ENUMERATION OF THE QUASISIMPLICIAL 3-SPHERES AND 4-POLYTOPES WITH EIGHT VERTICES

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A complete enumeration is given for quasisimplicial 3-spheres with eight vertices. It is found that there are 661 spheres, and it is proved that precisely 20 of these are not polytopal.

1. Introduction. This work is the first part of an enumeration of all the combinatorial 3-spheres and the 4-polytopes with 8 vertices. It is devoted to the quasisimplicial cases; that is, to those 3-spheres and 4-polytopes all of whose facets are simplicial.

In the last two decades extensive work has been done on the enumeration of simplicial 3-spheres and simplicial 4-polytopes with 8 vertices ([15], [10], [1]), with 9 vertices ([4], [6], [7], [8]) and with 10 vertices (the neighborly cases only, see [3]). With regard to the general (that is, not necessarily simplicial) cases, Kleinschmidt [16] proved that every d-sphere with up to $d + 4$ vertices is polytopal, Reif ([19]) prepared a detailed list of the 31 4-polytopes with 7 vertices and a partial list (containing some 1050 cases) of 3-spheres with 8 vertices, and Perles ([14, pages 114, 424]) determined the number of d-polytopes with $d + 3$ vertices for $d \leq 6$.

The simplicial 3-spheres with 8 vertices are well known ([15], [10], [1]). There are 39, exactly two of which are non-polytopal. The pyramidal 3-spheres with 8 vertices are also well known, based on the determination by Hermes, quoted in [13] and [14, page 424], of the 3-polytopes with 7 vertices. Obviously they all are polytopal, and their number is 34. Five of them are quasisimplicial.

In the present work we find all the non-simplicial, non-pyramidal quasisimplicial 3-spheres with 8 vertices. It turns out that there are 617, exactly 18 of which are not polytopal. Therefore the total number of quasisimplicial 3-spheres with 8 vertices is 661 of which 20 are not polytopal. Our 18 non-polytopal spheres are described in detail in Table 1. A detailed description of all the 617 cases is of course beyond the scope of a single paper. The full catalogue of these cases can be obtained upon request from the second author.

The non-polytopal spheres are of special interest because they might help to solve the classical "Steinitz problem", concerning the combinatorial characterization of the d-polytopes (for $d > 3$) among the $(d - 1)$-spheres. The first non-simplicial non-polytopal 3-sphere with 8 vertices was discovered in [11], Kleinschmidt [17] found two more such spheres. Recently Schulz ([20]) prepared a list of 27 non-polytopal 3-spheres with 8 vertices, which contains the 5 cases previously known. Of those 27, ten are quasisimplicial but not simplicial. They are among the 18 cases listed in Table 1.

In §2 we introduce the basic concepts involved in the present work. In §3 we describe the construction of the 617 quasisimplicial, non-simplicial, non-pyramidal 3-spheres with 8 vertices. In §4 we describe the theoretical background needed for the classification of the 617 spheres into polytopes and non-polytopes. In §5 we prove that the 18 spheres listed in Table 1 are not polytopal. In §6 we quote several theorems from [5] with which we prove the polytopality of all but three of the remaining 599 spheres. In §7 we prove that these three spheres are polytopal also and we conclude in §8 with some remarks.

Our terminology follows [14] and [7].

2. Basic concepts. A *d-cell complex* is a finite collection $\mathcal{C}$ of k-cells, $0 \leq k \leq d$, such that

1. each d-cell F is associated with a convex d-polytope $P(F)$ and with a (piecewise linear) homeomorphism $h(F)$ between F and $P(F)$;

2. each face of a d-cell in $\mathcal{C}$ is a member of $\mathcal{C}$, where a face of a d-cell F is the inverse image of a face of $P(F)$ under $h(F)$;

3. the intersection of any two members of $\mathcal{C}$ is a (possibly empty) face of both members.

A *combinatorial d-sphere* is a d-cell complex whose body (i.e. the union of its members) is homeomorphic to S^d. A *combinatorial d-ball* is similarly defined to be a d-cell complex whose body is homeomorphic to a topological d-ball.

A word of caution seems appropriate. The term *combinatorial sphere* appears in the literature with more than one meaning. For example in [12] the underlying complex is assumed to be geometric—that is, all the d-cells are assumed to be d-polytopes, properly embedded in a Euclidean space. We will refer to such a sphere as *geometric*. On the other hand, in [8], the underlying complex is assumed to be simplicial. (See also the concluding paragraph in [9, §1]). The definition given here is adopted basically from [10].

Obviously, every geometric d-sphere is a combinatorial d-sphere. Other examples of combinatorial d-spheres are provided by d-*diagrams*, as defined in [**14**, page 44]. The fact, established in [**21**], that there are geometric spheres that are not diagrams, and vice versa, shows that the concept of a combinatorial sphere is broader than the other two concepts. All three, however, agree in the case of the boundary complex of a convex polytope.

To simplify the notation, we refer to the cells of a combinatorial sphere as if they themselves were convex. Thus, using the notation of the above definitions, if F is a 3-cell such that $P(F)$ is the octahedron shown in Figure 1, we say that F is an octahedron, that 45 is an edge of F, that the diagonal 12 lies inside F, and so on. The convex hull of the points $a_1, a_2, \ldots, a_n$ is denoted by $a_1 \cdots a_n$. If not otherwise specified, all the spheres mentioned here are assumed to be combinatorial and 3-dimensional.

If a, b are vertices in a sphere S but the edge ab is not in S, we say that ab is a *missing edge* of S. For every two vertices a, $b \in S$, we define val ab, the *valence* of the edge (missing or not) ab, to be the number of 3-cells in S that contain both vertices a and b. Thus, if ab is an edge, val ab is as defined in [**7**] and is ≥ 3, while if ab is a missing edge, then $0 \leq$ val $ab \leq 2$, and val $ab = 2$ iff a, b belong to some 2-face in S, but ab is not an edge. The last case is impossible where S is quasisimplicial. Also, if ab, bc, ac are edges in some 3-cell A of S but the triangle abc is not a face of A (and therefore also not a face of any 3-cell of S), we say that abc is a missing 2-face in A and in S.

