PACIFIC JOURNAL OF MATHEMATICS
Vol 113, No. 2, 1984

CONTINUITY OF SPECTRAL FUNCTIONS AND THE LAKES OF WADA

DOMINGO A. HERRERO

The functions σ, mapping a Hilbert space operator T into its spectrum $\sigma(T)$, or σ_e (defined by $\sigma_e(T)$ = essential spectrum of T), or $\rho^h_{s-F}(T)$ mapping T into the set of complex numbers λ such that $\lambda - T$ is semi-Fredholm of index h, etc, have a very erratic behavior. They are continuous on a dense set of operators and discontinuous on another dense set of operators. It is not completely apparent, however, that all of them are simultaneously continuous on a certain dense subset and simultaneously discontinuous on another dense subset.

In order to prove these two assertions, we shall need some notation. In what follows, $\mathcal{L}(\mathcal{H})$ will denote the algebra of all (bounded linear) operators acting on the complex, separable, infinite dimensional Hilbert space $\mathcal{H}$. By a "spectral function" we shall mean a function mapping an operator T in $\mathcal{L}(\mathcal{H})$ to a certain "natural" subset of its spectrum $\sigma(T)$; more specifically, any of the following functions:

> σ (spectrum), σ_l (left spectrum), σ_r (right spectrum), σ_{lr} ($\sigma_{lr}(T) = \sigma_l(T) \cap \sigma_r(T)$), σ_e (essential spectrum, i.e., the spectrum of $T + \mathcal{K}(\mathcal{H})$ in the quotient Calkin algebra $\mathcal{A}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ denotes the ideal of all compact operators), σ_{le} (left essential spectrum), σ_{re} (right essential spectrum), σ_{lre} (Wolf spectrum; $\sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)$), $\bar{\sigma}_p$ (closure of the point spectrum of T), $\bar{\sigma}_0$ (closure of the set of all normal eigenvalues), σ_B (Browder spectrum; $\sigma_B(T) = \sigma(T)\setminus\sigma_0(T)$), σ_W (Weyl spectrum; $\sigma_W(T) = \bigcap\{\sigma(T+K)\colon K \in \mathcal{K}(\mathcal{H})\}$), $\bar{\rho}^h_{s-F}$ (defined by $\bar{\rho}^h_{s-F}(T) = \{\lambda \in \mathbf{C};\ \lambda - T$ is a semi-Fredholm operator of index $h\}^-$, $-\infty \leq y \leq \infty$, $h \neq 0$), or, more generally, ρ^{Σ}_{s-F} (defined by $\rho^{\Sigma}_{s-F}(T) = \{\lambda \in \mathbf{C};\ \lambda - T$ is a semi-Fredholm operator and $\operatorname{ind}(\lambda - T) \in \Sigma\}^-$, for each nonempty subset Σ of $\mathbf{Z}' = \mathbf{Z}^*\setminus\{0\}$, where $\mathbf{Z}^* = \mathbf{Z} \cup \{\pm\infty\}$).

All these functions naturally appear in many problems in Operator Theory. The reader is referred to [3], [4], [5], [6] or [8] for their precise definition and to [11] for the definition and properties of the semi-Fredholm operators. It is well-known that

$$\sigma_W(T) = \sigma(T)\setminus\{\lambda \in \mathbf{C}: \lambda - T \text{ is semi-Fredholm of index } 0\}$$
$$= \sigma_e(T) \cup \rho_{s-F}^{\mathbf{Z}'}(T)$$

[8], [11].

