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THE CLOSED IMAGE OF A HEREDITARY M_1-SPACE IS M_1

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We show that every closed image of a hereditary M_1-space is hereditarily M_1. This answers positively G. Gruenhage's question.

1. Introduction. J. Ceder [3] introduced the M_i-spaces, $i = 1, 2, 3$ and proved that $M_1 \Rightarrow M_2 \Rightarrow M_3$. He asked whether the converses hold. G. Gruenhage [4] and H. Junnila [8] independently proved that $M_2 = M_3$. Recently R. Heath and H. Junnila [6] showed that every M_3-space is the image of an M_1-space under a perfect retraction. Thus $M_1 = M_2$ if and only if for every M_1-space, every closed image of the space is M_1. However, in general, it is not known whether the closed image of an M_1-space is M_1.

G. Gruenhage [5] proved that the closed image of an M_1-space X with the property $(*)$ is M_1.

$(*)$ *Whenever H and K are closed subsets of X with $H \subset K$, then H has a σ-closure preserving outer base in K.*

If an M_1-space X has the property $(*)$, then every closed subspace of X is M_1. He then posed the following question.

If every closed subspace of a space X is M_1, is every closed image of X also M_1? ([5], *Question* 3.4.)

The aim of this paper is to give a positive answer to this question.

Secondly, we study the class of spaces with a σ-almost locally finite base which was introduced by K. Tamano and the author [7]. This class is contained in the class of M_1-spaces and contains every metrizable space and every M_0-space. Recently G. Gruenhage [5] proved that every F_σ-metrizable M_3-space is M_1. In §3 we shall show that every countable dimensional F_σ-metrizable M_3-space has a σ-almost locally finite base.

All spaces are assumed to be regular T_1 and maps to be continuous. The letter N denotes the positive integers. For undefined notion see [5].

2. Main results. Let X be a paracompact σ-space. If every closed subset of X has a σ-closure preserving outer base, then X is an M_1-space. However, it is not known whether every closed subset of an M_1-space has such a base. Our first theorem shows that every closed subset of a hereditary M_1-space has a closure preserving outer base. This result leads us to the main theorem. To prove the first theorem we start with the following lemma.

LEMMA 2.1. *Let X be a space, U a clopen set of X and $\mathfrak{B}$ a closure preserving family of subsets of X. Then $\{B \cap U: B \in \mathfrak{B}\}$ is closure preserving in X.*

Proof. Let $\mathfrak{B}' \subset \mathfrak{B}$ and $x \notin \bigcup\{\mathrm{Cl}(B \cap U): B \in \mathfrak{B}'\}$. If $x \notin U$, then obviously $x \notin \mathrm{Cl} \bigcup\{B \cap U: B \in \mathfrak{B}'\}$. Let $x \in U$. Then for every $B \in \mathfrak{B}'$, $x \notin \mathrm{Cl}\, B$. Hence $x \notin \mathrm{Cl} \bigcup\{B \cap U: B \in \mathfrak{B}'\}$.

The following results are well known and the proofs are omitted.

LEMMA 2.2. *Let X be a space, S a regular closed set of X and T a regular closed set of S. Then T is a regular closed set of X. Thus every closure preserving family of regular closed sets of S in S is a closure preserving family of regular closed sets of X in X.*

LEMMA 2.3. *Let X be a space. Then X is an M_1-space if and only if X has a σ-closure preserving quasi-base consisting of regular closed sets of X.*

THEOREM 2.4. *Let X be an M_1-space such that every regular closed subspace of X is M_1. Then every closed set of X has a closure preserving outer base.*

Proof. Let F be a closed set of X. Take a family $\{H_n: n \in N\}$ of regular closed sets such that

$$X = H_1 \supset \mathrm{Int}\, H_1 \supset H_2 \supset \cdots, \qquad \bigcap_{n \in N} H_n = F.$$

Set

$$S_1 = \mathrm{Cl}\big(\mathrm{Int}\, F \cup \big(\bigcup \{H_{2n-1} - H_{2n}: n \in N\}\big)\big); \text{ and}$$
$$S_2 = \mathrm{Cl}\big(\mathrm{Int}\, F \cup \big(\bigcup \{H_{2n} - H_{2n+1}: n \in N\}\big)\big).$$