With every 3-sphere S with n vertices $a_1, \ldots, a_n$, we associate an $n \times n$ matrix, the *edge-valence matrix* of S, in which the (i, j) element is the valence of the edge $a_i a_j$ (missing or not). This matrix, of course, depends on the particular ordering chosen for the vertices of S, but its determinant is independent of that ordering, (see [**3**, §2] and [**7**, §2]) and plays an important role in the identification of a sphere. Since our interest is in the combinatorial types of spheres, we do not distinguish between spheres that are isomorphic.

Finally, if $\mathcal{C}$ is a cell-complex and $A \in \mathcal{C}$, then st$(A, \mathcal{C})$, the *star* of A in $\mathcal{C}$, is the complex $\{C \in \mathcal{C}:$ There is a cell $B \in \mathcal{C}$ such that A and C are faces of $B\}$, and ast$(A, \mathcal{C})$, the *antistar* of A in $\mathcal{C}$, is the complex of all the cells in $\mathcal{C}$ disjoint from A.

3. Construction. Let $2S_8^3$ be the family of (combinatorial types of the) 3-spheres with 8 vertices which are quasisimplicial, but not simplicial and not pyramidal. Since each facet in such a sphere S has at most 6

vertices, it is either a simplex or one of the types A, B, C, shown in Figure
1. (We refer to a cell of type C as an octahedron, though its vertices 1, 2,
3, 4, or 3, 4, 5, 6, or 1, 2, 5, 6, do not necessarily lie in the same plane.)

A cell of type A (double tetrahedron) can be uniquely subdivided into
two tetrahedra, without adding any extra vertices or edges. A cell of type
B can be uniquely subdivided into three tetrahedra, without extra vertices
or edges. A cell of type C (octahedron), *after adding to it one of its three
main diagonals*, can be uniquely subdivided into four tetrahedra, without
extra vertices or edges.

By thus subdividing each facet of $S \in 2\mathfrak{S}_8^3$ which is not a simplex, a
simplicial 3-sphere S' with 8 vertices is obtained. There are 39 simplicial
3-spheres with eight vertices (see [15], [10]) which we denote by M_i^8,
$1 \leq i \leq 39$, where for $1 \leq i \leq 37$, M_i^8 is the polytopal sphere P_i^8 of [15],
M_{38}^8 is Grünbaum's non-polytopal sphere denoted $\mathfrak{M}$ in [15], and M_{39}^8 is
Barnette's non-polytopal sphere ([10]). Thus S' is one of these M_i^8's. Note
that if S contains no octahedron, then S' is uniquely determined by S,
while if S contains an octahedron, S' may also depend on the particular
diagonal chosen for the subdivision of the octahedron.

The idea used was to start with the 39 simplicial spheres M_i^8, and to
construct the family $2\mathfrak{S}_8^3$ from them by reversing the process of subdivi-
sion described above. The following lemma is useful:

LEMMA 1. *No $S \in 2\mathfrak{S}_8^3$ contains two facets with 6 vertices.*

Proof. If X, Y are facets of $S \in 2\mathfrak{S}_8^3$ with 6 vertices, then $X \cap Y$
contains at least 4 vertices. But this is impossible, since X and Y are
simplicial 3-polytopes. Thus, no $S \in 2\mathfrak{S}_8^3$ contains two facets of type B,
two facets of type C, or a facet of type B and a facet of type C.

The reversal of the process of subdivision mentioned above, depends
upon the following three operations, each of which applies to a 3-sphere
S.

Operation A. Choose two tetrahedra, *abcd* and *abce*, in S which share
a common triangle *abc*, such that *de* is a missing edge in S with val *de* = 0,
and "glue" them along the triangle *abc* to yield a 3-cell *abcde* of type A.

Operation B. Choose three tetrahedra *abdf*, *abef* and *acef*, in S such
that *bc*, *cd*, *de* are missing edges of valence 0 in S, and "glue" them along
the triangles *abf*, *aef*, to yield a 3-cell *abcdef* of type B.

Operation C. Choose an edge *ab* of valence 4 in S such that the four
3-cells containing it (*abce*, *abde*, *abdf* and *abcf*) are tetrahedra and such

that cd and ef are of valence 0 in S, and "glue" the four tetrahedra along the triangles abc, abd, abe, abf—removing the edge ab—to yield a 3-cell $abcdef$ of type C.

Using a CYBER-174 computer, we applied Operation A to each 3-sphere M_i^8, $1 \le i \le 39$, in all possible ways. To each resulting sphere S we again applied Operation A in all possible ways, and so on. By repeatedly applying Operation A to the spheres M_i^8 we obtained all the spheres in $2S_8^3$ whose facets are just simplices and double tetrahedra (type A). We then applied Operation B, followed by repeated application of Operation A, to each M_i^8 to obtain all the spheres in $2S_8^3$ which contain a facet of type B, in addition to simplices and double tetrahedra. Finally, Operation C, followed by Operations of type A, was applied to each M_i^8 to yield the remaining members in $2S_8^3$.

For each resulting sphere S, the edge-valence matrix and its determinant have been determined. The spheres were then checked for isomorphism. Here, as in other works (see [6], [7], [8]), the edge-valence matrix and its determinant proved to be very helpful. Also, as follows from a previous remark, two spheres in $2S_8^3$ that do not contain an octahedron (type C) can be isomorphic only if, beside sharing the same number of facets of each type and the same determinant, they result from the same simplicial 3-sphere M_i^8 (while two spheres that contain an octahedron may be isomorphic even if they result from different M_i^8's).

Altogether, the final list contains 617 non-isomorphic spheres, falling into three groups: first, 520 spheres S_i ($1 \le i \le 520$) that do not contain a cell of type B nor of type C, then 80 spheres S_i ($521 \le i \le 600$) that contain a cell of type B and finally 17 spheres S_i ($601 \le i \le 617$) that contain a cell of type C. The spheres within each group are ordered lexicographically by increasing number of cells, increasing number of double-tetrahedra and by increasing determinant. In the rare event of equal determinants the order is casual.