A spectral function τ maps $\mathcal{L}(\mathcal{H})$ into $\mathcal{C}(\mathbf{C})$, the family of all compact subsets of $\mathbf{C}$, the complex plane. As usual, we make $\mathcal{C}(\mathbf{C})$ a complete metric space by defining the (modified) Hausdorff distance d_H between two elements of $\mathcal{C}(\mathbf{C})$ by:

(1) If $X, Y \in \mathcal{C}(\mathbf{C})\setminus\{\varnothing\}$, then

$$d_H(X, Y) = \min\{1, \min\{\varepsilon > 0: X \subset Y_\varepsilon, Y \subset X_\varepsilon\}\},$$

where $X = \{\lambda \in \mathbf{C}: \text{dist}[\lambda, X] \leq \varepsilon\}$, and

(2) $d_H(X, \varnothing) = 1$ for all nonempty X in $\mathcal{C}(\mathbf{C})$.

The continuity points of the spectral function τ are then defined in terms of the norm topology of $\mathcal{L}(\mathcal{H})$ and the above mentioned metric structure of $\mathcal{C}(\mathbf{C})$.

In a sequence of remarkable papers, J. B. Conway and B. B. Morrel completely characterized those operators that are points of continuity for each of the functions listed above, except for $\bar{\sigma}_0$, σ_B and ρ_{s-F}^{Σ} (see [3], [4], [5]). On the other hand, it is not difficult to check, by using the results of these papers, that σ_B is continuous at T if and only if σ_W is continuous at T and $\sigma_B(T) = \sigma_W(T)$. Furthermore, by using the same kinds of arguments, it is possible to prove the following.

THEOREM 1. (i) *$\bar{\sigma}_0$ is continuous at $A \in \mathcal{L}(\mathcal{H})$ if and only if $\sigma(A) = [\rho_{s-F}^{\mathbf{Z}'}(A) \cup \sigma_0(A)]^-$ and $\bar{\sigma}_0(A) = \partial\sigma(A)$.*

(ii) *If Σ is a nonempty subset of $\mathbf{Z}'$, then ρ_{s-F}^{Σ} is continuous at $A \in \mathcal{L}(\mathcal{H})$ if and only if $\sigma_{lre}(A) \subset \rho_{s-F}^{\Sigma}(A)$.*

In each case the sufficiency of the given condition follows from standard arguments based only on the upper semicontinuity of separate parts of the spectrum [8, Chapter 1], [11, Chapter IV] and the stability properties of the semi-Fredholm operators (same references), so that these conditions are actually *sufficient* in any Banach space (not necessarily a Hilbert space!). The necessity is much more difficult to check and depends on results of approximation of operators developed strictly for the Hilbert space case, as the Apostol-Morrel simple models [2] (see also [1]).

It was observed in [**3**] that both, σ and σ_W, are continuous on a dense subset and discontinuous on a dense subset (this last result depending on the proof of Theorem 4 in [**7**]).

This note is devoted to the proof of the following (much stronger) results:

THEOREM 2. *$\mathcal{L}(\mathcal{H})$ contains a dense subset Γ_c such that all the spectral functions mentioned above are continuous at each point of Γ_c.*

THEOREM 3. *$\mathcal{L}(\mathcal{H})$ contains a dense subset Γ_d such that all the spectral functions mentioned above are discontinuous at each point Γ_d.*

1. The lakes of Wada. The name of the title is a classical construction of Point Set Topology, producing an indecomposable continuum. More precisely, the construction produces three nonempty disjoint open subsets, Ω_0, Ω_1 and Ω_2, of $\mathbf{C}$ with the property that the three of them have the same boundary, and this common boundary $X\,(=\partial\Omega_0 = \partial\Omega_1 = \partial\Omega_2)$ is a compact set. The details of the construction can be found, for example, in [**12**] [**10**, pp. 143–145]. The case of denumerably many open sets (instead of just three) follows by exactly the same argument and yields the following result:

LEMMA 4. *Let Δ be a nonempty compact connected subset of $\mathbf{C}$ such that $\Delta = (\text{interior } \Delta)^-$. Then there exists a denumerble family $\{\Omega_h\}_{h\in\mathbf{Z}^*}$ of pairwise disjoint simply connected open subsets of Δ such that $\bigcup_{h\in\mathbf{Z}^*}\Omega_h$ is dense in Δ and $\partial\Omega_h = \partial\Omega_0$ for all h.*

