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ON THE LIFTING THEORY OF FINITE GROUPS OF LIE TYPE

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Let G be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$ of characteristic $p > 0$, $q = p^a$. Let F be a corresponding Frobenius endomorphism such that $\mathbf{G}^{F^m} = \{g \in G: F^m(g) = g\}$ is a finite group of Lie type for a positive integer m. In this paper we discuss various aspects of the lifting theory of these finite groups.

0. Introduction. The paper is divided into four sections. In §1. N. Kawanaka's norm map is defined and admissible integers are discussed. In §2 the lifting theory of $G_2(q)$ is described. §3 is devoted to liftings of certain principal series representations of groups of adjoint type. Finally, we prove (in §4) that the duality operation defined by C. W. Curtis [7] commutes with lifting.

We use the notation $\mathbf{G}^{F^m} = G(q^m)$, $\operatorname{Irr} H =$ set of irreducible characters of a finite group H, $\overline{\mathbf{F}}_q =$ algebraic closure of $\mathbf{F}_q$, and $A = \langle F|_{G(q^m)} \rangle$. A acts on $G(q^m)$, and is a cyclic group of order m. Embed A and $G(q^m)$ into the semidirect product $A \cdot G(q^m)$. If $\chi \in \operatorname{Irr} G(q^m)$ is F-invariant, it extends to $\chi' \in \operatorname{Irr} A \cdot G(q^m)$. There is a norm map $\mathfrak{N}$ which yields a bijection $\{A \cdot G(q^m)\text{-conjugacy classes in } F \cdot G(q^m)\} \leftrightarrow \{\text{conjugacy classes of } G(q)\}$ (see §1).

DEFINITION. Let $\theta \in \operatorname{Irr} G(q)$. Then $\psi \in (\operatorname{Irr} G(q^m))^F$ is the *lift* of θ if ψ extends to $\psi' \in \operatorname{Irr} A \cdot G(q^m)$ and satisfies $\psi'(Fy) = C\theta(\mathfrak{N}(y))$ for some constant C and for all $y \in G(q^m)$.

In 1976 a paper of T. Shintani [16] was published which described the lifting theory of the finite groups $\mathrm{GL}(n, q)$. This marked the beginning of the lifting theory of finite groups of Lie type. Kawanaka subsequently developed much of the theory in his papers on $\mathrm{U}(n, q)$ [11] and on $\mathrm{Sp}(2n, q)$, $\mathrm{SO}(2n + 1, q)$, and $\mathrm{SO}^{\pm}(2n, q)$ [12]. We will consider the finite exceptional groups other than $G_2(q)$ in future papers (work is in progress).

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1. The norm map and admissible integers. The material in this section can be found in [**12**]. Let $\mathbf{G}$ and F be as above. For a finite group H and an element $h \in H$, let h^H denote the conjugacy class of h in H.

(1.1) LANG'S THEOREM. *The mapping f: $g \to F(g^{-1})g$ of $\mathbf{G}$ into $\mathbf{G}$ is surjective.*

For $x \in G(q)$, let $A_x = Z_{\mathbf{G}}(x)/Z_{\mathbf{G}}(x)^0$. A_x is a finite group. For $a \in A_x$, choose $y \in f^{-1}(r_a)$, where r_a is a representative of a in $Z_{\mathbf{G}}(x)$. Then $yxy^{-1} \in G(q)$, and its $G(q)$-conjugacy class depends only on x and a.

(1.2) DEFINITION. Denote the $G(q)$-conjugacy class of yxy^{-1} by $t_a(x)$ or $t_{r_a}(x)$.

For $y \in G(q^m)$, put $N(y) = y^{F^{m-1}}y^{F^{m-2}} \cdots y^F y$ (where $y^{F^i} = F^i(y)$), and choose $\alpha_y \in f^{-1}(y)$. Then $\alpha_y N(y)\alpha_y^{-1} \in G(q)$, and its $G(q)$-conjugacy class depends only on the class $(Fy)^{A \cdot G(q^m)}$.

(1.3) DEFINITION. Let $n_{\mathbf{G}}$: $\{(Fy)^{A \cdot G(q^m)}: y \in G(q^m)\} \to \{x^{G(q)}: x \in G(q)\}$ be defined by

$$n_{\mathbf{G}}\big((Fy)^{A \cdot G(q^m)}\big) = \big(\alpha_y N(y)\alpha_y^{-1}\big)^{G(q)}$$

for $y \in G(q^m)$. Then $n_{\mathbf{G}}$ is bijective.