4. Background for the classification.

The next step after constructing the 617 quasisimplicial spheres was to classify them into polytopes and non-polytopes.

Let S be any of our 617 spheres, with the vertices labelled $1, 2, \ldots, 8$. If S is polytopal, let P be a 4-polytope realizing S, and we assume the labelling of the vertices of P to be the same as in S. Let $x \in \{1, 2, \ldots, 8\}$ and let $Q = \operatorname{conv}(\operatorname{vert} P \backslash \{x\})$. Q is a 4-polytope with seven vertices. Denote by S^x the combinatorial 3-sphere isomorphic to bd Q. It is important to note that S^x depends not only on S, on x and on S being polytopal, but may also depend on the particular polytope P chosen to

realize S. In the terminology of [14, §5.2], the point x lies beneath some facets of Q, beyond some other facets of Q, and—unlike the simplicial case—may lie in the affine hulls of some facets of Q.

The relation between the facial structure of Q and that of P is determined by Grunbaum's theorem [14, Theorem 5.2.1] (or rather by its corrected version in [5]). That is, if $\mathcal{C} = \{A_1, \ldots, A_r\}$, $\mathcal{B} = \{B_1, \ldots, B_s\}$, $\mathcal{C} = \{C_1, \ldots, C_t\}$ is a partition of the facets of Q such that the point x lies beyond each A_i, $1 \le i \le r$, beneath each B_j, $1 \le j \le s$, and in aff C_k, $1 \le k \le t$, then the combinatorial structure of $P = \text{conv}(Q \cup \{x\})$ depends only on the partition $\mathcal{C}$, $\mathcal{B}$, $\mathcal{C}$ of the facets of Q. It does not depend on the actual point x chosen, provided, of course, that the set of points in R^4 that lie beyond each $\mathcal{C}_i$, beneath each B_j and in the affine hull of each C_k, is not empty.

So far, both the definition and the existence of the sphere S^x depends on S being polytopal. But it is the polytopality of S that we want to prove or disprove. Indeed, in the next section we will see that S^x can be defined and constructed combinatorially, without any previous knowledge of the polytopality of S.

If for some vertex x of S it can be shown that no sphere S^x does exist, then S is not polytopal. If for every vertex x of S there exists a sphere which might serve as an S^x then—fixing x and S^x and assuming Q to be a polytopal realization of S^x—the partition $\mathcal{C}$, $\mathcal{B}$, $\mathcal{C}$ of the facets of Q required for constructing a polytope P which is to realize S, is already determined by S, x and S^x. (Since we assume the labelling of the proper faces of Q to be the same as that of S^x, every partition of the facets of Q carries over to S^x, and vice versa). Namely, $\mathcal{B}$ is the set of 3-cells common to S and to S^x (that is, all the 3-cells in ast(x, S)), $\mathcal{C}$ is the set of 3-cells C of S^x such that vert $C \cup \{x\}$ is the vertex set of some 3-cell in S, and $\mathcal{C}$ is the set of all the remaining 3-cells in S^x. (Note that unlike $\mathcal{C}$, $\mathcal{B}$ depends on S and x only, and not on S^x. The situation for $\mathcal{C}$ is more complicated and we deal with it in §5.)

It follows from the last discussion, that the following sequence of steps is required for deciding the polytopality of a sphere S: Start with a vertex x in S and

1. construct a sphere S^x such that
2. S^x is polytopal, and
3. a polytope Q can be found which realizes S^x, such that the set of points that lie beyond each $A \in \mathcal{C}$, beneath each $B \in \mathcal{B}$ and in the affine hull of each $C \in \mathcal{C}$, is not empty. ($\mathcal{C}$, $\mathcal{B}$, $\mathcal{C}$ is the partition of the facets of S^x, uniquely determined by S, x and S^x.)

It also follows from Grünbaum's theorem that the possibility of carrying out these steps for *one* vertex x of S is sufficient to prove that S is polytopal, while the impossibility of carrying out these three steps for *one* vertex x of S is sufficient to prove that S is not polytopal.

In our case, since S has 8 vertices and S^x (if it exists) has 7 vertices, the second step does not pose any problem: Kleinschmidt ([16]) proved that very d-sphere with up to $d + 4$ vertices is polytopal. Actually, Kleinschmidt deals with geometric spheres, but we have checked independently that there are 31 combinatorial 3-spheres with 7 vertices altogether, and that they coincide with the 31 4-polytopes with 7 vertices listed in [19]. The proof is simple, technical and tedious, and we skip it.

As for the first step, we will see that in 18 of the 617 spheres under consideration there is a vertex x such that there exists no S^x, and therefore they are not polytopal. Concerning the third step, quite unexpectedly, we were able to show that for each of the remaining 599 spheres—all of which proved polytopal—there is a vertex x and there is a sphere S^x such that for *any* polytope Q chosen to realize S^x, the condition in the third step is satisfied.

We turn now to the strategy for constructing S^x or to prove its nonexistence.

5. Classification: 18 non-polytopal spheres.

Let S be a 3-sphere and x a vertex in S. Our purpose is to define the sphere S^x in a way that is independent of the question whether or not S is polytopal and to describe a method for finding it. Generally speaking, S^x will be obtained by removing the complex $\operatorname{st}(x, S)$ from S and refilling the "hole" thus created in S with a suitable complex that uses only vertices of S other than x. In the case where all the facets of S which contain x are simplices, this complex is actually the "refill" defined and studied in [7, page 100]. In the general case, however, there arise some complications which we are to explain now.