COROLLARY 5. *Let Δ and $\{\Omega_h\}_{h\in\mathbf{Z}^*}$ be as in Lemma 4. There exists L_Δ in $\mathcal{L}(\mathcal{H})$ such that $\sigma(L_\Delta)$ is the disjoint union of $\Delta\backslash\Omega_0$ and a sequence $\{\lambda_j\}_{j=1}^\infty$ contained in Ω_0 such that*

$$\{\lambda_j\}^- = \{\lambda_j\} \cup \partial\Omega_0, \quad \sigma_e(L_\Delta) = (\Omega_\infty \cup \Omega_{-\infty})^-,$$

each λ_j is a normal eigenvalue of L_Δ of multiplicity 1 and for each $h \in \mathbf{Z}'$ and each λ in Ω_h, $\lambda - L_\Delta$ is a semi-Fredholm operator such that

$$\operatorname{ind}(\Lambda - L_\Delta) = h \quad \textit{and} \quad \min\{\dim\ker(\lambda - L_\Delta), \dim\ker(\lambda - L_\Delta)^*\} = 0.$$

Proof. Decompose $\mathcal{H} = \bigoplus_{h\in\mathbf{Z}^*}\mathcal{H}_h$ (orthogonal direct sum), where $\mathcal{H}_h$ is an infinite dimensional subspace. Let $\{\lambda_j\}_{j=1}^\infty$ be a denumerable subset of Ω_0 such that $\operatorname{dist}[\lambda_j, \partial\Omega_0] \to 0$ $(j\to\infty)$ and $\{\lambda_j\}^- = \{\lambda_j\} \cup \partial\Omega_0$ and define $N_0 \in \mathcal{L}(\mathcal{H}_0)$ by the equations $N_0 e_j = \lambda_j e_j$ $(j \geq 1)$ with respect to some orthonormal basis $\{e_j\}_{j=1}^\infty$ of $\mathcal{H}_0$.

If Ω is a nonempty bounded open set such that $\Omega = \text{interior}(\Omega^-)$, then we define $M(\Omega) =$ "multiplication by λ" on the space $L^2(\Omega, dx\,dy)$ and $M_+(\Omega) = M(\Omega) \mid B^2(\Omega)$, where $B^2(\Omega)$ is the L^2-closure of the rational functions in λ with poles outside Ω^-. For each $h < 0$, we define $N_h = M_+(\Omega_h)^{(|h|)}$ (= direct sum of $|h|$ copies of $M_+(\Omega_h)$), acting in the usual fashion on the orthogonal direct sum $B^2(\Omega_h)^{(|h|)}$ of $|h|$ copies of the Hilbert space $B^2(\Omega_h)$. Similarly, for each $j > 0$, we define $N_h = [M_+(\Omega_h^*)^*]^{(h)}$, where $\Omega^* = \{\bar{\lambda}: \lambda \in \Omega\}$.

Clearly, $B^2(\Omega_h)^{(|h|)}$ is isometrically isomorphic to $\mathcal{H}_h$ for each $h < 0$, and $B^2(\Omega_h^*)^{(h)}$ is isometrically isomorphic to $\mathcal{H}_h$ for each $h > 0$. Thus, we can directly assume that $N_h \in \mathcal{L}(\mathcal{H}_h)$ for all $h \in \mathbf{Z}^*$. Now it is easy to check (by using, for instance, the results of [**8**, Chapter IV]) that equation

$$L = \bigoplus_{h \in \mathbf{Z}^*} N_h$$

actually defines an operator acting on $\mathcal{H}$ with the desired properties. □

The semi-Fredholm domain of $T \in \mathcal{L}(\mathcal{H})$ is the open set $\rho_{s-F}(T) = \{\lambda \in \mathbf{C}: \lambda - T \text{ is semi-Fredholm}\}$ $(= \mathbf{C} \setminus \sigma_{lre}(T))$. Assume that

$$\sigma(T) = (\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_r) \cup \{\mu_1, \mu_2, \ldots, \mu_n\} \cup \{\nu_1, \nu_2, \ldots, \nu_p\}$$