(1.4) DEFINITION. For $x \in G(q)$, let $\bar{x} =$ image of x in A_x. Then a positive integer l is admissible for $\mathbf{G}$ and F if $(\text{ord}(x), l) = 1$ for all $x \in G(q)$, where $\text{ord}(\bar{x}) =$ order of $\bar{x}$ in A_x. (It is known (see [**18**]) that the splitting of the class of $x \in \mathbf{G} \cap G(q^m)$ into classes in $G(q^m)$ is in 1-1 correspondence with the elements of $H^1(F^m, Z_{\mathbf{G}}(x)/Z_{\mathbf{G}}(x)^0)$ for any m. Thus the consideration of admissible integers is a natural one for our purposes.)

(*) Unless stated otherwise, we assume m is admissible. Thus, there is a unique element $x(m)$ of $\langle \bar{x} \rangle$ such that $x(m)^m = \bar{x}$.

(1.5) DEFINITION. The map $t_{\mathbf{G}}$: $\dot{G}(q) \to \{x^{G(q)}: x \in G(q)\}$ is given by

$$t_{\mathbf{G}}(x) = t_{x(m)}(x)$$

for $x \in G(q)$.

(1.6) REMARKS.

(i) $t_{\mathbf{G}}$ induces a permutation of the set of conjugacy classes of $G(q)$.

(ii) $t_{\mathbf{G}}$ is the identity map in case $Z_{\mathbf{G}}(x) = Z_{\mathbf{G}}(x)^0$ for all $x \in G(q)$.

(iii) $t_{\mathbf{G}}(x) = x$ in case $x \in Z_{\mathbf{G}}(x)^0$.

(1.7) DEFINITION. Let $\mathfrak{N}_{\mathbf{G}} = t_{\mathbf{G}}^{-1} \circ n_{\mathbf{G}}$ and $\mathfrak{M}_{\mathbf{G}} = \mathfrak{N}_{\mathbf{G}}^{-1}$.

(1.8) THEOREM.

(i)

$$\{x^{G(q)}\colon x \in G(q)\} \underset{\mathfrak{N}_{\mathbf{G}}}{\overset{\mathfrak{M}_{\mathbf{G}}}{\rightleftarrows}} \{(Fy)^{A\cdot G(q^m)}\colon y \in G(q^m)\}$$

are bijections.

(ii) For any $y \in G(q^m)$,

$$\left|(Fy)^{A\cdot G(q^m)}\right| \left|G(q^m)\right|^{-1} = \left|\mathfrak{N}_{\mathbf{G}}\left((Fy)^{A\cdot G(q^m)}\right)\right| \left|G(q)\right|^{-1}.$$

(iii) For $y \in G(q^m)$, if $Z_{\mathbf{G}}(N(y))^0 \ni N(y)\ (= (Fy)^m)$, then

$$\mathfrak{N}_{\mathbf{G}}\left((Fy)^{A\cdot G(q^m)}\right) = n_{\mathbf{G}}\left((Fy)^{A\cdot G(q^m)}\right).$$

In particular, this is the case if $N(y)$ is semisimple or $\mathbf{G} = \mathrm{GL}(n, \overline{\mathbf{F}}_q)$.

(iv) Let $\mathbf{T}$ be an F-stable torus in G, and let $T = \mathbf{T}^{F^m}$. For $t \in T$ we have

$$\mathfrak{N}_{\mathbf{G}}\left((Ft)^{A\cdot G(q^m)}\right) = N(t)^{G(q)}.$$

(v) Let $a \in G(q)$ such that $(\mathrm{ord}(a), m) = 1$. Then

$$\mathfrak{N}_{\mathbf{G}}\left((Fa)^{A\cdot G(q^m)}\right) = (a^m)^{G(q)}.$$

(1.9) Let $\mathbf{H}$ be an F-stable connected algebraic subgroup of $\mathbf{G}$ for which m is admissible. Put $H(q^m) = \mathbf{H}^{F^m}$. Then $\mathfrak{M}_{\mathbf{H}}(h^{H(q)}) \subset \mathfrak{M}_{\mathbf{G}}(h^{G(q)})$ for any $h \in H(q)$.

We will abbreviate $\mathfrak{N}_{\mathbf{G}}$ as $\mathfrak{N}$, and $\mathfrak{M}_{\mathbf{G}}$ as $\mathfrak{M}$.

Definitions (1.2)–(1.5) and (1.7) are due to Kawanaka.

2. The lifting theory of $G_2(q)$. The following theorem holds for $G_2(q)$, $q = p^a$, for sufficiently large p, depending on the rank of $\mathbf{G}_2$ (see **[9]**, **[13]**), and for $p = 2, 3$.