Assume for the moment that S is polytopal, that P is a 4-polytope realizing S with the same labelling of the vertices, that $Q = \operatorname{conv}(\operatorname{vert} P \setminus \{x\})$ and that S^x is a 3-sphere isomorphic to $\operatorname{bd} Q$, again with the same labelling of the vertices. (This convention allows us to treat a cell combinatorially—that is, as a member in S—or geometrically, as a cell in $\operatorname{bd} Q$, according to need.) As before, let $\mathcal{C}(\mathcal{B})$ denote the set of facets of Q—and of S^x—beyond (beneath) which x lies, and let $\mathcal{C}$ be the set of facets of Q in the affine hulls of which x lies. clearly, $\mathcal{B}$ is the set of 3-cells in $\operatorname{ast}\{x, S\}$.

Now let $C \in \mathcal{C}$. As a facet of Q, C has the property that $C' = $ conv(vert $C \cup \{x\}$) is a facet of P (and of S), and vert $C' = $ vert$C \cup \{x\}$. As a 3-cell in S^x, however, without reference to a particular polytope P chosen to realize S, all we can say about C is that vert $C \cup \{x\}$ is the vertex-set of some facet C' in S, and there is some realization of C' as a 3-polytope such that C is isomorphic to conv(vert $C' \setminus \{x\}$). Generally speaking, this does not define C uniquely, and every such C will be called *an admissible 3-cell arising from C' at x.*

As an example, let C' be the 3-cell of type B in Figure 1, with $x = 1$. Depending on the position of the vertices of C', there are three admissible candidates for C: the 3-pyramid 24356 with apex 6, the double tetrahedron 23645 (with order of vertices being "canonic", as in Figure 1, type A) and the double tetrahedron 45623.

We now define the concept of a *refill* (compare [8, page 100]) and the sphere S^x.

DEFINITION 1. Let S be a non-pyramidal 3-sphere, and let x be a vertex in S. A 3-complex R is called *a refill for S at x* if

(i) the union $|R|$ of the cells in R is a 3-ball,

(ii) all the vertices in R are vertices in ast(x, S),

(iii) $R \cap$ ast(x, S) is the boundary complex of both R and ast(x, S),

(iv) for every 3-cell C' in st(x, S) which is not a 3-pyramid with apex x, there is in R an admissible 3-cell C arising from C' at x, and

(v) ast(x, S) $\cup R$ is a 3-sphere.

REMARK. The Auxiliary Theorem in [3, page 407] can be used here, as in [3, pages 409, 410], to show that condition (v) in the last definition can be replaced by

(v') No missing edge of valence > 0 interior to ast(x, S) (resp. R) is an edge or a missing edge of valence > 0 in R (resp. ast(x, S)), and no missing 2-face in ast(x, S) (resp. R) is a 2-face or a missing 2-face in R (resp. ast(x, S)).

DEFINITION 2. In the notation of Definition 1, if R is a refill for S at x, we denote the 3-sphere ast(x, S) $\cup R$ by S^x, and call it a *contraction of S at x*. We also say that S is *directly obtainable* from S^x at x (Compare [8, page 100]) and that S is *contractible* at x.

Clearly, a polytopal 3-sphere S has a contraction at each of its vertices, the contraction at the vertex x being the boundary complex of conv(vert $P \setminus \{x\}$), where P is any 4-polytope realizing S. Thus:

THEOREM 2. *If a 3-sphere S has a vertex at which it is not contractible, then S is not polytopal.*

How can we tell if a 3-sphere S is contractible at a vertex x? In other words, how can we find a refill R for S at x? The first step is to find the set $C = C(R)$ of all the 3-cells C in R which are the admissible 3-cells arising from those facets C' of S which contain x but are not pyramidal at x. As already mentioned, in general, a facet C' of S which contains x (and is not pyramidal at x) does not define a unique admissible C. If, however, C' is a double tetrahedron, then C is unique, and it is a tetrahedron.

In each of the first 520 spheres S_i of our list of 617 all the facets are just tetrahedra and double tetrahedra, and therefore at each vertex x of S_i, the set C is uniquely and easily determined by S_i and x. If S_i has a refill R at x, let $\mathcal{Q}$ denote the set of all 3-cells in $R \backslash C$, and let C' ($\mathcal{Q}'$) denote the complex of all the faces of the members of C (resp. $\mathcal{Q}$).

Clearly $bd(ast(x, S_i) \cup C') = bd\mathcal{Q}'$. Since $bd(ast(x, S_i) \cup C')$ is simplicial, we conclude that $bd\,\mathcal{Q}'$ is simplicial. Thus even if $\mathcal{Q}'$ is not simplicial, it can be simplicially subdivided, without the addition of extra vertices. (We don't know if this assertion is true in general, but it is easy to check that it is true in our case, where $\mathcal{Q}'$ has at most seven vertices.) We may therefore assume that $\mathcal{Q}'$ (and therefore also the entire refill R) is simplicial. Since we know the boundary complex of $\mathcal{Q}'$, we may invoke the algorithm described in [1] to compute $\mathcal{Q}'$, and to actually compute *all* the possible simplicial complexes $\mathcal{Q}'$. The strong restriction that no face interior to $ast(x, S_i) \cup C'$ can be in $\mathcal{Q}'$, and the additional restrictions mentioned in Condition (v') make the process of computing $\mathcal{Q}'$ very efficient.

Thus we prepared a list of all the simplicial refills for each sphere S_i, $1 \leq i \leq 520$, at each of its vertices. For 18 of those 520 spheres there was at least one vertex at which the sphere is not contractible, and therefore those 18 spheres (listed in Table 1) are not polytopal.

As an example, consider the sphere $S = S_{137}$ of Table 1 at its vertex 4. Here the 3-cells in $ast(4, S)$ are 13567, 25718, 2356, 2358 and 3578. (In a quasisimplicial 3-sphere, a 3-cell with five vertices is necessarily a double-tetrahedron, and its representation is canonic, as in Figure 1, type A.) C is composed of 1256 and 2368. The 2-faces in $bd(ast(4, S) \cup C')$, and therefore also in $bd\,\mathcal{Q}'$, are all triangles: 136, 137, 127, 278, 378, 126, 268 and 368, and they define a triagulation of T of a 2-sphere. We are looking for a triangulation $\mathcal{Q}'$ of the 3-ball with boundary T, which uses only vertices from the set $\{1, 2, 3, 5, 6, 7, 8\}$.

