PACIFIC JOURNAL OF MATHEMATICS
Vol. 113, No. 2, 1984

DEFICIENCES OF IMMERSIONS

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Dedicated to Steve Warschawski

Let X and Y be manifolds of the same dimension $n \geq 2$ and let f: $X \to Y$ be an immersion with $p = \sup\{n(y): y \in Y\} < \infty$ where $n(y) =$ cardinality $f^{-1}(y)$. If Y is compact and X is not, then $n(y) < p$ for some $y \in Y$, see §2. If Y is compact and simply connected and $p \geq 2$, then Y contains a compact set E such that $Y - E$ is not simply connected and $n(y) \leq p - 2$ for all $y \in E$, see §5.

1. THEOREM. *Let X be a non-compact n-manifold, Y a compact n-manifold and f: $X \to Y$ an immersion. If $p = \max_{y \in Y} n(y) < \infty$, then $n(y) < p$ for some points $y \in Y$. In particular, if $y = \lim_{k \to \infty} f(x_k)$ for an infinite sequence of distinct points $x_k \in X$ which does not accumulate in X, then $n(y) < p$.*

Proof. Suppose that $n(y) = p$ with $f^{-1}(y) = \{a_1, \ldots, a_p\}$. Choose disjoint closed cells U_i in X such that $a_i \in \operatorname{int} U_i$ and such that $f | U_i$ is injective for $1 \leq i \leq p$. Then $x_k \notin \cup U_i$ for almost all k. Now choose a neighborhood V of y such that $V \subset \bigcap_{i=1}^{p} f(U_i)$ and let V_i denote the a_i component of $f^{-1}(V)$. Then f maps each V_i homeomorphically onto V and hence $n(y') = p$ for all $y' \in V_0$. It thus follows that $f(x_k) \notin V$ for all x_k in $X_0 = X - \cup V_i$, that is for almost all x_k. Hence $f(y_k) \nrightarrow y$, contradicting the assumption $f(x_k) \to y$, and thus $n(y) < p$.

2. REMARK. For compact manifolds X with boundary Theorem 1 says that $n(y) < p$ for every y in the cluster set of f on ∂X. This contains a result of Brannan and Kirwan [**1**, Theorem 1] as a special case.

3. Suppose that X is non-compact, that Y is compact and that $1 < p = \max n(y) < \infty$. We say that f has a *deficiency* at a point $y \in Y$ if $n(y) \leq p - 2$. The set $A = \{y \in Y: n(y) \leq p - 2\}$ will be called the *deficiency set* of f. It is not hard to construct immersions, for instance of $S^1 \times R$ into $S^1 \times S^1$ with empty deficiency set. The purpose of this note is to show that if Y is simply connected, then the deficiency set A is non-empty and, in fact, it is quite large.

4. THEOREM. *Let X be an n-manifold and Y a simply connected compact n-manifold, $n \geq 2$, and let f: $X \to Y$ be an immersion with $1 < p = \max n(y) < \infty$. Then the deficiency set A contains a compact subset E such that $Y - E$ is not simply connected.*

5. REMARK AND NOTATION. The proof is based on two elementary lemmas and on application of the monodromy theorem to a certain extension of f. The extension of f is essentially the same as in Lyzzaik and Styer [2, §2]. The following notation will be used: For $r > 0$ and $a \in R^n$, $B^n(a, r) = \{x \in R^n\colon |x - a| < r\}$, $B^n(r) = B^n(0, r)$, $B^n = B^n(1)$ and in particular $B^2 = \{z \in \mathbf{C}\colon |z| < 1\}$. We say that a compact set E in a simply connected space Y is *π_1-negligible* if $Y - E$ is simply connected. In this notation, Theorem 4 asserts that the deficiency set A has compact subsets which are not π_1-negligible in Y.

6. LEMMA. *Let H: $\bar{B}^2 \to R^n$ be a continuous function with $H(-1) \in B^n$ and $H(1) \notin \bar{B}^n$. Then $H^{-1}(\partial B^n)$ contains a continuum C which meets both components of $\partial B^2 - \{-1, 1\}$.*

Proof. By the Jordan separation theorem $F = H^{-1}(\partial B^n)$ separates the points -1 and 1 in $\bar{B}^2$. Let B_1 denote the connected component of $\bar{B}^2 - F$, which contains the point -1, and let B_2 be the connected component of $\mathbf{C} - B_1$, which contains the point 1. Then $C = \partial B_2 \cap \bar{B}^2$ is the desired continuum.

