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PRIME DIVISORS, ANALYTIC SPREAD AND FILTRATIONS

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We show that a Noetherian ring R is locally quasi-unmixed if and only if for every prime ideal $P \in \hat{A}^*(I)$, $\mathrm{ht}(P) = l(IR_P)$. The analytic spread of an e.p.f., $l(f)$ is also defined and many of the known results for the integral closures of powers of an ideal are proven for the weak integral closures of the ideals in a strong e.p.f. Several characterizations are given of when a Noetherian ring R is locally quasi-unmixed in terms of analytic spreads and integral closure of ideals. Several applications of these equivalences are given by showing when certain prime ideals are in $\hat{A}^*(f)$.

1. Introduction. Throughout this paper all rings will be assumed to be commutative with 1.

We will show some relationships between the analytic spread of an ideal and its prime divisors. The influence of these relationships on the behavior of $\hat{A}^*(I)$ (see (2.2.3) for the definition) is also studied. In (2.6) it is shown that if $f = \{I_n\}$ is an essentially powers filtration (e.p.f., see (2.1.2)), in a Noetherian ring, then for a fixed m', $l(I_{m'}) = l(f)$ (for all $n \geq 1$). (See (2.1.6) for the definition of l.) Our main Theorem (2.8), shows that a Noetherian ring R is locally quasi-unmixed if and only if for each ideal I in R, $P \in \hat{A}^*(I)$ implies $\mathrm{ht}(P) = l(IR_P)$. This is extended in (2.9) to e.p.f.'s. In (2.10)–(2.15), several equivalences are given to a Noetherian ring being locally quasi-unmixed; these are in terms of analytic spreads and integral closures of ideals. Several applications of these equivalences are given in (2.16)–(2.20) by showing certain prime ideals are in $\hat{A}^*(f)$.

2. Analytic spreads. We now give the definitions of the basic terms used in this paper.

(2.1) DEFINITION. Let R be a ring.

(2.1.1) A decreasing sequence of ideals $f = \{I_n\}_{n \geq 0}$ is said to be a *filtration* in case $I_0 = R$ and for all m and n, $I_n I_m \subseteq I_{n+m}$.

(2.1.2) (cf.) [**1**, Definition 2.14]. A filtration $f = \{I_n\}$ is said to be an *essentially powers of an ideal filtration* (*e.p.f.*) in case there exists $k > 0$ such that $I_n = \sum_1^k I_{n-i} I_i$, for all $n \geq 1$, where $I_i = R$ if $i \leq 0$.

(2.1.3) Let I be an ideal in R. Then the Rees ring of R *with respect to* I, $\mathcal{R}(R, I)$, is the ring $\mathcal{R}(R, I) = R[u, tI]$ where t is an indeterminant and $u = 1/t$. If $f = \{I_n\}$ is a filtration, then the *generalized Rees ring of R with respect to f*, $\mathcal{R}(R, f)$, is the ring $\mathcal{R}(R, f) = R[u, tI_1, t^2I_2, \ldots]$ (t, u, as above).

(2.1.4) The *integral closure* of an ideal I in R is $\{x \in R \mid x^n + b_1x^{n-1} + \cdots + b_ix^{n-i} + \cdots + b_n = 0$, where $b_i \in I^i\}$. This set will be denoted by I_a.

(2.1.5) The *weak integral closure* of an ideal I_k in a filtration $f = \{I_n\}$ is $\{x \in R \mid x^n + b_1x^{n-1} + \cdots + b_ix^{n-i} + \cdots + b_n = 0$, where $b_i \in I_{ki}\}$. This set will be denoted by $(I_k)_\alpha$.

(2.1.6) Let I be an ideal in R. Then the *analytic spread* of I, denoted $l(I)$, is defined as follows:

$$l(I) \equiv \max\{\operatorname{alt}(\mathcal{R}(R, I)/(u, M)\mathcal{R}(R, I)); \quad M \text{ is a maximal ideal in } R\}.$$

(2.1.7) Let $F = \{I_n\}$ be an e.p.f. in R. Then the *analytic spread* of f, denoted $l(f)$, is defined to be:

$$l(f) \equiv \max\{\operatorname{alt}(\mathcal{R}(R, f)/(u, M)\mathcal{R}(R, f)); \quad \text{where } M \text{ is a maximal ideal in } R\}.$$

(2.1.8) If I is an ideal in R, then $V(I)$ denotes the smallest number of elements that generate I.

(2.1.9) A local ring R is said to be *quasi-unmixed* in case, for every minimal prime ideal z in the completion R^* of R, $\operatorname{depth}(z) = \operatorname{alt}(R)$. A Notherian ring R is said to be *locally quasi-unmixed* in case, for each prime ideal P in R, R_P is quasi-unmixed.