(2.1) THEOREM. *Let m be a positive integer which satisfies $(m, 2) = (m, 3) = 1$. Let $\mathfrak{N}$ be the norm map defined in* (1.7). *Then*

(i) *For any irreducible character χ of $G_2(q)$, there is a unique $\psi_\chi \in (\operatorname{Irr} G_2(q^m))^F$ such that*

$$\chi(\mathfrak{N}(y)) = \varepsilon\delta\psi'_\chi(Fy)$$

for all $y \in G_2(q^m)$, where $\varepsilon = \pm 1$, δ is an mth root of 1 *(both independent of y), and ψ'_χ is an extension of ψ_χ to $A \cdot G_2(q^m)$.*

(ii) *The map $\chi \to \psi_\chi$ gives a bijection between* $\operatorname{Irr} G_2(q)$ *and* $(\operatorname{Irr} G_2(q^m))^F$.

Proof. Notation will be as in **[5]**, **[6]**, **[12]**. We first identify the admissible integers for $G_2(q)$.

(2.2) LEMMA. *The admissible integers for $G_2(q)$ are those integers m satisfying $(m, 2) = (m, 3) = 1$.*

Proof of Lemma (2.2). Assume m is admissible. We consider the cases $p \neq 2, 3$, $p = 2$, and $p = 3$ separately.

(i) $p \neq 2, 3$. If $q \equiv 1 \pmod 3$, consider the element $x = h(\omega, \omega, \omega)y$, where y is a regular unipotent element of $\mathrm{SL}(3, q)$. Then x has order $3p$, and $A_x = Z_{\mathbf{G}_2}(x)/Z_{\mathbf{G}_2}(x)^0$ is isomorphic to $\mathbf{Z}_3$, a cyclic group of order 3. Furthermore $\bar{x}$ has order 3 in A_x. A similar analysis for $q \equiv -1 \pmod 3$, together with the information above, yields $(m, 3) = 1$. To see that we must have $(m, 2) = 1$, consider $x = h(-1, -1, 1)x_b(1)x_c(1)$ if $q \equiv 1 \pmod 3$ (or $x = h(-1, -1, 1)x_b(1)x_c(\lambda)$ if $q \equiv -1 \pmod 3$). Then A_x is isomorphic to a cyclic group of order 2, and $\bar{x}$ has order 2 in A_x.

(ii) $p = 2$. As above we must have $(m, 3) = 1$. Since 2 is a bad prime for $\mathbf{G}_2$, $Z_{\mathbf{G}_2}(x)/Z_{\mathbf{G}_2}(x)^0 \cong \langle x \rangle \cong \mathbf{Z}_2$, a cyclic group of order 2, for a regular unipotent element x. This yields $(m, 2) = 1$.

(iii) $p = 3$. Arguments are the same as in (ii), with the roles of 2 and 3 interchanged.

(2.2) REMARKS.

(i) Since $\mathbf{G}_2$ is simply connected (as an algebraic group), the centralizers of semisimple elements are connected. These elements, then, will not impose restrictions on m.

(ii) For $p \neq 2, 3$ all unipotent elements u are contained in $Z_{\mathbf{G}_2}(u)^0$. Hence these too impose no restrictions on m. In any case, $|Z_{\mathbf{G}_2}(u)/Z_{\mathbf{G}_2}(u)^0|$ is divisible only by the primes 2 and 3.

(iii) Examination of the conjugacy classes of $G_2(q)$ ([5], [6]) shows that it is sufficient that $(m,2)=(m,3)=1$ for an integer m to be admissible. With this observation, the proof of Lemma (2.2) is complete.

Now the remainder of the proof proceeds much as that in [12]. We use a version of Brauer's characterization of characters (see Lemma (1.5) of [11]). After invoking (1.8) (i) and (ii), we see that it is sufficient to prove the following lemma:

(2.3) LEMMA. *Let $E = \langle g\rangle \times S$ be an elementary subgroup of $G_2(q)$. (So $g \in G_2(q)$ and S is an s-group for some prime number s satisfying $(\mathrm{ord}(g), s) = 1$.) Then there exists an F-stable subgroup $\mathbf{H} \subseteq G_2$ with $H(q^m)$ contained in $G_2(q^m)$ and a bijection*

$$\mathfrak{M}_{H(q^m)} = \{x^{H(q)}: x \in H(q)\} \to \{(Fy)^{A\cdot H(q^m)}: y \in H(q^m)\}$$

such that

(i) $E \subset H(q^m)$,

(ii) $\mathfrak{M}_{H(q^m)}(x^{H(q)}) \subset \mathfrak{M}(x^{G_2(q)})$ *for any* $x \in H(q)$,

(iii) *For any irreducible character ϕ of $A\cdot H(q^m)$, $\phi\circ\mathfrak{M}_{H(q^m)}$ is either zero or an irreducible character of $H(q)$ up to a constant multiple in $\mathbf{Z}(e^{2\pi i/m})$. (Note that for any class function η on $A\cdot G_2(q^m)$, $\eta\circ\mathfrak{M}$ is the class function on $G(q)$ defined by $\eta\circ\mathfrak{M}(x) = \eta(m_x)$ for $x\in G(q)$ and $m_x \in (x^{G_2(q)})$.)*