Then S_1 and S_2 are regular closed sets of X and cover X. For $i = 1, 2$, let $\bigcup_{n \in N} \mathfrak{B}_n(i)$ be a σ-closure preservig quasi-base of S_i such that for every $n \in N$, $\mathfrak{B}_n(i) \subset \mathfrak{B}_{n+1}(i)$ and every $B \in \mathfrak{B}_n(i)$ is a regular closed set of S_i. Set

$$\mathfrak{C}_{2n-1} = \{B \cap H_{2n-1}: B \in \mathfrak{B}_n(1)\}, n \in N; \text{ and}$$
$$\mathfrak{C}_{2n} = \{B \cap H_{2n}: B \in \mathfrak{B}_n(2)\}, n \in N.$$

Then for each $n \in N$, H_{2n-1} and H_{2n} are respectively clopen sets of S_1 and S_2, so by Lemma 2.1, $\mathcal{C}_{2n-1}$ and $\mathcal{C}_{2n}$ are respectively closure preserving families of regular closed sets in S_1 and S_2. By Lemma 2.2, every $\mathcal{C}_n$ is a closure preserving family of regular closed sets of X in X. Set

$$\{\mathcal{C}_\alpha : \alpha \in D\} = \left\{\mathcal{C} : \mathcal{C} \subset \bigcup_{n \in N} \mathcal{C}_n, F \subset \operatorname{Int} \bigcup \mathcal{C}\right\}; \quad \text{and}$$

$$\mathcal{U}' = \left\{U_\alpha = \bigcup \mathcal{C}_\alpha : \alpha \in D\right\}.$$

To prove that $\mathcal{U}'$ is closure preserving, let $\phi \neq D' \subset D$ and $x \notin \bigcup\{\operatorname{Cl} U_\alpha : \alpha \in D'\}$. Then $x \notin F$, so there exists a unique $n \in N$ such that $x \in H_n - H_{n+1}$. Then

$$x \notin \operatorname{Cl}\Big(H_{n+1} \cap \Big(\bigcup \{U_\alpha : \alpha \in D'\}\Big)\Big); \quad \text{and}$$

$$\Big(\bigcup \{U_\alpha : \alpha \in D'\}\Big) - H_{n+1} \subset \bigcup \left\{A : A \in \Big(\bigcup_{\alpha \in D'} \mathcal{C}_\alpha\Big) \cap \Big(\bigcup_{i=1}^{n} \mathcal{C}_i\Big)\right\}.$$

For each $A \in (\bigcup_{\alpha \in D'} \mathcal{C}_\alpha) \cap (\bigcup_{i=1}^n \mathcal{C}_i)$, $x \notin \operatorname{Cl} A$. Since $\bigcup_{i=1}^n \mathcal{C}_i$ is closure preserving,

$$x \notin \operatorname{Cl}\left(\bigcup \left\{A : A \in \Big(\bigcup_{\alpha \in D'} \mathcal{C}_\alpha\Big) \cap \Big(\bigcup_{i=1}^{n} \mathcal{C}_i\Big)\right\}\right); \quad \text{and}$$

$$x \notin \operatorname{Cl}\Big(\Big(\bigcup \{U_\alpha : \alpha \in D'\}\Big) - H_{n+1}\Big).$$

Therefore $x \notin \operatorname{Cl}(\bigcup\{U_\alpha : \alpha \in D'\})$.

To prove that $\mathcal{U}'$ is a quasi-outer base of F, suppose $F \subset W$ and W is open. For each $x \in F$ we define $\mathcal{C}_x \subset \bigcup_{n \in N} \mathcal{C}_n$ as follows. If $x \in S_1 \cap S_2$, then there exist $n \in N$, $B_x(1) \in \mathcal{B}_n(1)$ and $B_x(2) \in \mathcal{B}_n(2)$ such that $x \in \operatorname{Int}_{S_1} B_x(1) \subset B_x(1) \subset W$ and $x \in \operatorname{Int}_{S_2} B_x(2) \subset B_x(2) \subset W$. Define

$$\mathcal{C}_x = \{B_x(1) \cap H_{2n-1}, B_x(2) \cap H_{2n}\}.$$

If $x \in S_1 - S_2$, there exist $n \in N$ and $B_x(1) \in \mathcal{B}_n(1)$ such that $x \in \operatorname{Int}_{S_1} B_x(1) \subset B_x(1) \subset W$. Define

$$\mathcal{C}_x = \{B_x(1) \cap H_{2n-1}\}.$$

If $x \in S_2 - S_1$, then we define analogously $\mathcal{C}_x$. Let $\mathcal{C} = \bigcup\{\mathcal{C}_x : x \in F\}$ and $U = \bigcup \mathcal{C}$. Then $U \in \mathcal{U}'$ and $F \subset \operatorname{Int} U \subset U \subset W$.