(2.1.10) An ideal I in R is said to be of the principal class in case $V(I) = \operatorname{ht}(I)$.

(2.1.11) An ideal I is said to be *integrally dependent* on an ideal B in case, $B \subseteq I \subseteq B_a$. An ideal I is said to be weakly integrally dependent on an ideal B_n in a filtration $f = \{B_n\}$ in case, $B_n \subseteq I \subseteq (B_n)_\alpha$

(2.2) REMARK. We will now state a few results on filtrations. Throughout, R is a ring and $f = \{I_n\}$ is a filtration in R. The proofs are given in the cited references.

(2.2.1) [**1**, Proposition 2.18 and Theorem 2.20] and [**10**, (2.4.3)]. If R is Noetherian, then f is an e.p.f. if and only if there exist an integer $m > 0$ such that for all $j \geq m$, $I_{m+j} = I_mI_j$.

(2.2.2) [**6**, Corollary (2.8)]. Let R be Noetherian. Fix $k > 0$. If f satisfies the condition:

($*$) For all integers $m \geq m'$ and $j > m'$, $I_{m+j} = I_m I_j$, then the sets $B_k(n) = \text{Ass}(I_n/I_{n+k})$ are equal for all large n.

(2.2.3) [**6**, Corollary (2.8), Proposition (2.14) and Proposition (3.7)] and [**2**, Corollary 5]. If f is a strong e.p.f. (in particular if f is powers of an ideal) in a Noetherian ring R, then the sets $\hat{A}(n) = (\text{Ass}(R/I_n)_\alpha)$ ($= \text{Ass}(R/(I_n)_\alpha)$ for powers of an ideal) and $A(n) = \text{Ass}(R/I_n)$ are constant for all large n. We denote the constant values by $\hat{A}^*(f)$ and $A^*(f)$ respectively. If $f = \{I^n\}$ then we will denote the sets by $\hat{A}^*(I)$ and $A^*(I)$ respectively.

(2.2.4) [**5**, Corollary (3.9)]. Let R be a Noetherian ring and let $f = \{I_n\}$ be a strong e.p.f. Then for all $n \geq 1$, $\hat{A}^*(I_{nm'}) = \hat{A}^*(f)$.

(2.3) Remark. Let R be a Noetherian ring and let $f = \{I_n\}$ be an e.p.f. in R. The smallest integer m satisfying $I_{m+j} = I_m I_j$ (see (2.2.1)) will be of particular importance throughout the paper. This integer will be denoted by m'. A filtration satisfying condition ($*$) in (2.2.2) will be denoted as a strong e.p.f. The powers of an ideal $f = \{I^n\}$ is a strong e.p.f. with $m' = 1$.

Our first lemma proves a useful inequality relating the analytic spread of an ideal I to the minimal number of generators of I. This inequality will be used throughout the chapter.

(2.4) Lemma. *Let R be a Noetherian ring, let I, P be ideals in R such that P is prime, and let $\mathfrak{R} = \mathfrak{R}(R_P, IR_P)$. Then, $l(IR_P) \leq V(IR_P) \leq V(I)$* (see (2.1.6) and (2.1.8)).

Proof. Since R_P is local, $l(IR_P) = \text{alt}(\mathfrak{R}/(u, PR_P)\mathfrak{R})$. However,

$$\mathfrak{R}/(u, PR_P)\mathfrak{R} \cong (R_P/PR_P)[t(IR_P + (u, PR_P)/(u, PR_P))],$$

so

$$\text{alt}(\mathfrak{R}/(u, PR_P)\mathfrak{R}) \leq V(IR_P).$$

Finally, it is clear that $V(IR_P) \leq V(I)$, so $l(IR_P) \leq V(IR_P) \leq V(I)$. □

The next lemma shows that the Rees ring is integral over a certain subring. This will be used in the proof of (2.6).

(2.5) Lemma. *Let R be a Noetherian ring and let $f = \{I_n\}$ be an e.p.f. in R. Let m be large enough so that $\mathfrak{R}(R, f) = R[u, tI_1, \ldots, t^m I_m]$ and let n be an arbitrary positive integer. Then $\mathfrak{R}(R, f)$ is integral over $R[u^{nm'}, t^{nm'} I_{nm'}]$.*

Proof. It is sufficient to show that the elements of t^iI_i are integral over $R[u^{nm'}, t^{nm'}I_{nm'}]$, for $1 \leq i \leq m$. For this, let $at^i \in t^iI_i$. Then $(at)^{nm'} = a^{nm'}t^{inm'}$ with $a^{nm'} \in (I_i)^{nm'} \subseteq I_{inm'} = (I_{nm'})^i$. Thus $(at^i)^{nm'} \in (t^{nm'}I_{nm'})^i$. This shows that at^i is integral over$R[u^{nm'}, t^{nm'}I_{nm'}]$. Therefore, t^iI_i is integral over $R[u^{nm'}, t^{nm'}I_{nm'}]$ for $1 \leq i \leq m$, and so $\mathfrak{R}(R, f)$ is integral over $R[u^{nm'}, t^{nm'}I_{nm'}]$. □

We now show that for a local ring, the analytic spread of an ideal of the form $I_{nm'}$ in an e.p.f. $f = \{I_n\}$ is equal to the analytic spread of f.