Proof of Lemma (2.3). First assume $p \neq 2, 3$. Let $E = \langle g\rangle\times S$ be an elementary subgroup of $G_2(q)$. Let $g = g_s g_u$ be the Jordan decomposition of g, where g_s is semisimple and g_u is unipotent. Since the center of $\mathbf{G}_2$ is trivial, we have $Z_{\mathbf{G}_2}(x) \lneqq G_2$ for all $x \in \mathbf{G}_2$. In many cases, then, we appeal to the lifting theory of $Z_{\mathbf{G}_2}(x)$. The following cases will be considered:

(1) $g_s \neq 1$,

(2) $g = g_u$, $s = 2$,

(3) $g = g_u$, $s = 3$,

(4) $g = g_u$, $s = p$,

(5) $g = g_u$, $s \neq 2, 3, p$.

Case 1. If $g_s \neq 1$, then $1 \neq Z_{\mathbf{G}_2}(g_s) = Z_{\mathbf{G}_2}(g_s)^0$. In view of the results in [3], [5], [6], [10] it suffices to prove (ii) and (iii) above for $\mathbf{H} = \mathrm{SL}(2,\overline{\mathbf{F}}_q)$, $\mathrm{SL}(3,\overline{\mathbf{F}}_q)$, $\mathrm{GL}(2,\overline{\mathbf{F}}_q)$, and F-stable maximal tori $\mathbf{T} \le \mathbf{G}_2$. (Then $\mathbf{H}^F = \mathrm{SL}(2,q)$, $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q)$, $\mathrm{GL}(2,q)$, $\mathrm{U}(2,q)$.) It is easy to see that all odd integers are admissible for $\mathrm{SL}(2,\overline{\mathbf{F}}_q)$ and that all integers relatively prime to 3 are admissible for $SL(3, \bar{\mathbf{F}}_q)$. It is also straightforward that irreducible characters of $SL(2, q)$ (respectively $SL(3, q)$) always lift to F-invariant irreducibles of $SL(2, q^m)$ (respectively $SL(3, q^m)$) if and only if m is admissible, for q odd. (If $q = 2^a$, irreducibles of $SL(2, q)$ always lift.) All integers are admissible for $GL(2, \bar{\mathbf{F}}_q)$ and F-stable maximal tori $\mathbf{T}$, and the lifting theory for these groups is known. If $\operatorname{ord}(g_s) \neq 2, 3$, $H = Z_{\mathbf{G}_2}(g_s)^0$, $H(q^m) = \mathbf{H}^{F^m}$, and $\mathfrak{M}_{H(q^m)} = \mathfrak{M}_{\mathbf{H}}$ satisfy conditions (i)–(iii) above. In case $\operatorname{ord}(g_s) = 2$, we may use $\mathbf{H} = SL(2, \bar{\mathbf{F}}_q) \circ SL(2, \bar{\mathbf{F}}_q)$ (central product) instead of the full centralizer of g_s, and (i)–(iii) are satisfied with $H(q^m) = SL(2, q^m) \circ SL(2, q^m)$ and $\mathfrak{M}_{H(q^m)} = \mathfrak{M}_{\mathbf{H}}$. In case g_s is an element of order 3 whose centralizer has order $q(q \pm 1)(q \mp 1)^2$, an appropriate subgroup of $Z_{\mathbf{G}_2}(g_s)$ may be used for $\mathbf{H}$ in a similar manner.

In the remaining cases, $g = g_u$ and, since $g_u \in Z_{\mathbf{G}_2}(g_u)^0$, the mapping t_{G_2} is trivial on g_u.

Case 2. S nontrivial $\Rightarrow$ there is an element $1 \neq x \in Z(S)$ (the center of S). Then $\langle g_u \rangle \times S \leq Z_{\mathbf{G}_2}(x)$ and this is considered in case 1.

Case 3 and *Case* 5 are proved similarly.

Case 4. This can only occur if $E = \langle g_u \rangle$ or $E = S$, where S is a p-group, since $(\operatorname{ord} g_u, s) = 1$. If $E = \langle g_u \rangle$, we invoke Gyoja's result on the lifting theory of exponential unipotent subgroups, and the case $E = S$ follows similarly. (This is where the constraint on p is needed.)

This completes the proof of Lemma 2.3, and thus completes the proof of Theorem (2.1) in case $p \neq 2, 3$.