Since the vertex 5 and the edge 23 are interior in $\operatorname{ast}(4, S) \cup \mathcal{C}'$, they should not appear in $\mathcal{C}'$. Also, 18 and 67 are missing edges of valence 1 in $\operatorname{ast}(4, S) \cup \mathcal{C}'$, and they too should not appear in $\mathcal{C}'$. Now, since the triangle 136 is in T, there must be some tetrahedron $136y$ in $\mathcal{C}'$, where $y \in \{2, 5, 7, 8\}$, but because of the above restrictions, no such y is possible. We thus conclude that S_{137} is not contractible at the vertex 4, and therefore it is not polytopal.

TABLE 1

The non-polytopal, quasisimplicial non-simplicial 3-spheres with eight vertices

Case S_i i =	3-cells					Non contractible at
125	24718	13467	3468	2578		1,8
	15723	1356	2568	3578		
det. = 1200	12645	2468	3568	3478		
137	13567	23468	1347	3578		4,5
	25718	1247	2356	2478		
det. = 4087	12645	1346	2358	3478		
194	15723	1247	3468	2578	3478	1,8
	12645	1356	2568	3578		
det. = 244	13467	2468	3568	2478		
199	34718	1256	1356	2578	2478	4,5
	12467	1257	1357	3578		
det. = 1100	23568	1346	2346	2348		
348	24718	1257	2346	3568	2348	4,5
	12645	1356	2368	2578	3478	
det. = 31008	13467	1357	2568	3578		
349	24718	1257	1357	2368	2348	2,3
	12645	1346	1347	3568	3478	
det. = 31775	25867	1356	2346	3578		
354	16748	2367	1237	1234	1358	5,7
	24657	1456	1345	1378	1568	
det. = 41888	35628	1247	2345	3678		
355	35628	2467	1456	1345	3678	5,7
	16748	2367	1247	2345	1568	
det. = 43208	13857	2456	1237	1234		
356	13658	2367	1237	1234	1468	3,4
	24657	2356	1345	1378	4678	
det. = 51847	14728	1456	2345	3678		
401	12645	1247	1347	2568	2348	6,7
	35618	1346	2346	2578	2478	
det. = 29760	1257	1357	2368	3578	3478	
416	16748	2367	1247	2345	3678	5,7
	35628	2456	1237	1234	1358	
det. = 40768	2467	1456	1345	1378	1568	

TABLE 1 (*Continued*)

Case S_i i =			3-cells				Non contractible at
487	24657	1467	1456	1278	2348	3458	1,2
	23768	2356	1247	1378	1358		
det. = 211023	1367	1356	2345	1248	1458		
488	23768	1367	2356	1247	1248	3458	6,8
	13857	1467	1356	2345	2348		
det. = 214660	2467	2456	1456	1278	1458		
489	34718	1237	1245	2345	2378	4678	2,4
	23658	1267	1457	3456	2678		
det. = 241808	1234	1256	1567	4567	3468		
490	16728	1237	1347	2356	1568	4678	5,7
	45638	1256	1457	2367	1578		
det. = 262300	1234	1245	2345	3467	4578		
506	13857	1367	2356	1247	2378	1458	6,8
	2467	1467	1356	2345	1248	3458	
det. = 236288	2367	2456	1456	1278	2348		
519	16758	1347	2356	3456	1348	2378	2.7
	1256	1457	2367	4567	2348	1268	
det. = 897117	1245	2345	3467	1248	1378	2678	
520	23678	1256	3467	1257	1248	1268	7,8
	1234	1237	4567	1457	3568	1568	
det. = 924672	3456	1347	2567	2348	3458	1458	

det. is the determinant of the edge-valence matrix. Every 3-cell with five vertices is a double-tetrahedron.

6. Classification: 596 polytopal spheres.

The following three theorems are particular cases of more general theorems stated and proved in [5], and are used here to show that 596 of our 617 spheres are polytopal.

THEOREM 3. *Let $Q \subset R^4$ be a 4-polytope, let L be an edge of Q and let $A_1, \ldots, A_m$ be all the facets of Q in st(L, Q) in their natural cyclic order, that is, $A_i \cap A_{i+1}$, $1 \le i \le m$ (where $A_{m+1} = A_1$) is the 2-face common to A_i, A_{i+1} and contains L. Then for every $n \le m$ there are points x_0, x_1, x_2 in R^4 such that, with respect to Q:*

x_0 *lies beyond $A_1, \ldots, A_n$ and beneath all the other facets of Q;*

x_1 *lies in aff A_1 beyond $A_2, \ldots, A_n$ and beneath all the other facets of Q (provided $n \ge 2$);*

x_2 *lies in aff $A_1 \cap$ aff A_n, beyond $A_2, \ldots, A_{n-1}$ and beneath all the other facets of Q (provided $n \ge 3$);*

and, for each $0 \le i \le 2$, vert conv($Q \cup \{\dot{x}_i\}$) = vert $Q \cup \{x_i\}$.

REMARK. Note that if in Theorem 3, case x_0, we take $n = 1$, we get the frequently used (and trivial) assertion that for very facet F of a 4-polytope Q there is a point which lies beyond F and beneath all the other facets of Q.

THEOREM 4. *With the conventions of Theorem 3, let $n \leq m$, let y be a vertex of L and let $\mathfrak{D}$ be the set of all the facets of Q other than $A_1, \ldots, A_n$ which contain y. Then there are points x_0', x_1', x_2' in R^4 such that, with respect to Q:*

x_0' *lies beyond every $F \in \mathfrak{D}$ and beneath all the other facets of Q;*

x_1' *lies in aff A_1, beyond every $F \in \mathfrak{D}$ and beneath all the other facets of Q;*

x_2' *lies in aff $A_1 \cap$ aff A_n, beyond every $F \in \mathfrak{D}$ and beneath all the other facets of Q (provided $n \geq 2$);*

and, for each $0 \leq i \leq 2$, vert $\operatorname{conv}(Q \cup \{x_i'\}) = \operatorname{vert} Q \cup \{x_i'\}$.