(2.6) LEMMA. *Let (R, M) be a local ring and let $f = \{I_n\}$ be an e.p.f. in R. Then for all positive integers n, $l(I_{nm'}) = l(f)$.*

Proof. Let $\mathfrak{R} = \mathfrak{R}(R, f)$ and let $\mathcal{O} = \mathrm{R}[u^{nm'}, t^{nm'}I_{nm'}]$, so $\mathfrak{R}$ is integral over $\mathcal{O}$, by (2.5). By definition, $l(f) = \mathrm{depth}(M, u)\mathfrak{R}$ and

$$l(I_{nm'}) = \mathrm{depth}(M, u)R[u, tI_{nm'}] = \mathrm{depth}(M, u^{nm'})\mathcal{O}.$$

Let P be a minimum prime divisor of $(u, M)\mathfrak{R}$ such that $\mathrm{depth}(P) = l(f)$. Then $l(f) = \mathrm{alt}(\mathfrak{R}/P) = \mathrm{alt}(\mathcal{O}/(P \cap \mathcal{O}))$, since $\mathfrak{R}$ is integral over $\mathcal{O}$. But $(M, u^{nm'})\mathcal{O} \subseteq P \cap \mathcal{O}$, so

$$\mathrm{alt}(\mathcal{O}/ (P \cap \mathcal{O})) = \mathrm{depth}(P \cap \mathcal{O}) \leq \mathrm{depth}(M, u^{nm'})\mathcal{O}' = l(K_{nm'}).$$

Thus $l(f) \leq l(I_{nm'})$. Now pick a minimal prime divisor Q of $(M, u^{nm'})\mathcal{O}$ such that $\mathrm{depth}(Q) = l(I_{nm'})$. Let Q' be a prime ideal in $\mathfrak{R}$ such that $Q' \cap \mathcal{O} = Q$. Then $Q' \supseteq (M, u)\mathfrak{R}$, and so

$$\begin{aligned} l(I_{nm'}) &= \mathrm{alt}(\mathcal{O}/Q) = \mathrm{alt}(\mathfrak{R}/Q') = \mathrm{depth}(Q') \\ &\leq \mathrm{depth}(M, u)\mathfrak{R} = l(f). \end{aligned}$$

Thus $l(I_{nm'}) \leq l(f)$, and so $l(I_{nm'}) = l(f)$. □

(2.7) REMARK. With the notation of (2.6), it would be interesting to know if $l(I_n) = l(f)$, for all $n \geq 1$.

The main theorem in this section shows that, for an ideal I in a locally quasi-unmixed Noetherian ring R, if $P \in \hat{A}^*(I)$, then $\mathrm{ht}(P) = l(IR_P)$. This is a generalization of a result proved by Ratliff, for integral domains satisfying the altitude formula.

(2.8) THEOREM. (Cf. [**12**, Theorem 1].) *Let R be a Noetherian ring. R is locally quasi-unmixed if and only if for every ideal I in R and for every prime ideal $P \in \hat{A}^*(I)$, $\mathrm{ht}(P) = l(IR_P)$.*

Proof. Assume that for every ideal I in R, if $P \in \hat{A}^*(I)$, then $\mathrm{ht}(P) = l(IR_P)$. Then to show that R is locally quasi-unmixed it is sufficient to show that for each ideal I of the principal class, $(I^n)_a$ is height unmixed for every integer $n \geq 1$, by [**8**, (2.29)]. For this, let I be an ideal of the principal class and let P be a prime divisor of $(I^n)_a$ for some $n \geq 1$. This implies $P \in \hat{A}^*(I)$, by [**6**, (3.6.1)] and (2.2.3), so by the hypothesis and (2.4),

$$\mathrm{ht}(P) = l(IR_P) \leq V(IR_P) \leq V(I) = \mathrm{ht}(I) \leq \mathrm{ht}(P).$$

This proves the implication.

For the reverse implication, assume R is locally quasi-unmixed. Let I be an ideal in R and assume $P \in \hat{A}^*(I)$. Then, with $\mathfrak{R} = \mathfrak{R}(R, I)$, there exists a prime divisor Q' of $u\mathfrak{R}'$ such that $Q' \cap R = P$, by [**11**, Corollary 3]. Since Q' contains the regular element u, $\mathrm{ht}(Q') = 1$. Let z' be a minimal prime divisor in $\mathfrak{R}'$ such that $z' \subset Q'$, let $Q = Q' \cap \mathfrak{R}$, and let $z = z' \cap \mathfrak{R}$. Also, let $z_0 = z \cap R$. We now show that $\mathrm{ht}(Q/z) = 1$.