If $p = 2$ or 3, the lemma, hence the theorem, is proved essentially as above, except that now regular unipotent elements $u \notin Z_{\mathbf{G}_2}(u)^0$. It is necessary then to reconsider the case $g = g_u$, $s = 2$ (if $p = 3$), and $g = g_u$, $s = 3$ (if $p = 2$) in this context. Letting $\mathbf{H} = \langle g_u \rangle \times S$ and $\mathfrak{M}_{H(q^m)}^{-1}(Fxy) = (x^m y^m)^{H(q)}$ for $x \in \langle g_u \rangle$ and $y \in S$ proves the lemma in case $p = 2$ or 3, and the theorem is established.

Theorem (2.1) holds for all p, in case we require that $(m, p) = 1$. One would use Theorem (1.8)(v) instead of [**9**, Prop. 4.4] in the proof above.

3. Principal series representations. Let $\mathbf{G}$ be a simple adjoint algebraic group, and let F be a Frobenius endomorphism such that $\mathbf{G}^{F^n} = G(q^n)$ is a finite untwisted group of adjoint type. Let (W, R) be the Coxeter system of G, and for $J \subseteq R$, denote by W_J the parabolic subgroup of W corresponding to J. Fix a positive integer m and assume lifting from

$\operatorname{Irr} G(q)$ to $\operatorname{Irr} G(q^m)$ occurs for all $\phi \in \operatorname{Irr} G(q)$, and similarly for all parabolics of $G(q)$.

Let $T(q)$ be the standards torus of $G(q)$. For any $\lambda \in \operatorname{Irr} T(q)$, $W(\lambda) = \{w \in W: \lambda^w = \lambda\}$ is a reflection group with fundamental system S, and $(W(\lambda), S)$ is a Coxeter system. See **[14]** for details. For each $J \subseteq R$, $W(\lambda) \cap W_J$ is a parabolic subgroup of $W(\lambda)$. (See **[15]**.) Fix $\lambda \in \operatorname{Irr} T(q)$ so $W(\lambda)$ satisfies the following: as J ranges over all subsets of R, $J \cap S$ ranges over all subsets of S. Then $\{W(\lambda) \cap W_J: J \subseteq R\}$ consists of all parabolic subgroups of $W(\lambda)$. (This occurs, for example, when $W(\lambda)$ is a parabolic subgroup of W.)

Composing λ with N_m (the usual map on tori), it is evident that $\lambda \circ N_m \in \operatorname{Irr} T(q^m)$, and $W(\lambda) = W(\lambda \circ N_m)$. The constituents of $(\lambda \circ N_m)_{B(q^m)}^{G(q^m)}$ are parametrized by the irreducible characters of $W(\lambda)$, as are the constituents of $\lambda_{B(q)}^{G(q)}$. We denote by $\zeta_{\psi,m}$ (respectively ζ_ψ) the unique constituent of $(\lambda \circ N_m)_{B(q^m)}^{G(q^m)}$ (respectively $\lambda_{B(q)}^{G(q)}$) corresponding to $\psi \in \operatorname{Irr} W(\lambda)$. If $W(\lambda)$ is of type G_2, E_7, or E_8, we consider only those irreducible characters ψ of $W(\lambda)$ which are uniquely determined by the multiplicities $\{(\psi, 1_{W_J \cap W(\lambda)}^{W(\lambda)}): J \subseteq R\}$. This occurs for almost all the irreducibles (see **[4]**).

Let $A(\lambda)$ be the generic algebra associated with $(W(\lambda), S)$. Then $A(\lambda)$ is an associative $C[t]$-algebra with generators $\{a_w: w \in W(\lambda)\}$ and satisfies:

(i) $a_w a_{w_i} = a_{ww_i}$, $l(ww_i) = l(w) + 1$,

(ii) $a_w a_{w_i} = t a_{ww_i} + (t-1) a_w$, $l(ww_i) = l(w) - 1$,

for $w \in W(\lambda)$, $w_i \in S$.

For a subset $J \subseteq R$, $A_J(\lambda)$ is a subalgebra of $A(\lambda)$ with generators $\{a_w: w \in W_J \cap W(\lambda)\}$ since $W_J \cap W(\lambda)$ is a parabolic subgroup of $W(\lambda)$. Let $\mathcal{H}(G(q^m), B(q^m), \lambda \circ N_m)$ be the Hecke algebra of $G(q^m)$ corresponding to $\lambda \circ N_m$. There are isomorphisms $f_m: t \to q^m$ and $f_0: t \to 1$ with extensions f_m^* and f_0^* such that the specialized algebra $A(\lambda)_{f_m^*} \cong \mathcal{H}(G(q^m), B(q^m), \lambda \circ N_m) = \mathcal{H}(\lambda \circ N_m)$ and $A(\lambda)_{f_0^*} \cong W(\lambda)$. Then $\{a_{w,f_m^*}\}$ forms a basis for $\mathcal{H}(\lambda \circ N_m)$; $\{a_{w,f_0^*}\} = W(\lambda)$. For any irreducible character χ of $A(\lambda)_K$ ($K =$ splitting field for $A(\lambda)$), $\chi_{f_0^*} \in \operatorname{Irr} W(\lambda)$ and $\chi_{f_m^*} \in \operatorname{Irr} \mathcal{H}(\lambda \circ N_m)$. We sometimes say $\zeta_{\psi,m} = \chi_{f_m^*}$, though actually $\zeta_{\psi,m}|_{\mathcal{H}(\lambda \circ N_m)} = \chi_{f_m^*}$.