THEOREM 5. *Let $Q \subset R^4$ be a 4-polytope, and let A_0 be a facet of Q which contains a vertex p whose valence in A_0 is 3 and whose valence in Q is ≥ 5. Let A_1, A_2, A_3 be the facets of Q adjacent to A_0 (that is, sharing a common 2-face with A_0) and containing p, and let $D = \{1, 2, 3\}$. Then for every two disjoint subsets S, T of D (including the cases $S = \varnothing$, $T = \varnothing$) there are points $x_1, x_2 \in R^4$ such that, with respect to Q:*

x_1 *lies in $\bigcap_{t \in T} \operatorname{aff} A_t$, beyond A_0, beyond every A_s, $s \in S$, and beneath all the other facets of Q;*

x_2 *lies in $\bigcap_{t \in T} \operatorname{aff} A_t$, beyond all the facets of Q other than A_0, A_1, A_2, A_3, which contain p, beyond A_r ($r \in D \backslash (S \cup T)$), and beneath all the other facets of Q;*

and for $i = 1, 2$, vert$(\operatorname{conv}(Q \cup \{x_i\})) = \operatorname{vert} Q \cup \{x_i\}$

REMARK. In Theorem 5, the condition that the valence of p in Q is ≥ 5 is superfluous in case $S \cup T \neq D$.

As already mentioned, for each S_i, $1 \leq i \leq 520$, and for each vertex x of S_i, we computed all the simplicial refills for S_i at x. As an example, consider the sphere S_{195} (see Table 2). At its vertex 2, it has a refill R_1, the 3-cells of which are 1567, 3568, 4678, 5678, –1467, –3468 (the negative sign before a 3-cell denotes that that 3-cell is in $\mathcal{C} = \mathcal{C}(R_1)$). All those 3-cells contain the vertex 6. The 3-cells in ast$(2, S_{195})$ which contain the vertex 6 are 1346 and 1356. We now use the second part of Theorem 5 to show that S_{195} is polytopal.

TABLE 2

Some polytopal quasisimplicial 3-spheres with eight vertices

Case S_i $i =$	3-cells				
15	12768	12546	2345	1378	
	13657	23847	2356		
det. = 112	13458	2367	1248		
127	23748	26857	1345	3467	
	35618	1245	1346	2358	
det. = 1426	24617	1256	2345	3678	
141	12378	1345	1367	2356	2348
	1256	1356	2367	1248	
det. = 0	1245	1267	2345	1348	
195	34718	1256	1356	2358	2478
	12467	1257	1357	2578	
det. = 500	23468	1346	2356	3578	
325	13658	2367	1456	1234	1238
	24657	1467	1345	1678	2378
det. = 14736	12748	2356	2345	3678	
617	578631	1467	1456	1278	2348
	2467	2456	1247	2378	1458
det. = 53760	2367	2356	2345	1248	3458

det. is the determinant of the edge-valence matrix. Every 3-cell with five vertices is a double-tetrahedron. The 3-cell with six vertices (in Case 617) is an octahedron.

The contraction $S^2_{195} = \text{ast}(2, S_{195}) \cup R_1$ is a 3-sphere with seven vertices, and therefore it is polytopal. Let Q be any 4-polytope which realizes S^2_{195}, with the same labelling of the vertices. To conform to the notation of Theorem 5, let $A_0 = 1346$, $A_1 = 1467$, $A_2 = 3468$, $A_3 = 1356$, $\mathfrak{D} = \{1567, 3568, 4678, 5678\}$, $T = \{1, 2\}$, $S = \{3\}$ and $p = 6$.

The second part of Theorem 5 and Grünbaum's theorem now guarantee the existence of a point x_2 in R^4 and the existence of a combinatorial equivalence φ, such that the polytope $P = \text{conv}(Q \cup \{x_2\})$ has eight vertices $1, x_2, 3, 4, \ldots, 8$ and φ is a combinatorial equivalence between S_{195} and bd P such that $\varphi(i) = i$ for every $1 \le i \le 8$, $i \ne 2$, and $\varphi(2) = x_2$. Therefore S_{195} is polytopal.

S_{195} has another simplicial refill R_2 at the vertex 2, the 3-cells of which are 1567, 3568, 4568, 4578, 4567, -1467, -3468. But this refill is not useful for our purpose, since there is no vertex common to all the 3-cells in R_2, and therefore none of the Theorems 3, 4, 5 can be used here. One would like to correct the situation by gluing together the 3-cells 4578 and 4567, that is, by replacing in R_2 these two 3-cells by the type A cell 45768,

thus obtaining a refill all of whose 3-cells contain the vertex 6. But this will not yield a refill, since the edge 68 would appear here both as an edge in 3468 and as a missing (interior) edge in the new 3-cell 45768.

On the other hand, S_{195} has two simplicial refills at vertex 8, the 3-cells of one of which are 2347, 2357, -1347, -2346. Here 2347 and 2357 can be glued together to yield the (type A) cell 23745, and now the third part of Theorem 3 can be evoked, with $L = 34$, $n = 3$, $A_1 = 1347$, $A_2 = 23745$, $A_3 = 2346$, to yield once again that S_{195} is polytopal.

In a similar manner we proved that 499 of the 520 spheres S_i, $1 \leq i \leq 520$, are polytopal. In each of those cases there was a *simplicial* refill which, together with one of the Theorems 3, 4, 5, did the job. Since we have already seen that 18 of the spheres S_i, $1 \leq i \leq 520$, are not polytopal. There remain just three undecided spheres in this group and we deal with them in the next section. They are S_{15}, S_{127} and S_{325}.