For this, since $z = z_0 R[t, u] \cap \mathfrak{R}$, we have

$$\mathfrak{R}/z = \mathfrak{R}/\left(z_0 R[t, u] \cap \mathfrak{R}\right) \cong \mathfrak{R}\left(R/z_0, (I + z_0)/z_0\right),$$

by [**13**, Lemma 1.1]. Therefore $\mathfrak{R}/z$ satisfies the altitude formula, since R/z_0 does (since R is locally quasi-unmixed). Thus, since $(Q'/z') \cap (\mathfrak{R}/z) = Q/z$, $\mathrm{ht}(Q'/z') = \mathrm{ht}(Q/z)$, by [**7**, Theorem 3.8], so $\mathrm{ht}(Q/z) = 1$.

Now, since R/z_0 satisfies the altitude formula,

$$\begin{aligned}\mathrm{ht}(Q/z) &+ \mathrm{trd}(((\mathfrak{R}/z)/(Q/z))/((R/z_0)/(P/z_0))) \\ &= \mathrm{ht}(P/z_0) + \mathrm{trd}((\mathfrak{R}/z)/(R/z_0)).\end{aligned}$$

Now

$$\mathrm{ht}(Q/z) = 1 \quad \text{and} \quad \mathrm{trd}((\mathfrak{R}/z)/(R/z_0)) = 1,$$

since $\mathfrak{R}/z \cong \mathfrak{R}(R/z_0, (I + z_0)/z_0)$, so

$$\mathrm{ht}(P/z_0) = \mathrm{trd}((\mathfrak{R}/Q)/(R/P)).$$

Since R is locally quasi-unmixed, $\mathrm{ht}(P) = \mathrm{ht}(P/z_0)$. Let $S = R - P$, so $\mathfrak{R}_S = \mathfrak{R}(R_P, IR_P)$. Then

$$\begin{aligned}\mathrm{alt}(\mathfrak{R}_P) - 1 = \mathrm{alt}(R_P) &= \mathrm{ht}(P) = \mathrm{trd}((\mathfrak{R}/Q)/(R/P)) \\ &= \mathrm{trd}\left((\mathfrak{R}_S/Q\mathfrak{R}_S)/(R_P/PR_P)\right).\end{aligned}$$

However, $R_P/PR_P \cong (R/P)_{P/P}$ and

$$\mathfrak{R}_S/Q\mathfrak{R}_S = \mathfrak{R}_{(R-P)}/Q\mathfrak{R}_{(R-P)} \cong (\mathfrak{R}/Q)_{(R/P)-(P/P)};$$

so

$$\text{ht}(P) = \text{trd}((\mathfrak{R}_S/Q\mathfrak{R}_S)/(R_P/PR_P)) = \text{alt}(\mathfrak{R}_S/Q\mathfrak{R}_S),$$

by [**4**, (14.6)],

$$\begin{aligned} &= \text{depth}\, Q\mathfrak{R}_S \\ &\le \text{depth}(P, u)\mathfrak{R}_S \\ &= l(IR_P) \le \text{alt}\, R_P = \text{ht}(P). \end{aligned}$$

Therefore $\text{ht}(P) = l(IR_P)$. □

The next corollary extends (2.8) to the case of an arbitrary e.p.f.

(2.9) COROLLARY. *Let R be a Noetherian ring. Then R is locally quasi-unmixed if and only if for every strong e.p.f. $f = \{I_n\}$ in R and for every prime ideal $P \in \hat{A}^*(f)$,* $\text{ht}(P) = l(fR_P)$.

Proof. Let R be locally quasi-unmixed and let $f = \{I_n\}$ be a strong e.p.f. in P. Assume $P \in \hat{A}^*(f)$. By (2.2.4) $\hat{A}^*(f) = \hat{A}^*(I_{nm'})$ for every integer $n > 0$. From (2.8), $\text{ht}(P) = l(I_{nm'}, R_P)$. However, (2.6) shows that $l(I_{nm'}, R_P) = l(fR_P)$. Thus $\text{ht}(P) = l(fR_P)$.

For the converse, let I be an ideal in R and let $f = \{I^n\}$. Assume $P \in \hat{A}^*(f)$, so $\text{ht}(P) = l(fR_P)$. By (2.6), $l(fR_P) = l(IR_P)$. Thus $\text{ht}(P) = l(IR_P)$, so, by (2.8), R is locally quasi-unmixed. □

Our next result, (2.10), together with (2.12) extends a theorem of Ratliff, (cf. [**12**, Theorem 1]) from the case of an integral domain satisfying the altitude formula, to that of a locally quasi-unmixed Noetherian ring. In (2.11) and (2.13), this is further extended by consideration of arbitrary e.p.f.'s

(2.10) PROPOSITION. *The following statements are equivalent for a Noetherian ring R:*

(2.10.1) *R is locally quasi-unmixed.*

(2.10.2) *If $I \subseteq P$ are ideals in R such that $P \in \hat{A}^*(I)$, then* $\text{ht}(P) = l(IR_P)$.