Since F acts trivially on W, each $\zeta_{\psi,m}$ may be regarded as an irreducible character of $A \cdot G(q^m)$. Call the extended character $\tilde{\zeta}_{\psi,m}$. From Shintani's work **[16]** and Lemma 1.1.9 in **[2]** we see that $(\lambda \circ N_m)_{B(q^m)}^{G(q^m)}(Fy) = c\lambda_{B(q)}^{G(q)}(\mathfrak{N}(y))$ for all $y \in G(q^m)$ and some constant c, where

$(\lambda \circ N_m)^{G(q^m)}_{B(q^m)}$ is regarded as a character of $A \cdot G(q^m)$ and $\mathfrak{N}$ is the norm map in [2]. In [2, §2], T. Asai characterized the lifting theory of constituents of $1^{G(q)}_{B(q)}$, where $G(q)$ is $\mathrm{GL}(n, q)$, $\mathrm{U}(n, q)$, $\mathrm{Sp}(2n, q)$, $\mathrm{SO}(2n+1, q)$, or $\mathrm{SO}^{\pm}(n, q)$. Because of the nature of the decomposition of $(\lambda \circ N_m)^{G(q^m)}_{B(q^m)}$ and $\lambda^{G(q)}_{B(q)}$, the general ideas in [2, §2] also apply to our setting. In particular, for $\chi' \in \mathrm{Irr}(W(\lambda) \cap W_J)$, let $\chi'^{W(\lambda)} = \sum_\phi n_{\chi',\phi}\phi$, as ϕ varies over $\mathrm{Irr}\, W(\lambda)$, for nonnegative integers $n_{\chi',\phi}$. Then $\zeta^{G(q^m)}_{\chi',m} = \sum_\phi n_{\chi',\phi}\zeta_{\phi,m}$ and $\tilde{\zeta}^{A\cdot G(q^m)}_{\chi',m} = \sum_\phi n_{\chi',\phi}\tilde{\zeta}_{\phi,m}$, where $\zeta_{\chi',m}|_{A_J(\lambda)_{f^*_m}} = \chi_{f^*_m}$, for $\chi \in \mathrm{Irr}\, A_J(\lambda)_K$ and $\tilde{\zeta}_{\chi',m}$ is the extension of $\zeta_{\chi',m}$ to $A \cdot P_J(q^m)$.

(3.1) THEOREM. *Notation as above. For $\zeta_\psi \in \lambda^{G(q)}_{B(q)}$ and $\psi \in \mathrm{Irr}\, W(\lambda)$, one has* $\mathrm{lift}(\zeta_\psi) = \zeta_{\psi,m}$.

Proof. We use induction on the rank of $(W(\lambda), S)$. For rank $(W(\lambda), S) = 1$, $\lambda^{G(q)}_{B(q)} = \mathrm{St}_{G(q),\lambda} + \zeta(\lambda)$, where $\mathrm{St}_{G(q),\lambda}$ is the generalized Steinberg character and $\zeta(\lambda)$ is the generalized identity in $\mathrm{Irr}\, G(q)$ corresponding to λ. Then Gyoja's work implies that $\mathrm{St}_{G(q),\lambda}$ lifts to $\mathrm{St}_{G(q^m),\lambda \circ N_m}$ and $\zeta(\lambda)$ lifts to $\zeta(\lambda \circ N_m)$ (see §6 of [9], and [14]).

Now assume $\mathrm{rank}(W(\lambda), S) \geq 2$, and the theorem holds for all proper parabolics $P_J(q)$ of $G(q)$. Since $(\lambda \circ N_m)^{G(q^m)}_{B(q^m)}(Fy) = c\lambda^{G(q)}_{B(q)}(\mathfrak{N}(y))$, it follows that $\mathrm{lift}(\zeta_\psi) = \zeta_\phi$ for some $\phi \in \mathrm{Irr}\, W(\lambda)$, by Lemma 1.1.8 of [2]. Restricting ζ_ψ and ζ_ϕ to the appropriate parabolic subgroups gives $\psi|_{W_J \cap W(\lambda)} = \phi|_{W_J \cap W(\lambda)}$ (as J ranges over all subsets of R), using the induction hypothesis.