(disjoint union), where

(1) Γ_t is the closure of the bounded open set interior $\Gamma_t \subset \rho_{s-F}(T)$; $\text{ind}(\lambda - T) \neq 0$ and $\min\{\dim\ker(\lambda - T), \dim\ker(\lambda - T)^*\} = 0$ for all $\lambda \in \text{interior } \Gamma_t$ $(t = 1, 2, \ldots, r;\ r < \infty)$;

(2) $\partial(\bigcup_{t=1}^r \Gamma_t)$ is the union of finitely many pairwise disjoint smooth Jordan curves $\gamma_1, \gamma_2, \ldots, \gamma_m$;

(3) $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is a finite set of isolated points of $\sigma_e(T)$; and

(4) $\{\nu_1, \nu_2, \ldots, \nu_p\} = \sigma_0(T)$.

Let $\eta > 0$ be small enough to guarantee that $\sigma(T)_\eta$ has exactly the same number of components as $\sigma(T)$ and let $\Delta_j = (\gamma_j)_\eta \setminus (\text{interior } \sigma(T))$ for $j = 1, 2, \ldots, m$, and $\Delta_j = \{\lambda: |\lambda - \mu_{j-m}| < \eta\}$ for $j = m + 1, m + 2, \ldots, m + n$. For each j, $j = 1, 2, \ldots, m + n$, we choose an operator L_{Δ_j} defined exactly as in Corollary 5 (with Δ replaced by Δ_j, $j = 1, 2, \ldots, m + n$) and define $L_\eta = \bigoplus_{j=1}^{m+n} L_{\Delta_j}$.

Combining the "easy part" of the results of [3], [4], [5] with Theorem 1, we conclude that

PROPOSITION 6. *Let T and L_η be as above; then all the spectral functions considered here are continuous at $T \oplus L_\eta$.*

2. Simultaneous continuity. Let $A \in \mathcal{L}(\mathcal{H})$ and let $\varepsilon > 0$ be given. The main result of [2] says that there exists an operator $A_\varepsilon \in \mathcal{L}(\mathcal{H})$ such that $\|A - A_\varepsilon\| < \varepsilon$ and

$$A_\varepsilon \simeq \begin{pmatrix} S_+ & * & * \\ 0 & M & * \\ 0 & 0 & S_- \end{pmatrix},$$

where $\simeq$ denotes unitary equivalence, $\sigma(S_+)$, $\sigma(M)$ and $\sigma(S_-)$ are pairwise disjoint, A_ε is similar to $S_+ \oplus M \oplus S_-$, M is a normal operator with finite spectrum, $\sigma(S_+) = \{\lambda \in \rho_{s-F}(S_+)\colon \operatorname{ind}(\lambda - S_+) < 0\}^-$, $\rho_{s-F}(S_+) \cap \sigma(S_+) = \text{interior } \sigma(S_+)$ and $\dim\ker(\lambda - S_+) = 0$ for all $\lambda \in \rho_{s-F}(S_+)$, $\sigma(S_-) = \{\lambda \in \rho_{s-F}(S_-)\colon \operatorname{ind}(\lambda - S_-) > 0\}^-$, $\rho_{s-F}(S_-) \cap \sigma(S_-) = \text{interior } \sigma(S_-)$ and $\dim\ker(\lambda - S_-) = 0$ for all $\lambda \in \rho_{s-F}(S_-)$, and $\partial[\sigma(S_+) \cup \sigma(S_-)]$ consists of finitely many pairwise disjoint smooth Jordan curves.

