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REGULARITY OF SOLUTIONS TO ELLIPTIC ISOPERIMETRIC PROBLEMS

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In an earlier paper by the author, the existence theory for solutions of elliptic isoperimetric problems was developed in the context of rectifiable and integral currents. The regularity problem for solutions of those isoperimetric problems was essentially left open. Examples were given to show that, in certain cases, regularity may be totally absent. In this note we derive positive regularity results for solutions of elliptic isoperimetric problems.

1. Introduction. Let Φ and Ψ be class 2 elliptic integrands. The problem studied in [**PH**] was that of finding a surface, S, minimizing the integral of the integrand Φ over S subject to the requirements that S have a given boundary and that the integral of the integrand Ψ over S equal a specified value, ψ. In [**PH**] it was shown that two parameters, ψ_0 and ψ_1, were crucial to understanding the problem:

> ψ_0 is the absolute minimum for the Ψ integral among surfaces which satisfy the boundary condition, ψ_1 is the minimum for the Ψ integral among surfaces which satisfy the boundary condition and for which the first variation of the Φ integral vanishes.

In this paper, also, ψ_0 and ψ_1 are crucial. Using the first variation to estimate the integrals over deformed surfaces, we are able to show that the regularity results of F. J. Almgren, Jr., [**AF**], in the form recently derived by E. Bombieri, [**BE**], yield the following (see 6):

THEOREM. *If $\psi_0 \leq \psi < \psi_1$ and T is a rectifiable current which minimizes the Φ integral among rectifiable currents which satisfy the boundary condition and have Ψ integral equal to ψ, then the regular points are dense in* $\operatorname{spt} T \sim \operatorname{spt} \partial T$.

2. Preliminaries. We will use the notation and terminology of [**PH**]. Additionally, we

(1) assume Φ and Ψ are class 2 elliptic integrands of degree m on Z,

(2) choose λ with $1 \le \lambda < \infty$ such that

$$\lambda^{-1} \le \Phi(z, \alpha) \le \lambda \quad \text{and}$$
$$\lambda^{-1} \le \Psi(z, \alpha) \le \lambda$$

hold for $z \in A$ and $\alpha \in \wedge_m \mathbf{R}^n$ with $|\alpha| = 1$,

(3) fix ψ with $\psi_0 \le \psi < \psi_1$,

(4) fix $T \in \mathfrak{R}_{m,A}(Z)$ which satisfies $\operatorname{spt}(R - \partial T) \subset B$, $\langle \Psi, T \rangle = \psi$,

$$\langle \Phi, T \rangle = \inf\{\langle \Phi, S \rangle : S \in \mathfrak{R}_{m,A}(Z),$$
$$\operatorname{spt}(R - \partial S) \subset B, \langle \Psi, S \rangle = \psi\},$$

(5) denote by G the set of

$$x \in (\operatorname{Int} A \cap \operatorname{spt} T) \sim (B \cup \operatorname{spt} \partial T)$$

such that there exist $r > 0$ and a class 3 isotopic deformation

$$h: I \times W \to Z,$$

of an open subset, W, of Z in Z, for I an open interval with $0 \in I$, satisfying

(i) $A \subset W$,

(ii) $h(I \times \operatorname{spt} T) \subset A$,

(iii) $h[I \times (B \cap \operatorname{spt} \partial T)] \subset B$,

(iv) $h(t, z) = z$ for $t \in I$ and $z \in \mathbf{B}^n(x, r) \cup (\operatorname{spt} \partial T \sim B)$,

(v) $h(t, z) \notin \mathbf{U}^n(x, r)$ for $t \in I$ and $z \notin \mathbf{U}^n(x, r)$,

(vi) $\mathbf{U}^n(x, r) \subset \operatorname{Int} A \sim (B \cup \operatorname{spt} \partial T)$,

(vii) $\delta^{(1)}(T, \Phi, h) \neq 0$.

3. Lemma. *There is at most one point in*

$$[(\operatorname{Int} A \cap \operatorname{spt} T) \sim (B \cup \operatorname{spt} \partial T)] \sim G.$$