Then since $\{W_J \cap W(\lambda) \colon J \subseteq R\}$ is the set of all parabolic subgroups of $W(\lambda)$, the proof of the theorem is completed by using a theorem of Benson and Curtis [4]:

Let (W, S) be an irreducible Coxeter system of rank ≥ 2. Let χ_1, $\chi_2 \in \mathrm{Irr}\, W$ (with exceptions in types G_2, E_7, and E_8 as noted above). Then $\chi_1|_{W_J} = \chi_2|_{W_J}$ for all $J \subset S$ implies $\chi_1 = \chi_2$.

4. Duality and lifting. Let G be a finite group of Lie type, and (W, R) the Coxeter system of G. C. W. Curtis has defined two operations in $\mathrm{char}_{\mathbf{Z}}(G)$, the ring of complex valued characters of G [7].

(1.4) DEFINITION. Let ζ be a character of G and let M be the module affording ζ. For any subset $J \subseteq R$, let $\zeta_{(P_J)}$ be the character of P_J afforded by $\mathrm{inv}_{V_J}(M) = \{m \in M \colon vm = m \;\forall\, v \in V_J\}$.

(4.2) DEFINITION. Let G, (W, R), ζ be as above. The *dual* ζ^* of ζ is defined to be

$$\zeta^* = \sum_{J \subseteq R} (-1)^{|J|} \zeta^G_{(P_J)}.$$

Extending the definition of $\zeta_{(P_J)}$ by additivity, there are well-defined maps $\mathrm{char}_{\mathbf{Z}}(G) \to \mathrm{char}_{\mathbf{Z}}(P_J)$ via $\zeta \to \zeta_{(P_J)}$ and $\mathrm{char}_{\mathbf{Z}}(G) \to \mathrm{char}_{\mathbf{Z}}(G)$ via $\zeta \to \zeta^*$. The duality operation is a generalization of the construction of the Steinberg character St_G. In fact, $1_G^* = \mathrm{St}_G$. D. Alvis has proved that $\pm\zeta^* \in \mathrm{Irr}\, G$ if $\zeta \in \mathrm{Irr}\, G$, and that $\zeta^{**} = \zeta$ [**1**].

(4.3) THEOREM. *Suppose $\zeta \in \mathrm{Irr}\, G(q)$ lifts to $\psi \in \mathrm{Irr}\, G(q^m)$. Then $\varepsilon_\zeta \zeta^* \in \mathrm{Irr}\, G(q)$ also lifts, and* $\mathrm{lift}(\varepsilon_\zeta\zeta^*) = \varepsilon_\psi \psi^* \in \mathrm{Irr}\, G(q^m)$, *for constants ε_ζ, ε_ψ. The converse is also true, since * is an involution.*

Proof. For a character θ of G, $\theta^* = \sum_{J \subseteq R}(-1)^{|J|}\theta^G_{(P_J)}$, where $\theta_{(P_J)} = \sum a_i \tilde{\phi}_i$, summed over all $\phi_i \in \mathrm{Irr}\, L_J$, and $\tilde{\phi}_i$ is the extension of ϕ_i to P_J obtained by putting V_J in its kernel.

Assume ζ lifts to ψ_ζ. We must show (i) $\varepsilon_\psi \psi^*$ is F-invariant and (ii) $(\varepsilon_\psi\psi^*)'(Fy) = \varepsilon_\zeta \zeta^*(\mathfrak{N}(y))$ for all $y \in G(q^m)$ (where $(\varepsilon_\psi\psi^*)'$ is an extension of $\varepsilon_\psi\psi^*$ to $A \cdot G(q^m)$). That $\varepsilon_\psi\psi^*$ if F-invariant follows from the F-invariance of ψ and from the properties of ψ^*.

The original proof of (ii) is given now, and holds only in case F is of untwisted type. We first show that $\zeta_{(P_J(q))}(\mathfrak{N}(lv)) = \psi'_{(A \cdot P_J(q^m))}(Flv)$ for $l \in L_J(q^m)$ and $v \in V_J(q^m)$. By definition of the norm map $\mathfrak{N}$ we have $\mathfrak{N}(lv) = \mathfrak{N}(l)v'$ for some $v' \in V_J(q)$ since $V_J(q^m)$ is normalized by $L_J(q^m)$. Thus

$$\zeta_{(P_J(q))}(\mathfrak{N}(lv)) = \zeta_{(P_J(q))}(\mathfrak{N}(l)v') = \zeta\big|_{L_J(q)}(\mathfrak{N}(l)).$$

Now we may assume

$$\psi'_{(A \cdot P_J(q^m))}(Fl) = \psi'\big|_{A \cdot L_J(q^m)}(Fl).$$

Since lift $\zeta = \psi$, it follows that

$$\varepsilon_\psi \psi'|_{A \cdot L_J(q^m)}(Fl) = \zeta|_{L_J(q)}(\mathfrak{N}(l)),$$

and hence

$$\zeta_{(P_J(q))}(\mathfrak{N}(lv)) = \varepsilon_\psi \psi'_{(A \cdot P_J(q^m))}(Flv).$$

