C_t$ has modulus $|D_{E/F}|^{-1/2} \cdot |D(t)^{1/2}$, the result is clear. □

7. The last ingredient is the orbital integral of f over the conjugacy class of u_0. To calculate it we need to specify the measure on the centralizer $Z(u_0)$.

Now $Z(u_0)$ consists of all upper triangular matrices p which are constant along diagonal lines, i.e. $p_{ij} = p_{i+1,j+1}$, for all i, j. We can write $Z(u_0) = Z \cdot U$ where $Z \cong F^\times$ is the centre of G and U consists of all upper triangular unipotent matrices with $u_{ij} = u_{i+1,j+1}$. For the measure on $Z(u_0)$ we take the product of the standard $F^\times$ measure $1/|x|\,dx$ on $Z \cong F^\times$ and the standard F measure on each parameter of U (i.e. $du_{12}du_{13} \cdots du_{1n}$).

We write $G = BK$, where B is the upper triangular subgroup, so the integral over $Z(u_0)\backslash G$ can be replaced by an integral over $(Z(u_0)\backslash B) \cdot K$, and $Z(u_0)\backslash B$ can be represented by B_1, the group of all elements of B whose last column is all zeroes except for a 1 in the bottom right corner. We write db_1 for the obvious measure on B_1, that is write $b = au$ with a diagonal and u unipotent and let db_1 be the product of the standard $F^\times$ measure on each non-trivial entry of a and the standard F measure on each non-trivial entry of u.

The quotient measure $d\dot{g}$ on $Z(u_0)\backslash G$ is obtained by writing $g = b_1 k$ with $b_1 \in B_1$, $k \in K$, and putting

$$d\dot{g} = (1 - 1/q)^{-n}\Delta_1(b_1)/\Delta(b_1)\, db_1\, dk.$$

Here Δ, Δ_1 are the modular functions of B, B_1, respectively, and the constant is needed because of our normalization of Haar measure dg. (The constant is easily evaluated by computing the measure of K, since the modular functions are both trivial on $B_1 \cap K$.)

PROPOSITION 4. *With measures normalized as above,*

$$\int_{Z(u_0)\backslash G} f(g^{-1}u_0 g)\, d\dot{g} = q^{-(n-1)n}.$$

Proof. Consider u_0^{bk}, with $b \in B_1$, $k \in K$. An easy calculation shows that $u_0^{bk} \in S$ if and only if $b \in B_1 \cap K$ and $k = b^{-1}k'$, with $k' \in (Z(u_0) \cap K) \cdot K_1$. The modular functions are both 1 for $b \in K$, so the orbital integral is $(1 - 1/q)^{-n}$ times the product of the measures of $B_1 \cap K$ and $(Z(u_0) \cap K) \cdot K_1$. The former is $(1 - 1/q)^{n-1}$ and the latter is $(1 - 1/q)q^{-(n-1)n}$. □

8. We now combine everything.

THEOREM. *For an elliptic torus T, the germ associated to the largest unipotent u_0 is*

$$\Gamma_0(t) = c \cdot |D(t)|^{-1/2}.$$

If the measures on T and $Z(u_0)$ are normalized as in §§6 *and* 7, *then the constant c is given by*

$$c = |D_{E/F}|^{1/2},$$

where E is the extension of F such that $T \cong E^\times$, $D_{E/F}$ is its discriminant, and the absolute value is the natural absolute value on F.

Proof. Recall that by Proposition 1 the right side of equation (3.1) reduces to the single term with $i = 0$. Propositions 3 and 4 give the two integrals. □

REMARKS 1. Of course other normalizations are possible. In addition to using different measures, a factor of $|D(t)|^{1/2}$ is often included with the orbital integral over the class of t (cf. Harish-Chandra [**1**]). Another convention when forming this orbital integral (cf. Rogawski [3]) is to integrate over $Z\backslash G$ rather than $T\backslash G$, with the result that the orbital integral (and hence the germ) is multiplied by the measure of $Z\backslash T$. This is the measure of $F^\times \backslash E^\times$, which turns out to be $e(1 - q^{-n/e})/(1 - 1/q)$, where e is the ramification index of E/F.

2. It is perhaps worth emphasizing that neither the square root of $D_{E/F}$ nor the square root of $1/D(t)$ need be in F (as examples in GL(2) show), but their product, the Jacobian of a map defined over F, is in F.

3. If T is a non-elliptic torus, it is possible to find a parabolic subgroup $P = MN$ so that T is an elliptic torus in the Levi factor M, and by descent to P the result of this paper implies one about germs of T.

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