As for the remaining 97 spheres S_i, $521 \leq i \leq 617$, each of which has a 3-cell of either type B or C, the process of finding *all* the refills at *all* the vertices is slightly more complicated—because of the non-uniqueness of the admissible 3-cell arising from a cell of type B or C—and we did not attempt it. However, for each of these spheres we found a vertex and a refill at that vertex, for which one of the Theorems 3, 4, 5 could be used to prove that the sphere is indeed polytopal. We now describe one such example.

Consider the last sphere S_{617} (see Table 2). We will find a refill at the vertex 1. Besides tetrahedra, st$(1, S_{617})$ contains the octahedron 578631 (the notation is canonic, as in Figure 1, type C). This octahedron admits three admissible 3-cells at its vertex 1: a type B cell 35768, a type B cell 36857, and a pyramid 56783 with apex 3. Now, *each* of these admissible
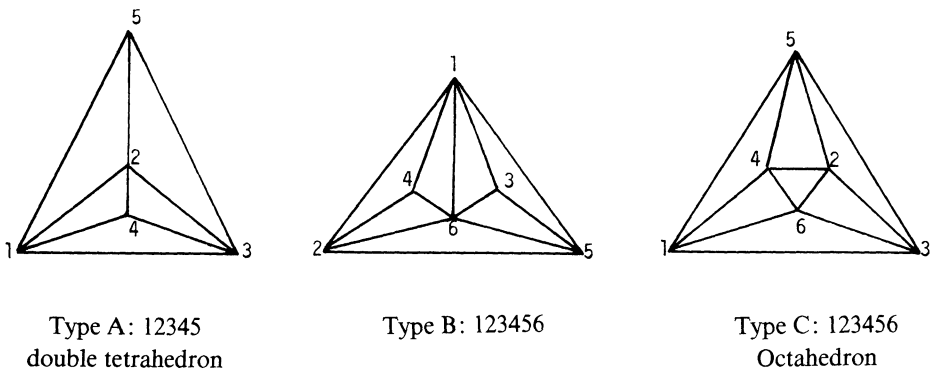

Type A: 12345   Type B: 123456   Type C: 123456
double tetrahedron   Octahedron

FIGURE 1

3-cells can be completed into a refill from which Theorems 3, 4 and 5 establish the polytopality of S_{617}. For the refill constructed from the type B cell $A_1 = 35768$, the remaining 3-cells are $A_0 = 4578$, $A_2 = 2478$, $A_3 = 4567$, which conforms to the first part of Theorem 5 with $p = 7$, $T = \{1\}$, $S = \{2,3\}$. For the refill constructed from the type B cell $A_1 = 36857$, the remaining 3-cells are $A_0 = 4678$, $A_2 = 2478$, $A_3 = 4568$, and this too conforms to the first part of Theorem 5, with $p = 8$, $T = \{1\}$, $S = \{2,3\}$. Finally, for the refill constructed from the pyramid $A_1 = 56783$ (with apex 3), the other 3-cells are a pyramid $A_3 = 56784$ with apex 4 and a tetrahedron $A_2 = 2478$. This conforms to the second part of Theorem 3 with $L = 78$, $m = 4$, $n = 3$.

7. Classification: The last three cases.

We now turn our attention to the three remaining spheres S_{15}, S_{127}, S_{325} (see Table 2) which are undecided, and we show that they are polytopal.

The sphere S_{15}. To see that none of the Theorems 3, 4, 5 can be used here, we note that for every vertex x other than 5, 6, and for every refill R of S_{15} at x, the 3-cells in $\mathcal{C} = \mathcal{C}(R)$ (that is, the 3-cells in R whose existence is required by part (iv) of Definition 1) do not share a common vertex. At vertex 5, the only refill is composed of the 3-cells 1346, 2346, -1367, -1348, -1246, but these five tetrahedra do not share a common vertex, and 1346 cannot be glued to 2346 to change the situation. A similar situation occurs at vertex 6, where the only refill consists of the 3-cells 1257, 2357, -1278, -1357, -1245.

With the refill for S_{15} at 5 mentioned above, let Q be any 4-polytope realizing S_{15}^5, the contraction of S_{15} at 5, and containing the origin in its interior. Recall that every 5-vertex facet in S_{15}, and therefore also in Q, is a double tetrahedron. The facets of Q are as follows:

$$A_1 = 1346 \quad C_1 = 1246 \quad C_3 = 1367 \quad B_2 = 1248 \quad B_4 = 23847$$
$$A_2 = 2346 \quad C_2 = 1348 \quad B_1 = 1378 \quad B_3 = 2367 \quad B_5 = 12768$$

Our goal is to prove the existence of a point x in R^4 which, with respect to Q, lies in $\bigcap_{k=1}^3 \operatorname{aff} C_k$, beyond each A_i, $i = 1, 2$ and beneath each B_j, $1 \le j \le 5$. Let Q^* be the 4-polytope polar to Q, and for every i-face $(0 \le i \le 3)$ F of Q denote by F^* the corresponding $(3-i)$-face of Q^*. Our goal will be achieved by finding a hyperplane H in R^4 which contains the vertices C_k^*, $1 \le k \le 3$, of Q^* and strictly separates the vertices A_i^*, $i = 1, 2$ from the vertices B_j^*, $1 \le j \le 5$, since x can be taken as the point polar to H.
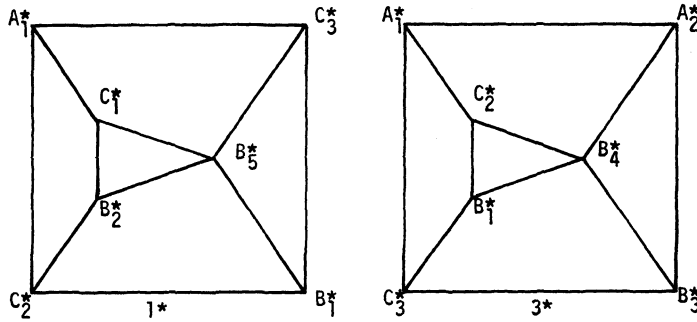
The facets 1*, 3* of $Q^* \approx \left(S_{15}^5 \right)^*$

FIGURE 2

Figure 2 depicts Schlegel diagrams of the facets 1* and 3* of Q^*. The hyperplane aff 1* supports Q^* and therefore does not contain the vertex A_2^*. Let y be any point interior to the edge $A_2^* B_3^*$, which is not in $F = \text{aff}\{C_1^*, C_2^*, C_3^*\}$. We claim that $H = \text{aff}\{y, C_1^*, C_2^*, C_3^*\}$ is the desired hyperplane.