(2.10.3) *If I is an ideal in R and P is a prime divisor of $(I^n)_a$ for some $n \ge 1$, then* $\text{ht}(P) = l(IR_P)$.

(2.10.4) *If I is an ideal in R such that* $\text{ht}(I) = V(I)$, *then $(I^n)_a$ is height unmixed for every integer $n \ge 1$.*

Proof. (2.10.1) is equivalent to (2.10.2) by (2.8). (2.10.2) and (2.10.3) are equivalent, by [**9**, (2.5)]. Finally, (2.10.1) and (2.10.4) are equivalent by [**8**, (2.29)], □

(2.11) COROLLARY. *The following statements are equivalent for a Noetherian ring* R;

(2.11.1) R *is locally quasi-unmixed.*

(2.11.2) *For every strong e.p.f.* $f = \{I_n\}$ *in* R, *if* P *is a prime divisor of* $(I_n)_\alpha$ *for some* $n \geq 1$, *then* $\mathrm{ht}(P) = l(fR_P)$.

(2.11.3) *For every strong e.p.f.* $f = \{I_n\}$ *in* R *and for each fixed* $n \geq 1$, *if* $\mathrm{ht}(I_{nm'}) = V(I_{nm'})$, *then* $(I_{nm'})_\alpha$ *is height unmixed.*

Proof. By [**6**, (3.6.1)] if f is an e.p.f. in R and if $P \in \hat{A}(k)$ for some $k > 0$, then $P \in \hat{A}(n)$ for all large n. Thus if $P \in \hat{A}(n)$ for some $n > 1$, then $P \in \hat{A}^*(f)$. Therefore, by (2.9), (2.11.1) and (2.11.2) are equivalent.

Next we show (2.11.2) implies (2.11.3). So assume (2.11.2) holds, let f be a strong e.p.f. in R, fix $n \geq 1$, let $\mathrm{ht}(I_{nm'}) = v(I_{nm'})$, and let $P \in \mathrm{Ass}((I_{nm'})_\alpha)$. Then, by (2.11.2), $\mathrm{ht}(P) = l(fR_P)$, and (2.6) implies that $l(fR_P) = l(I_{nm'}R_P)$. Now we have

$$\mathrm{ht}(P) = l(I_{nm'}R_P) \leq v(I_{nm'}) = \mathrm{ht}(I_{nm'}) \leq \mathrm{ht}(P),$$

since $l(I_{nm'}R_P) \leq v(I_{nm'})$, by (2.4). Therefore $\mathrm{ht}(P) = \mathrm{ht}(I_{nm'})$ and so $(I_{nm'})_\alpha$ is height unmixed. This shows (2.11.2) implies (2.11.1).

Finally, let I be an ideal in R such that $\mathrm{ht}(I) = v(I)$ and let $f = \{I^n\}$. Then $m' = 1$, so (2.11.3) says that $(I^n)_a = (I^n)_\alpha$ is height unmixed. Thus, by (2.10.4) ⇒ (2.10.1), R is locally quasi-unmixed, and so (2.11.3) ⇒ (2.11.1). □

It will now be shown that the equivalent conditions in (2.10) imply the converse of (2.10.2). As already noted, this generalizes [**13**, Theorem 1].

(2.12) PROPOSITION. *Let* R *be a locally quasi-unmixed Noetherian ring and let* $I \subseteq P$ *be ideals in* R *with* P *prime. If* $\mathrm{ht}(P) = l(IR_P)$, *then* $P \in \hat{A}^*(I)$.

Proof. The proof is the same as the proof for the corresponding result in [**13**, Theorem 1], except for the last sentence: Then Q is a prime divisor of $(u\mathfrak{R})_a$, so the proof of (2.8.1) shows Q is a prime divisor of $(I_n)_\alpha = (I^n)_a$ for all large n, and so $P \in \hat{A}^*(I)$, by (2.2.3). □

We now extend (2.12) to the case of an arbitrary e.p.f.