Let $\mathcal{U} = \{\operatorname{Int} U_\alpha : \alpha \in D\}$. It is easy to show that for every $\alpha \in D$, $\operatorname{Cl} U_\alpha = \operatorname{Cl}(\operatorname{Int} U_\alpha)$. Then clearly $\mathcal{U}$ is a closure preserving outer base of F and the proof is completed.

The proofs of the following two theorems are straightforward, and are thus omitted.

THEOREM 2.5. *Let X be an M_1-space with* $\dim X = 0$. *Then every closed set of X has a closure preserving outer base.*

THEOREM 2.6. *Let X be a space and $\{S_\alpha: \alpha \in D\}$ a locally finite cover of X consisting of regular closed M_1-subspaces. Then X is an M_1-space.*

COROLLARY 2.7. *Let $\{X_\alpha: \alpha \in D\}$ be a family of M_1-spaces such that each X_α satisfies one of the following conditions.*

(1) *Every regular closed subspace of X_α is M_1.*

(2) $\dim X_\alpha = 0$.

(3) *X_α is first countable.*

Then for every $p \in B_\alpha X_\alpha$, Ξ_p is M_1. (Here $B_\alpha X_\alpha$ is the box product space of $\{X_\alpha: \alpha \in D\}$ and Ξ_p is the subspace $\{x \in B_\alpha X_\alpha: x_\alpha \neq p_\alpha$ for at most finitely many $\alpha\}$ of $B_\alpha X_\alpha$.)

Proof. This follows from Theorem 2.4, 2.5 and [**10**], Theorem 3.1.

Before stating the main theorem of this paper, we note the following lemma holds. Then a space X is hereditarily M_1 if and only if every closed subspace of X is M_1. Therefore Theorem 2.9 is a positive answer to G. Gruenhage's question ([**5**], Question 3.4).

LEMMA 2.8. *Every dense subspace of an M_1-space is M_1.*

Proof. This follows from the fact that the closure of an open set is equal to the closure of the intersection with a dense subset.

THEOREM 2.9. *Let X be a hereditary M_1-space. Then every closed image of X is hereditarily M_1.*

Proof. Let $f: X \to Y$ be a closed onto map. It is enough to show that Y is M_1. Let H and K be closed sets of X with $H \subset K$. Then K is hereditarily M_1 and H is closed in K. So by Theorem 2.4, H has a closure preserving outer base in K. Then X satisfies the property of [**5**], Theorem 3.2. Hence by [**5**], Theorem 3.2, Y is M_1.

Problem 2.10. Is the countable product of hereditary M_1-spaces hereditarily M_1?

More basically:

Problem 2.11. If X and Y are hereditary M_1-spaces, is $X \times Y$ hereditarily M_1?

If the answer to Problem 2.10 is positive, then the class of hereditary M_1-spaces is one giving a positive answer to [5], Problem 3.6.

3. Maps of spaces with a σ-almost locally finite base. Recently K. Tamano and the author [7] introduced the class of spaces with a σ-almost locally finite base. This class is contained in the class of M_1-spaces and contains every metrizable space and every M_0-space. In this section we shall prove that the class of spaces with a σ-almost locally finite base is closed under finite to one closed maps. As a corollary of this result, we have every countable dimensional F_σ-metrizable M_3-space has a σ-almost locally finite base.

DEFINITION 3.1. Let X be a space, $x \in X$ and $\mathfrak{A}$ a family of subsets of X. $\mathfrak{A}$ is said to be *almost locally fiite at* x if there exist a neighborhood U of x and a finite family $\mathfrak{B}$ of subsets of X such that

$$\{A \cap U: A \in \mathfrak{A}\} \subset \{B \cap V: B \in \mathfrak{B},\ V \text{ is a neighborhood of } x\}.$$

$\mathfrak{A}$ is said to be *almost locally finite in* X if $\mathfrak{A}$ is almost locally finite at every $x \in X$. Note that we can take X as above U.

Every locally finite family is of course almost locally finite and every almost locally finite family is closure preserving. For other fundamental results concerning almost locally finite families see [7].

LEMMA 3.2. *Let X be a space and $\mathfrak{A}$ an almost locally finite family at $x \in X$. Then both* $\{\operatorname{Int} A: A \in \mathfrak{A}\}$ *and* $\{\operatorname{Cl} A: A \in \mathfrak{A}\}$ *are almost locally finite at x.*

Proof. By Definition 3.1, there exist a neighborhood U of x and a finite family $\mathfrak{B}$ of subsets of X such that

$$\{A \cap U: A \in \mathfrak{A}\} \subset \{B \cap V: B \in \mathfrak{B},\ V \text{ is a neighborhood of } x\}.$$

Let $A \cap U = B \cap V$ with $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ and V is a neighborhood of x.