Proof. Fix

$$x_1 \in [(\operatorname{Int} A \cap \operatorname{spt} T) \sim (B \cup \operatorname{spt} \partial T)] \sim G,$$
$$x_2 \in (\operatorname{Int} A \cap \operatorname{spt} T) \sim (B \cup \operatorname{spt} \partial T)$$

with $x_1 \neq x_2$. We will show

$$x_2 \in G.$$

Since $\psi < \psi_1$, T is not Φ stationary with respect to (A, B) and, hence, there is a class 3 isotopic deformation

$$k: J \times V \to Z$$

of an open subset, V, of Z in Z, for J an open interval with $0 \in J$, satisfying

$$\begin{aligned}
&A \subset V,\\
&k(J \times \operatorname{spt} T) \subset A,\\
&k[J \times (B \cap \operatorname{spt} \partial T)] \subset B,\\
&k(t, z) = z \quad \text{for } t \in J \quad \text{and} \quad z \in \operatorname{spt} \partial T \sim B,\\
&\delta^{(1)}(T, \Phi, k) \neq 0.
\end{aligned}$$

Choose $r > 0$ such that

$$\mathbf{U}^n(x_1, 5r) \subset \operatorname{Int} A \sim (\{x_2\} \cup B \cup \operatorname{spt} \partial T).$$

Choose an open interval I, with $0 \in I \subset J$, and an open W, with $A \subset W \subset V$, such that

$$|k(t, z) - z| < r \quad \text{holds for } t \in I \quad \text{and} \quad z \in \mathbf{B}^n(x_1, 3r),$$

$$k[I \times (W \sim \mathbf{U}^n(x_1, 2r))] \cap \mathbf{B}^n(x_1, r) = \varnothing.$$

Choose $\phi\colon \mathbf{R}^n \to \mathbf{R}$ of class ∞ such that

$$\begin{aligned}
&\phi(z) = 0 \quad \text{if } |z| \leq 2,\\
&\phi(z) = 1 \quad \text{if } |z| \geq 3,\\
&0 \leq \phi(z) \leq 1 \quad \text{for all } z \in \mathbf{R}^n.
\end{aligned}$$

Define

$$l\colon I \times W \to Z$$

by setting

$$l(t, z) = \phi(r^{-1}(z - x_1))k(t, z) + (1 - \phi(r^{-1}(z - x_1)))z$$

for $(t, z) \in I \times W$. We note that r and l satisfy the conditions 2(5i-vi) with x replaced by x_1. Thus we have

$$\delta^{(1)}(T, \Phi, l) = 0.$$

Define

$$h\colon I \times W \to Z$$

by setting

$$h(t, z) = (1 - \phi(r^{-1}(z - x_1)))k(t, z) + \phi(r^{-1}(z - x_1))z$$

for $(t, z) \in I \times W$. We have

$$\begin{aligned}
0 &\neq \delta^{(1)}(T, \Phi, k)\\
&= \delta^{(1)}(T, \Phi, l) + \delta^{(1)}(T, \Phi, h) = \delta^{(1)}(T, \Phi, h).
\end{aligned}$$

Since r and h satisfy the conditions 2(5i-vi) with x replaced by x_2, we have $x_2 \in G$.

4.1. Preliminaries. In 4.2 we will use the following notation:

(1) Fix $x_0 \in G$ and let $r_0 > 0$ and $h: I \times W \to Z$ satisfy the conditions of 2(5).

(2) Set

$$\alpha = \delta^{(1)}(T, \Phi, h), \qquad \beta = \delta^{(1)}(T, \Psi, h).$$

By hypothesis we have $\alpha \neq 0$, and without loss of generality we may assume $\alpha > 0$.

(3) Define the elliptic integrand

$$\Xi: Z \times \wedge_m \mathbf{R}^n \to \mathbf{R}$$

by setting

$$\Xi(z, \mu) = \left(|\alpha| + |\beta|\right)^{-1}\left[|\beta|\Phi(z, \mu) + |\alpha|\Psi(z, \mu)\right]$$

for $(z, \mu) \in Z \times \wedge_m \mathbf{R}^n$.

4.2. THEOREM. *There exist* r_1, c_1, *with* $0 < r_1, c_1 < \infty$, *such that for each* r, *with* $0 < r \leq r_1$,

$$T \llcorner \mathbf{U}^n(x_0, r)$$

is $(\Xi, \omega, 2r)$ *minimal in* $\mathbf{U}^n(x_0, r)$ (*see* [**BE**; *Definition* 1]), *where*

$$\omega(t) = c_1 \sup\left\{\|T\|\mathbf{B}^n(x, t): x \in \mathbf{U}^n(x_0, r_1)\right\}.$$

Proof. We will assume

$$\delta^{(1)}(T, \Psi, h) \neq 0;$$

the proof in case $\delta^{(1)}(T, \Psi, h) = 0$ is similar. By [**PH**; 1.4] and the argument used in the proof of [**PH**; 1.3], we see that, in fact,

$$\delta^{(1)}(T, \Psi, h) < 0$$

must hold.