Thus, $S_+ \oplus M \oplus S_-$ has exactly the same form as the operator T of Proposition 6. For each $\eta > 0$ small enough, we define $(S_+ \oplus M \oplus S_-) \oplus L_\eta$ as in Proposition 6. It readily follows that $S_+ \oplus M \oplus S_- \oplus L_\eta$ is a point of continuity for all spectral functions,. A fortiori, so is every operator *similar* to $S_+ \oplus M \oplus S_- \oplus L_\eta$.

Following [1], let us write $R \underset{\text{sim}}{\to} B$ to indicate that the operator B is the norm limit of operators similar to R. According to the same reference, there is a normal operator M such that $\sigma(M_\eta) = \sigma(L_\eta)$ and $L_\eta \underset{\text{sim}}{\to} M_\eta$. It follows that $S_+ \oplus M \oplus S_- \oplus L_\eta \underset{\text{sim}}{\to} S_+ \oplus M \oplus S_- \oplus M_\eta$; moreover, $S_+ \oplus M \oplus S_-$ can be uniformly approximated by operators similar to $S_+ \oplus M \oplus S_- \oplus M_\eta$. (Consider a sequence of such operators corresponding to a decreasing sequence $\{\eta_n\}_{n=1}^\infty$ such that $\eta_n > 0$, as $n \to \infty$.)

Hence, $S_+ \oplus M \oplus S_-$ is the limit of a sequence of points of continuity. Therefore, so is the operator A_ε (similar to $S_+ \oplus M \oplus S_-$).

Since ε can be chosen arbitrarily small we conclude that A can be uniformly approximated by a sequence $\{A_n\}_{n=1}^\infty$ such that all the spectral functions considered in the Introduction are simultaneously continuous at A_n, for each $n \geq 1$.

The proof of Theorem 2 is now complete. □

REMARKS. (i) The spectral radius and the essential spectral radius are also continuous at each point of the set Γ_c described by Theorem 2 (see [3]).

(ii) It is tempting to think that any "natural" spectral function is necessarily continuous at the points of Γ_c. However, this is not true at all.

Namely, the spectral function defined by

$$\overline{\sigma_e^0}(T) = \{\{\lambda\}: \{\lambda\} \text{ is a component of } \sigma_e(T)\}^-$$

(that is, the closure of the union of those components of $\sigma_e(T)$ which consist of a single point), which plays a very important role in the work of Conway and Morrel, *is continuous nowhere*! (This can be deduced, for example, from the main result of [**1**].)

3. Simultaneous discontinuity.

LEMMA 7. *Let R be a normal operator such that $\sigma(R) = \{\lambda: |\lambda| \leq 1\}$ and $\sigma_p(R) = \varnothing$. All the spectral functions considered in the Introduction are discontinuous at R.*

Proof. Let Q be a quasinilpotent operator such that Q^k is compact for no value of $k \geq 1$, and let N_0 be defined as in the proof of Corollary 5 with $\lambda_j = 2^{-j}$, $j = 1,2\ldots$. According to [**1**], $Q \oplus N_0 \underset{\text{sim}}{\to} R$, whence it readily follows that all "σ-functions" (that is, σ, σ_B, σ_l, σ_e, etc.) are discontinuous at R because every neighborhood of R contains an operator similar to $Q \oplus N_0$.

Similarly, if S is a semi-Fredholm operator such that $\sigma(S) = \sigma(R)$ and ind $S = h \in \mathbf{Z}'$, then $S \underset{\text{sim}}{\to} R$, whence we conclude that ρ_{s-F}^{Σ} is discontinuous at R for all possible sets Σ. □

According to [7] (Theorem 4 and its proof), given A in $\mathcal{L}(\mathcal{H})$, a point $\lambda \in \sigma_e(A) \cap \partial\sigma(A)$, $\varepsilon > 0$ and $R \in \mathcal{L}(\mathcal{H})$, $\|R\| \leq 1$, there exists $A(\lambda, \varepsilon, R)$ unitarily equivalent to

$$\begin{pmatrix} \mu + \delta R & * \\ 0 & A' \end{pmatrix}$$

such that $\sigma_e(A') = \sigma_e(A)$, $0 < \text{dist}[\mu, \sigma(A')] < \varepsilon$, $|\lambda - \mu| < \varepsilon$, $0 < \delta < \varepsilon$ and $\|A - A(\lambda, \varepsilon, R)\| < \varepsilon$; furthermore, $A(\lambda, \varepsilon, R)$ is similar to $(\mu + \delta R) \oplus A'$, so that if the spectral function τ is discontinuous at R, then τ is also discontinuous at $A(\lambda, \varepsilon, R)$.