Then

$$\operatorname{Int} A = \operatorname{Int}(B \cup (X - U)) \cap \operatorname{Int}(A \cup (U \cap V)); \quad \text{and}$$
$$\operatorname{Cl} A = \operatorname{Cl}(B \cup (X - U)) \cap (\operatorname{Cl} A \cup \operatorname{Int}(U \cap V)).$$

Therefore

$$\{\operatorname{Int} A : A \in \mathcal{A}\} \subset \{V \cap \operatorname{Int}(B \cup (X - U)) : B \in \mathcal{B},\ V \text{ is a neighborhood of } x\};$$

and

$$\{\operatorname{Cl} A : A \in \mathcal{A}\} \subset \{V \cap \operatorname{Cl}(B \cup (X - U)) : B \in \mathcal{B},\ V \text{ is a neighborhood of } x\}.$$

That completes the proof.

LEMMA 3.3. *Let f: $X \to Y$ be a finite to one closed onto map and $\mathcal{A}$ an almost locally finite family of subsets of X. Then $\{f(A)$: $A \in \mathcal{A}\}$ is an almost locally finite family of Y.*

Proof. Let $y \in Y$ and $f^{-1}(y) = \{x_1, \ldots, x_n\}$. For each x_i there exist a neighborhood U_i of x_i and a finite family $\mathcal{B}_i$ of subsets of X such that

$$\{A \cap U_i : A \in \mathcal{A}\} \subset \{B \cap V : B \in \mathcal{B}_i,\ V \text{ is a neighborhood of } x_i\}.$$

We may assume $\{U_i$: $i = 1, \ldots, n\}$ is disjoint and $\cup \mathcal{B}_i \subset U_i$. Set

$$\mathcal{B}_y = \left\{B[B_1, \ldots, B_n] = f\left(\left(\bigcup_{i=1}^{n} B_i\right) \cup \left(X - \bigcup_{i=1}^{n} U_i\right)\right) : B_i \in \mathcal{B}_i,\ i = 1, \ldots, n\right\}.$$

Then $|\mathcal{B}_y| < \aleph_0$. Let $A \in \mathcal{A}$. Then for each x_i, there exist $B_i \in \mathcal{B}_i$ and a neighborhood V_i of x_i such that $A \cap U_i = B_i \cap V_i$. There exists a neighborhood V of y such that $f^{-1}(V) \subset \cup_{i=1}^{n}(U_i \cap V_i)$. Then

$$V \cap B[B_1, \ldots, B_n] \subset f(A) \subset B[B_1, \ldots, B_n].$$

Set $V_y = V \cup (F(A) - (V \cap B[B_1, \ldots, B_n]))$. Then

$$f(A) = V_y \cap B[B_1, \ldots, B_n];$$
$$B[B_1, \ldots, B_n] \in \mathcal{B}_y; \quad \text{and} \quad V_y \text{ is a neighborhood of } y.$$

Therefore $\{f(A): A \in \mathcal{A}\}$ is an almost locally finite family of Y and the proof is completed.

THEOREM 3.4. *Let X be a space with a σ-almost locally finite base and f: $X \to Y$ a finite to one closed onto map. Then Y has a σ-almost locally finite base.*

Proof. Let $\bigcup_{n\in N} \mathfrak{B}_n$ be a σ-almost locally finite base of X such that for each $n \in N$, $\mathfrak{B}_n \subset B_{n+1}$ and if $\mathfrak{B} \subset \mathfrak{B}_n$, then $\bigcup \mathfrak{B} \in \mathfrak{B}_n$. For each $n \in N$, let $\mathfrak{U}_n = \{\text{Int } f(B): B \in \mathfrak{B}_n\}$. Then by Lemma 3.2 and 3.3, each $\mathfrak{U}_n$ is almost locally finite in Y. It is easy to check that $\bigcup_{n\in N} \mathfrak{U}_n$ is a base of Y and the proof is completed.

COROLLARY 3.5. *Let X be a F_σ-metrizable M_3-space with countable dimension. Then X has a σ-almost locally finite base.*

Proof. By **[9]**, Corollary 3, there exist a paracompact F_σ-metrizable space Z with $\dim Z = 0$, and a closed onto map f: $Z \to X$ such that for every $x \in X$, $|f^{-1}(x)| < \aleph_0$. Since, in this case, X is M_3, so is Z. Then by **[5]**, Theorem 3.1, Z is an M_0-space and has a σ-almost locally finite base. Hence by Theorem 3.4, X has a σ-almost locally finite base.

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