We can find δ, c, positive real numbers, and

$$f, g: \mathbf{R} \cap \{t: |t| < \delta\} \to \mathbf{R} \cap \{t: |t| < c\delta\},$$

of class 2, such that for t with $|t| < \delta$ the following hold:

$$\begin{gathered}(1+c)t \in I,\\ \left|\langle \Phi, h_{t\#}T\rangle - \langle \Phi, T\rangle - t\delta^{(1)}(T,\Phi,h)\right| \le ct^2,\\ \left|\langle \Psi, h_{t\#}T\rangle - \langle \Psi, T\rangle - t\delta^{(1)}(T,\Psi,h)\right| \le ct^2,\\ \langle \Phi, h_{f(t)\#}T\rangle = \langle \Phi, T\rangle + t,\\ \langle \Psi, h_{g(t)\#}T\rangle = \langle \Psi, T\rangle + t,\\ \left|f(t) - t\left[\delta^{(1)}(T,\Phi,h)\right]^{-1}\right| \le ct^2,\\ \left|g(t) - t\left[\delta^{(1)}(T,\Psi,h)\right]^{-1}\right| \le ct^2.\end{gathered}$$

Let r_1 be such that

$$0 < r_1 \le 3^{-1}r_0, \qquad \|T\|\mathbf{U}^n(x_0, 3r_1) < \lambda^{-3}\delta.$$

Set

$$c_1 = 5\lambda^4(1+\alpha)(1+|\beta|)(1+\alpha^{-1})^2\left(1+|\beta|^{-1}\right)^2(1+\delta)^2(1+c)^3.$$

Fix r with $0 < r \le r_1$. Let K be compact with $K \subset \mathbf{U}^n(x_0, r)$. Suppose $X \in \mathcal{R}_m(\mathbf{R}^n)$ satisfies $\partial X = 0$, $\operatorname{spt} X \subset K$. Set $d = \operatorname{diam}(\operatorname{spt} X)$ and fix $x_1 \in \operatorname{spt} X$. By [**PH**; 1.4] and the argument used in the proof of [**PH**; 1.3], we may suppose either

$$(*) \qquad \langle \Psi, T\rangle < \langle \Psi, T+X\rangle, \quad \langle \Phi, T\rangle > \langle \Phi, T+X\rangle$$

or

$$(**) \qquad \langle \Psi, T\rangle > \langle \Psi, T+X\rangle, \quad \langle \Phi, T\rangle < \langle \Phi, T+X\rangle$$

Case I. *Assume* $(*)$ *holds*. We have

$$\begin{aligned}\langle \Psi, T+X\rangle - \langle \Psi, T\rangle &\le \langle \Psi, (T+X)\llcorner\mathbf{U}^n(x_1,d)\rangle\\ &\le \lambda\|T+X\|\mathbf{U}^n(x_1,d) \le \lambda^2\langle \Phi, (T+X)\llcorner\mathbf{U}^n(x_1,d)\rangle\\ &\le \lambda^2\langle \Phi, T\llcorner\mathbf{U}^n(x_1,d)\rangle \le \lambda^3\|T\|\mathbf{U}^n(x_1,d) < \delta.\end{aligned}$$

Set

$$t = \langle \Psi, T\rangle - \langle \Psi, T+X\rangle.$$

Then, by 2(5iv, v),

$$\langle \Psi, h_{g(t)\#}(T+X)\rangle = \langle \Psi, T\rangle$$

holds, so we have

$$\left\langle \Phi, h_{g(t)\#}(T+X)\right\rangle \geq \langle \Phi, T\rangle.$$

But also we have, again using 2(5iv, v)

$$\begin{aligned}\left\langle \Phi, h_{g(t)\#}(T+X)\right\rangle &\\ &\leq \langle \Phi, T+X\rangle + \delta^{(1)}(T,\Phi,h)g(t) + c[g(t)]^2\\ &\leq \langle \Phi, T+X\rangle + \alpha\left(\beta^{-1}t+ct^2\right) + c\left(\beta^{-1}t+ct^2\right)^2.\end{aligned}$$

It follows that

$$\begin{aligned}|\beta|\langle \Phi, T\rangle \leq |\beta|\langle \Phi, T+X\rangle &+ \alpha(\langle \Psi, T+X\rangle - \langle \Psi, T\rangle)\\ &+|\beta|(\langle \Psi, T+X\rangle - \langle \Psi, X\rangle|t|)\\ &\cdot\left(\alpha c + \beta^{-2}c + 2c^2|\beta|^{-1}|t| + c^3t^2\right).\end{aligned}$$