Now we use Lemma 1.1.9 in [2] to finish the proof. Combining it with the above, we have

$$\varepsilon_\psi \psi'^{A\cdot G(q^m)}_{(A\cdot P_J(q^m))}(Fy) = \zeta^{G(q)}_{(P_J(q))}(\mathfrak{N}(y)) \quad \text{for all } y \in G(q^m).$$

Thus

$$\begin{aligned}(\varepsilon_\zeta \zeta^*)(\mathfrak{N}(y)) &= \sum_{J\subseteq R} (-1)^{|J|} \varepsilon_\zeta \zeta^{G(q)}_{(P_J(q))}(\mathfrak{N}(y)) \\ &= \sum_{J\subseteq R} (-1)^{|J|} \varepsilon_\psi \psi'^{A\cdot G(q^m)}_{(A\cdot P_J(q^m))}(Fy),\end{aligned}$$

and the proof is complete.

We wish to heartily thank Professor Kawanaka for providing us with the following explicit proof. In (II), below, he generalizes the theorem to include the case of F acting nontrivially on $P_J(q^m)$.

(I) If F fixes $P_J(q^m)$, then

$$\begin{aligned}\zeta_{(P_J(q))}(\mathfrak{N}(lv)) &= |V_J(q)|^{-1} \sum_{v'} \zeta(\mathfrak{N}(lv)v') \qquad (v' \in V_J(q)) \\ &= |V_J(q)|^{-1} \sum_{v'} \zeta(\mathfrak{N}(vl)v') \\ &= |V_J(q)|^{-1} \left(\frac{|V_J(q)|}{|V_J(q^m)|} \sum_v \zeta(\mathfrak{N}(lv)) \right) \qquad (v \in V_J(q^m)) \\ &= |V_J(q^m)|^{-1} \sum_v \varepsilon_\psi \psi'(Flv) = \chi_J(Flv),\end{aligned}$$

where χ_J is a character of $A \cdot P_J(q^m)$ afforded by $\mathrm{inv}_{V_J(q^m)}(M_{\psi'})$ (and $M_{\psi'}$ affords ψ'). Then

$$\zeta^*(\mathfrak{N}(y)) = \sum_{J\subseteq R} (-1)^{|J|} \zeta^{G(q)}_{(P_J(q))}(\mathfrak{N}(y)) = \sum_{J\subseteq R} (-1)^{|J|} \chi_J^{A\cdot G(q^m)}(Fy)$$

(by using Lemma 1.1.9 on [2]).

But this is an extension of ψ^* to $A \cdot G(q^m)$, and we may conclude that $(\varepsilon_\zeta \zeta^*)(\mathfrak{N}(y)) = \varepsilon_\psi \psi^*(Fy)$.

(II) If F acts nontrivially on $P_J(q^m)$, let $\{P_J(q^m) = P_{J_0}(q^m), P_{J_1}(q^m), \ldots, P_{J_n}(q^m)\}$ be the F-orbit of $P_J(q^m)$. Let χ_J be the character of $A \cdot G(q^m)$ satisfying

$$\chi_J|_{G(q^m)} = \sum_{i=0}^{n} \left((\zeta \circ F^i)_{(P_{J_i}(q^m))} \right)^{G(q^m)}$$

and F permutes the $(n+1)$ characters which are summands of $\chi_J|_{G(q^m)}$. Then $\chi_J(Fx) = 0$ for all $x \in G(q^m)$, and the result follows by summing over all $J \subseteq R$ as above.

(4.4) COROLLARY. *Let* $\lambda \in \operatorname{Irr} T(q)$ *and* $\zeta_\phi \in \lambda_{B(q)}^{G(q)}$ *(notation as in §3). Then* ζ_ϕ *lifts to* $\zeta_{\phi,m} \in (\lambda \circ N_m)_{B(q^m)}^{G(q^m)}$ *if and only if* $\zeta_{\varepsilon|_{W(\lambda)}\cdot\phi}$ *lifts to* $\zeta_{\varepsilon|_{W(\lambda)}\cdot\phi,m}$.

Proof. In **[15]** it is proved that $\zeta_\phi^* = \zeta_{\varepsilon|_{W(\lambda)}\cdot\phi}$.

(4.5) COROLLARY. *The Steinberg character* St_G *always lifts.*

Proof. $1_G^* = \mathrm{St}_G$.

Corollary (4.5) has been proved independently by Gyoja [**9**, Lemma 6.2(3)], in case the algebraic group giving rise to $G(q)$, $G(q^m)$ has a connected center.

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