(2.13) COROLLARY. *Let R be a locally quasi-unmixed Noetherian ring, let $f = \{I_n\}$ be a strong e.p.f. in R, and let P be a prime ideal in R such that $f \subseteq P$. If* $\mathrm{ht}(P) = l(fR_P)$, *then $P \in \hat{A}^*(f)$.*

Proof. Let $\mathrm{ht}(P) = l(fR_P)$. Then, by (2.6), $\mathrm{ht}(P) = l(I_{nm'}RP)$ for all integers $n \geq 1$. Therefore, by (2.12), $P \in \hat{A}^*(I_{nm'})$ for all $n \geq 1$, so $P \in \hat{A}^*(f)$ by (2.2.4). □

(2.14) gives three more conditions which are equivalent to a Notherian ring R being locally quasi-unmixed. (2.14) generalizes [**12**, Theorem 2] from the case of a Noetherian domain to that of a locally quasi-unmixed Noetherian ring.

(2.14) PROPOSITION. *The following statements are equivalent for a Noetherian ring R.*

(2.14.1) *R is locally quasi-unmixed.*

(2.14.2) *If $I \subseteq M$ are ideals in R such that M is maximal,* $\mathrm{ht}(I) = V(I)$ *and* $\mathrm{ht}(M/I) = 1$, *then, for every $n \geq 1$, M is not a prime divisor of $(I^n)_a$.*

(2.14.3) *If $I \subseteq M$ are ideals in R with M maximal such that* $\mathrm{ht}(M/I) = 1$ *and if I is integrally dependent on an ideal B of the principal class, then, for every integer $n \geq 1$, M is not a prime divisor of $(I^n)_a$.*

(2.14.4) *If I is an ideal in R which can be generated by h elements, then, for every integer $n \geq 1$, $(I^n)_a$ has no prime divisors of height strictly greater than h.*

Proof. (2.14.1) and (2.14.2) are equivalent by [**8**, (2.29)]. It is clear that (2.14.3) implies (2.14.2). We now show that (2.14.2) implies (2.14.3). For this, let $I \subseteq M$ be ideals in R such that $\mathrm{ht}(M/I) = 1$ and assume B is an ideal of the principal class such that I is integrally dependent on B. It suffices to show that M is not a prime divisor of $(B^n)_a$ for $n \geq 1$, since $B \subseteq I \subseteq B_a$ implies $(B^n)_a = (I^n)_a$ for all $n \geq 1$. To insure that $B \subseteq I \subseteq M$ satisfies the hypothesis of (2.14.2), we need only show that $\mathrm{ht}(M/B) = 1$. But this holds, since $B \subseteq I \subseteq (B)_a \subseteq \mathrm{rad}(B)$. Thus (2.14.2) implies (2.14.3).

Next it will be shown that (2.14.1) and (2.14.4) are equivalent. To see that (2.14.4) implies (2.14.1), it suffices, by [**8**, (2.29)], to show that for every ideal I of the principal class, $(I^n)_a$ is height unmixed for $n \geq 1$. Let I be an ideal in R such that $\mathrm{ht}(I) = V(I)$ and let P be a prime divisor of

$(I^n)_a$ for some $n \geq 1$. Then, by (2.14.4), $h = V(I) = \mathrm{ht}(I) = \mathrm{ht}(I^n)_a \leq \mathrm{ht}(P) \leq h$, so $\mathrm{ht}(P) = V(I)$ and $(I^n)_a$ is height unmixed. Thus R is locally quasi-unmixed, so (2.14.4) implies (2.14.1).

Finally, assume R is locally quasi-unmixed and let I be an ideal in R such that I is generated by h elements. Let P be a prime divisor of $(I^n)_a$ for some integer $n \geq 1$. Then by (2.10.1) $\Leftrightarrow$ (2.10.3), $\mathrm{ht}(P) = l(IR_P) \leq V(IR_P) \leq V(I) \leq h$. Thus (2.14.1) implies (2.14.4). □

(2.15) generalizes (2.14) to the case of an arbitrary e.p.f..

(2.15) COROLLARY. *The following statements are equivalent for a Noetherian ring R:*

(2.15.1) *R is locally quasi-unmixed.*

(2.15.2) *If $f = \{I_n\}$ is an e.p.f. in R and M is a maximal ideal in R such that $f \subseteq M$, $\mathrm{ht}(I_{m'}) = v(I_{m'})$, and if $\mathrm{ht}(M/I_{m'}) = 1$, then, for all large n, M is not a prime divisor of $(I_n)_\alpha$.*

(2.15.3) *If $f = \{I_n\}$ is a strong e.p.f. in R and M is a maximal ideal in R such that $f \subseteq M$, $\mathrm{ht}(M/I_{m'}) = 1$, and $I_{m'}$ is weakly integrally dependent on an ideal B of the principal class, then, for all large n, M is not a prime divisor of $(I_n)_\alpha$.*

(2.15.4) *If $f = \{I_n\}$ is a strong e.p.f. in R such that $I_{m'}$ is generated by h elements, then, for all large n, $(I_n)_\alpha$ has no prime divisors of height strictly greater than h.*

Proof. Assume R is locally quasi-unmixed, and let f and M satisfy the hypothesis of (2.15.2). Then, by (2.14.1) $\Rightarrow$ (2.14.2), M is not a prime divisor of $(I_{km'})_\alpha = ((I_{m'})^k)_a$. But, for large k and n, $\mathrm{Ass}((I_{km'})_a) = \mathrm{Ass}((I_n)_\alpha) = \hat{A}^*$, by (2.2.4). Thus (2.15.1) implies (2.15.2).