By Lemma 7, R can be chosen so that all the spectral functions are discontinuous. The proof of Theorem 3 is now complete. □

REMARKS. (i) If λ is chosen so that $|\lambda| = \max\{|\xi| : \xi \in \sigma_e(A)\}$ (= the essential spectral radius of A), then we obtain a bonus result: The set Γ_d of Theorem 3 can be chosen so that the essential spectral radius is also discontinuous at each point of Γ_d.

Can we choose Γ_d so that the spectral radius, $\text{sp}(A) = \max\{|\lambda|: \lambda \in \sigma(A)$, is also discontinuous at the points of Γ_d? Definitely: NO. Indeed, if $\mu \in \sigma_0(A)$, $|\mu| = \text{sp}(A)$ and $\varepsilon > 0$, then the upper semicontinuity of separate parts of the spectrum and the continuity properties of the Riesz-Dunford functional calculus (see, for example, [**8**, Corollary 1.6]) imply that $\sigma_0(B) \cap \{\lambda \in \mathbf{C}: |\lambda - \mu| < \varepsilon\} \neq \varnothing$ and therefore $\text{sp}(B) > \text{sp}(A) - \varepsilon$ for all B close enough to A. Since the spectral radius is an upper semicontinuous function of its argument, we conclude that sp is continuous at A.

Combining this observation with the main result of [**9**], we obtain the following

PROPOSITION 8. *For each complex Banach space* $\mathcal{X}$, $\mathcal{L}(\mathcal{X})$ *contains an open dense subset* Φ *such that* sp *is continuous at every point of* Φ.

REFERENCES

[1] C. Apostol, D. A. Herrero and D. Voiculescu, *The closure of the similarity orbit of a Hilbert space operator*, Bull. Amer. Math. Soc. (New Series), **6** (1982), 421–426.

[2] C. Apostol and B. B. Morrel, *On uniform approximation of operators by simple models*, Indiana Univ. Math. J., **16** (1977), 427–442.

[3] J. B. Conway and B. B. Morrel, *Operators that are points of spectral continuity*, Integral Equations and Operator Theory, **2** (1979), 174–198.

[4] ______, *Operators that are points of spectral continuity. II*, Integral Equations and Operator Theory, **4** (1981), 459–503.

[5] ______, *Operators that are points of spectral continuity. III*, Integral Equations and Operator Theory, **6** (1983), 319–344.

[6] P. R. Halmos, *A Hilbert Space Problem Book*, D. Van Nostrand, Princeton. New Jersey, 1967.

[7] D. A. Herrero, *On multicyclic operators*, Integral Equations and Operator Theory, **1** (1978), 57–102.

[8] ______, *Approximation of Hilbert Space Operators*, Pitman Publ. Inc., London-Boston-Toronto, 1982.

[9] D. A. Herrero and N. Salinas, *Operators with disconnected spectra are dense*, Bull. Amer. Math. Soc., **78** (1972), 525–526.

[10] J. G. Hocking and G. S. Yound, *Topology*, Addison Wesley Publ. Co., London, 1961.

[11] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.

[12] K. Yoneyama, *Theory of continuous sets of points I*, Tohoku Math. J., **12** (1917), 43–153.

Received August 31, 1982 and in revised form January 5, 1983. This research has been partially supported by a Grant of the National Science Foundation.

ARIZONA STATE UNIVERSITY
TEMPE, AZ 85287