Consequently, we have

$$\begin{aligned}\langle \Xi, T\rangle \leq \langle \Xi, T+X\rangle &+ |\beta|(\alpha+|\beta|)^{-1}(\langle \Psi, T+X\rangle - \langle \Psi, T\rangle)\\ &\cdot|t|\cdot 5(1+\alpha)(1+c)^3(1+\delta)^2\left(1+|\beta|^{-1}\right)^2.\end{aligned}$$

We note that

$$\langle \Psi, T+X\rangle - \langle \Psi, T\rangle \leq \langle \Psi, T \llcorner K + X\rangle \leq \lambda \mathbf{M}[T \llcorner K + X]$$

and

$$|t| \leq \lambda^3 \sup\{\|T\|\mathbf{B}^n(x,d) : x \in \mathbf{U}^n(x_0, r_1)\}.$$

So we have

$$\langle \Xi, T \llcorner K\rangle \leq \langle \Xi, T \llcorner K + X\rangle + \omega(d)\mathbf{M}[T \llcorner K + X].$$

Case II. *Assume* (**) *holds*. The argument used in this case is similar to that used in Case I. The difference is that we set

$$t = \langle \Phi, T\rangle - \langle \Phi, T+X\rangle$$

and consider

$$h_{f(t)\#}(T+X).$$

5. LEMMA. *There exists $\mu > 0$, which depends only on m and λ, with the following property: For each $r_2 > 0$ there exists $c_2 = c_2(r_2, T, m, \lambda) > 0$ such that if $0 < \rho \leq r_2$ and $x \in \operatorname{spt} T$ satisfies $\operatorname{dist}(x, B \cup \operatorname{spt} \partial T) \geq r_2$, then $\|T\|\mathbf{U}^n(x,\rho) \leq c_2\rho^\mu$ holds.*

Proof. Set $\mu = \lambda^{-2}m$. Let $r_2 > 0$ be given. Set $c_2 = \mathbf{M}(T)(r_2)^{-\mu}$. Fix $x \in \operatorname{spt} T$ which satisfies $\operatorname{dist}(x, B \cup \operatorname{spt} \partial T) \geq r_2$.

Define $u: \mathbf{R}^n \to \mathbf{R}$ by setting

$$u(z) = |z - x|.$$

Define $v: \{\rho: 0 < \rho \leq r_2\} \to \mathbf{R}$ by setting

$$v(\rho) = \|T\|\mathbf{U}^n(x, \rho).$$

Define $H: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ by setting

$$H(t, z) = x + t(z - x).$$

For $0 < \rho < r_2$, set

$$S_\rho = T \llcorner \mathbf{U}^n(x, \rho) + H_\#\big[(\mathbf{E}^1 \llcorner \{t: 0 \leq t \leq 1\})\langle T, u, \rho +\rangle\big].$$

By [**PH**; 1.4], for each $0 < \rho < r_2$ we have either

$$\langle \Psi, T \rangle \leq \langle \Psi, S_\rho \rangle \quad \text{or} \quad \langle \Phi, T \rangle \leq \langle \Phi, S_\rho \rangle.$$

For $\mathfrak{L}^1$ almost every $0 < \rho < r_2$ we have

$$\mathbf{M}\big[H_\#\big[(\mathbf{E}^1 \llcorner \{t: 0 \leq t \leq 1\}) \times \langle T, u, \rho +\rangle\big]\big] \leq m^{-1}\rho v'(\rho).$$

We conclude that for $\mathfrak{L}^1$ almost every $0 < \rho < r_2$

$$\lambda^{-1}v(\rho) \leq \lambda m^{-1}\rho v'(\rho)$$

and, hence,

$$(*) \qquad \lambda^{-2}m\rho^{-1} \leq (v(\rho))^{-1}v'(\rho)$$

holds. The conclusion of 5 is now easily obtained by integrating $(*)$.

6. THEOREM. *Suppose Φ and Ψ are class 2 elliptic integrands and $\psi_0 \leq \psi < \psi_1$. If $T \in \mathfrak{R}_{m,A}(Z)$ satisfies*

$$\operatorname{spt}(R - \partial T) \subset B, \quad \langle \Psi, T \rangle = \psi,$$
$$\langle \Phi, T \rangle = \inf\{\langle \Phi, S \rangle : S \in \mathfrak{R}_{m,A}(Z), \operatorname{spt}(R - \partial S) \subset B, \langle \Psi, S \rangle = \psi\},$$

then the set of regular points is dense in

$$(\operatorname{Int} A \cap \operatorname{spt} T) \sim (B \cup \operatorname{spt} \partial T).$$

Proof. The theorem follows from 3, 4.2, 5 and the main result of [**BE**].

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