7. LEMMA. *Let A be a closed set in R^n. If every compact subset E of A such that $R^n - E$ is connected is π_1-negligible then*
(i) int $A = \varnothing$.
(ii) $U = R^n - A$ *is connected.*

Proof. (i) is trivial.

(ii) Suppose that U is not connected. Choose points a_1 and a_2 which belong to different connected components of U. Since A is closed there is $r > 0$ such that $B^n(a_i, 2r) \subset U$, $i = 1, 2$. Let

$$G = \bigcup_{0 \leq t \leq 1} B^n(ta_1 + (1 - t)a_2, r)$$

and $E = A \cap \partial G$. Now choose points $b_i \in \partial B(a_i, 2r)$, $i = 1, 2$, so that a_1, a_2, b_1, b_2 are vertices of a rectangle R. Since $R^n - E$ is simply connected, there is a continuous function $H\colon \bar{B}^2 \to R^n - E$ mapping ∂B^2 homeomorphically onto R. We may assume that $H(-1) = a_1$, $H(1) = b_1$, $H(i) \in \partial B^n(a_1, r)$ and $H(-i) \in \partial B^n(a_2, r)$. By Lemma 6 there exists a continuum C in $H^{-1}(\partial G)$ joining the components of $\partial B^2 - \{-1, 1\}$. Hence $C' = H(C)$ is a continuum in ∂G joining $\partial^n B(a_1, r)$ and $\partial^n B(a_2, r)$. Hence a_1 and a_2 can be joined by a continuum in U, contradicting the assumption that U is not connected.

8. *Proof of Theorem* 4.. Let $A_k = \{y \in Y\colon n(y) = k\}$. Then A_p and $A_p \cup A_{p-1}$ are open and hence the deficiency set $A = Y - (A_p \cup A_{p-1})$ is compact. Consider the disjoint union $\tilde{X} = X \cup A_{p-1}$ with the topology containing the topology of X and the topology of int A_{p-1}, which makes the extension $\tilde{f}\colon \tilde{X} \to Y$ of f, $\tilde{f}(x) = f(x)$ for $x \in X$ and $\tilde{f}(x) = x$ for $x \in A_{p-1}$, a local homeomorphism. Obviously, $\tilde{f}$ is a local homeomorphism in $X \cup \text{int } A_{p-1}$. For $y \in \bar{A}_p \cap A_{p-1}$ with $f^{-1}(y) = \{x_1, \ldots, x_{p-1}\}$ choose disjoint cells U_i in X with $x_i \in \text{int } U_i$ and such that each $f | U_i$ is injective, $1 \le i < p$. Now let V be an open set in $\bigcap f(U_i)$ containing y. Then $\tilde{f}$ maps $U_0 = f^{-1}(V) - \bigcup U_i$ homeomorphically onto $V \cap A_p$ and $\tilde{f}$ maps $U = U_0 \cup (V \cap A_{p-1})$ injectively onto V. Such sets U form a base of neighborhoods of $y \in \bar{A}_p \cap A_{p-1}$.

Suppose now that Theorem 4 is false, i.e., all compact subsets E of A such that $Y - E$ is connected are π_1-negligible in Y. Then obviously int $A = \phi$. Also, if D is an open cell in Y, then, by Lemma 7, $D - A$ is connected. Since every two points a and b in Y can abe connected by a chain of open cells $D_1, \ldots, D_k$ such that $a \in D_1$, $b \in D_k$ and $D_i \cap D_{i+1} \neq \varnothing$ for $1 \le i < k$, it follows that $Y - A$ is connected and hence so is $X_0 = \tilde{X} - f^{-1}(A)$. Now X_0 is a manifold and $f_0 = \tilde{f} | X_0$ is a p to 1 covering map of X_0 onto $Y - A$. The assumption that $Y - A$ is simply connected implies, by the monodromy theorem, that f_0 is injective and hence that $p = 1$. This contradiction completes the proof.

9. Remark. For $n = 2$ Theorem 4 says that the deficiency set of an immersion of a non-compact surface into S^2 has at least two points. This contains a reuslt of Brannan and Kirwan [**1**, Theorem 2] as a particular case.

10. Acknowledgement. I wish to thank Mike Freedman, Brit Kirwin and Joe Wolf for helpful discussions.

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Received May 21, 1982.

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