(2.15.2) implies (2.15.1), by (2.14.2) implies (2.14.1). It is clear that (2.15.3) implies (2.15.2). For the converse, let M, f, and B be as in (2.15.3). Then, by (2.14.3), M is not a prime divisor $((I_{m'})^k)_a = (I_{km'})_\alpha$, for $k \geq 1$. Since, by (2.2.3) and (2.2.4), $\mathrm{Ass}((I_{km'})_\alpha) = \mathrm{Ass}((I_n)_\alpha) = \hat{A}^*$, for large n and k, M is not a prime divisor of $(I_n)_\alpha$ for all large n. Therefore (2.15.2) implies (2.15.3).

(2.15.4) implies (2.15.1), by, (2.11.3) $\Rightarrow$ (2.11.1). To show (4.13.1) implies (2.15.4), assume that R is locally quasi-unmixed and let $f = \{I_n\}$ be an e.p.f. in R such that $I_{m'}$ is generated by h elements. Then $((I_{m'})^k)_a = (I_{m'k})_\alpha$ has no prime divisor of height $> h$, by (2.14.1) $\Rightarrow$ (2.14.4). Therefore, since $(I_n)_\alpha$ and $(I_{m'k})_\alpha$ have the same prime divisors for large n and k, by (3.9), (2.15.1) $\Rightarrow$ (2.15.4). □

(2.16.1) is a corollary to (2.8) and (2.12). (2.16.2) gives a condition for the analytic spread of an ideal to be equal to its height. (2.16) is extended to the case of an e.p.f. in (2.17).

(2.16) PROPOSITION. (C.f. [**12**, *Corollary* 4].) *Let R be a locally quasi-unmixed Noetherian ring and let $I \subseteq P$ be ideals in R such that P is a prime ideal. Then the following statements are true*:

(2.16.1) $P \in \hat{A}^*(I)$ *if and only if* $\mathrm{ht}(P) = l(IR_P)$

(2.16.2) *Assume R is local with maximal ideal P. If* $\mathrm{ht}(P/I) = 1$, *then* $\mathrm{ht}(I) = l(I) = \mathrm{alt}(R) - 1$ *if and only if* $P \notin \hat{A}^*(I)$.

Proof. For (2.16.1), if $P \in \hat{A}^*(I)$, then $\mathrm{ht}(P) = l(IR_P)$, by (2.8), and if $\mathrm{ht}(P) = l(IR_P)$ then (2.12) implies $P \in \hat{A}^*(I)$.

To prove (2.16.2), assume $\mathrm{ht}(P/I) = 1$. Let $\mathrm{ht}(I) = l(I) = \mathrm{alt}(R) - 1$ and suppose $P \in \hat{A}^*(I)$. Then, by (2.16.1), $\mathrm{ht}(P) = l(I)$. But $\mathrm{ht}(P) > \mathrm{ht}(I) = l(I)$, so this contradiction shows $P \notin \hat{A}^*(I)$. For the converse, assume $p \notin \hat{A}^*(I)$. Suppose $\mathrm{depth}(I) = 1$, $\mathrm{ht}(I) = \mathrm{alt}(R) - 1$, by [**4**, (34.5)]. Also, $\mathrm{ht}(P) > l(I)$, by hypothesis and (2.12), so P is the maximal ideal in R. Thus $l(I) \leq \mathrm{ht}(P) - 1 = \mathrm{ht}(I)$. But, there exists a prime divisor Q of I such that $\mathrm{ht}(Q) = \mathrm{ht}(I)$. Then $Q \in \hat{A}^*(I)$, and so $\mathrm{ht}(I) = \mathrm{ht}(Q) = l(IR_Q) \leq l(I)$ by ([**12**, comments preceding Corollary 9]) $\leq \mathrm{ht}(I)$. Therefore $\mathrm{ht}(I) = l(I)$. Therefore $\mathrm{ht}(I) = l(I) = \mathrm{alt}(R) - 1$, so (2.16.2) holds. □

(2.17) COROLLARY. *Let R be a locally quasi-unmixed Noetherian ring, let $f = \{I_n\}$ be a strong e.p.f. in R and let P be a prime ideal in R such that $I_1 \subseteq P$. Then the following statements are true*:

(2.17.1) $P \in \hat{A}^*(f)$ *if and only if* $\mathrm{ht}(P) = l(fR_P)$.

(2.17.2) *Assume R is local with maximal ideal P. If* $\mathrm{ht}(P/I_{m'}) = 1$, *then* $\mathrm{ht}(f) = l(f) = \mathrm{alt}(R) - 1$ *if and only if* $P \notin \hat{A}^*(f)$.

Proof. (2.17.1) is immediate from (2.9) and (2.13).

For (2.17.2), assume $\mathrm{ht}(P/I_{m'}) = 1$ and let $\mathrm{ht}(f) = l(f) = \mathrm{alt}(R) - 1$. Then, since $\mathrm{ht}(f) = \mathrm{ht}(I_{m'})$ and $l(f) = l(I_{m'})$ (by (2.6)), $P \notin \hat{A}^*(I_{m'})$, by (2.8), and so $P \notin \hat{A}^*(f)$. For the converse, assume $P \notin \hat{A}^*(f)$, so $P \notin \hat{A}^*(I_{m'})$, by (2.2.3). By (2.16.2), $\mathrm{ht}(I_{m'}) = l(I_{m'}) = \mathrm{alt}(R) - 1$, and (2.17.2) readily follows from this. □

We conclude this section with three interesting applications of (2.8) and (2.14).

(2.18) COROLLARY. (Cf. [**12**, *Corollary* 6].) *Let R be a Noetherian ring and let M be a maximal ideal in R. Then R_M is quasi-unmixed if and only if there does not exist a system of parameters $b_1,\ldots,b_d$ in R_M such that MR_M is a prime divisor of $((b_1,\ldots,b_{d-1})^n)_a$ for some integer $n > 0$.*

Proof. Assume R_M is quasi-unmixed (hence locally quasi-unmixed), and let $b_1,\ldots,b_d$ be a system of parameters in R_M. Then $\mathrm{ht}(b_1,\ldots,b_{d-1}) = d-1$, by [**3**, (12.K)] and [**4**, (34.5)]. Therefore, for each $n \geq 1$, MR_M is not a prime divisor of $((b_1,\ldots,b_{d-1})^n)a$ by (2.14.1) $\Rightarrow$ (2.14.2).

For the reverse implication assume there does not exist a system of parameters $b_1,\ldots,b_d$ in R_M such that MR_M is a prime divisor of $((b_1,\ldots,b_{d-1})^n)_a$ for some $n > 0$. Then it readily follows from (2.14.2) $\Rightarrow$ (2.14.1) that R_M is quasi-unmixed. □

(2.19) is essentially a restatement of (2.18).

(2.19) COROLLARY. *Let R be a Noetherian ring and let M be a maximal ideal in R. Then R_M is quasi-unmixed if and only if there does not exist a system of parameters $b_1,\ldots,b_d$ in R_M such that $MR_M \in \hat{A}^*(b_1,\ldots,b_{d-1})$.*

Proof. This follows immediately form (2.18) and [**9**, (2.5)]. □

This chapter will be closed with the following corollary.

(2.20) COROLLARY. *Let (R, M) be a quasi-unmixed local ring, let* $\mathrm{alt}(R) = d \geq 1$, *and let I be an ideal in R such that* $\mathrm{ht}(I) < d$. *Then the following statements are true*:

(2.20.1) *$M \notin \hat{A}^*(I)$ if and only if $l(I) < d$.*

(2.20.2) *If there exists a prime ideal $P \in \hat{A}^*(I)$ such that* $\mathrm{ht}(I) = \mathrm{ht}(P)$ *and $l(I) = l(IRp)$, then* $\mathrm{ht}(I) = l(I)$.

(2.20.3) *If* $\mathrm{ht}(I) = l(I)$, *then, for each $n \geq 1$, all prime divisors of $(I^n)_a$ have the same height.*

Proof. We first prove (2.20.1). From (2.16.1), $M \notin \hat{A}^*(I)$ if and only if $\mathrm{ht}(M) \neq l(IR_M)$. However, it is clear that $\mathrm{ht}(M) \neq l(IR_M)$ if and only if $l(I) < d$, so (2.20.1) holds.

For (2.20.2), let $P \in \hat{A}^*(I)$ such that $\mathrm{ht}(I) = \mathrm{ht}(P)$ and $l(I) = l(IR_P)$. Then, by (4.6), $\mathrm{ht}(P) = l(IR_p)$, so $\mathrm{ht}(I) = \mathrm{ht}(P) = l(IR_P) = l(I)$, proving (2.10.2).

To see (2.20.3), assume $\mathrm{ht}(I) = l(I)$. Let $n \geq 1$ and let $P \in \mathrm{Ass}((I^n)_a)$. Then $\mathrm{ht}(I) \leq \mathrm{ht}(P) = l(IR_P) \leq l(I) = \mathrm{ht}(I)$. Thus, for every integer $n \geq 1$, $(I^n)_a$ is height unmixed. □

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