Indeed, let H^+, H^- be the open halfspaces of R^4 determined by H so that $A_2^* \in H^-$, and let $G = H \cap \text{aff} \, 3^*$. A study of the 2-face $A_1^* A_2^* B_3^* C_3^*$ reveals that, in aff 3*, G strictly separates A_1^* and A_2^* from B_3^*. A study of the 2-face $A_1^* C_2^* B_1^* C_3^*$ reveals that, in aff 3*, G strictly separates A_1^* from B_1^*, and a study of the 2-face $B_1^* B_4^* B_3^* C_3^*$ reveals that, in aff 3*, B_1^*, B_3^*, and B_4^* lie on the same side of G. Thus A_1^*, $A_2^* \in H^-$ and B_1^*, B_3^*, $B_4^* \in H^+$. Now consider the facet 1* of Q^*. Since $F = H \cap \text{aff} \, 1^*$ and, in aff 1*, F strictly separates A_1^* from B_1^*, B_2^* and B_5^*, it follows that B_2^*, $B_5^* \in H^+$.

We thus conclude that S_{15} is polytopal.
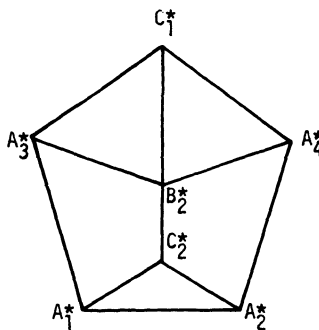

The facet 8* of $Q^* \approx \left(S_{127}^3 \right)^*$

FIGURE 3

The sphere S_{127}. A refill for S_{127} at vertex 3 consists of the 3-cells $A_1 = 1458$, $A_2 = 1468$, $A_3 = 2458$, $A_4 = 4678$, $C_1 = 2478$, $C_2 = 1568$. In addition, S_{127}^3 contains the 3-cells $B_1 = 24617$, $B_2 = 26857$, $B_3 = 1245$, $B_4 = 1256$. Let Q be any 4-polytope realizing S_{127}^3 with the origin of R^4 in its interior, and let Q^* be the polytope polar to Q. Figure 3 depicts a Schlegel diagram of the facet 8^* of Q^*.

As in the case of S_{15}, it is sufficient to find a hyperplane H which contains C_1^*, C_2^* and strictly separates A_1^*, A_2^*, A_3^*, A_4^* from B_1^*, B_2^*, B_3^*, B_4^*. Let $F = \mathrm{aff}\{C_1^*, C_2^*, A_4^*\}$ and let F^+, F^- be the open 3-dimensional half-spaces determined by F in aff 8^* so that $A_3^* \in F^-$. Then clearly A_1^*, $A_2^* \in F^-$. As F separates A_2^* from B_2^*, $B_2^* \in F^+$. Thus a slight rotation in aff 8^* of F about the line $\mathrm{aff}\{C_1^*, C_2^*\}$ yields a plane G which, in aff 8^*, contains C_1^*, C_2^* and strictly separates B_2^* from A_1^*, A_2^*, A_3^* and A_4^*. Finally, a slight rotation of aff 8^* about G yields the desired hyperplane H. S_{127} is therefore polytopal.

The sphere S_{325}. A refill for S_{325} at vertex 8 consists of the 3-cells $A_1 = 1237$, $A_2 = 1367$, $C_1 = 1356$, $C_2 = 1247$. In addition, S_{325}^8 contains the 3-cells $B_1 = 24657$, $B_2 = 2367$, $B_3 = 1467$, $B_4 = 2356$, $B_5 = 1456$, $B_6 = 1345$, $B_7 = 2345$, $B_8 = 1234$. Let Q be any 4-polytope realizing S_{325}^8 with the origin of R^4 in its interior, and let Q^* be the polytope polar to Q. Figure 4 depcits a Schlegel diagram of the facet 1^* of Q^*.

To prove the polytopality of S_{325} it is sufficient here, as before, to find a hyperplane H which contains C_1^*, C_2^* and strictly separates A_1^*, A_2^* from B_j^*, $1 \le j \le 8$. Let $F = \mathrm{aff}\{C_1^*, C_2^*, B_3^*\}$ and let F^+, F^- be the two open 3-dimensional half-spaces determined by F in aff 1^*, so that $A_2^* \in F^-$. Then clearly $B_5^* \in F^+$. But, in aff 1^*, B_5^*, B_6^*, B_8^* are on the same side of F, hence B_6^*, $B_8^* \in F^+$. Since A_2^*, A_1^* are on the same side of F, $A_1^* \in F^-$. Thus, a slight rotation in aff 1^* of F about the line $\mathrm{aff}\{C_1^*, C_2^*\}$ yields a
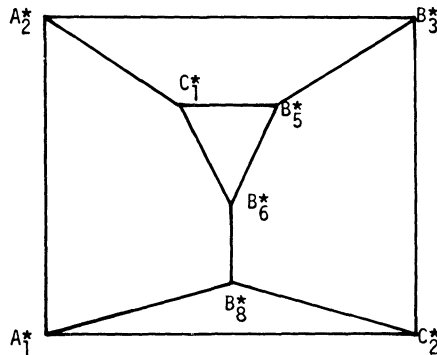
The facet 1^* of $Q^* \approx \left(S_{325}^8\right)^*$

FIGURE 4

plane G which, in aff 1^*, contains C_1^*, C_2^* and strictly separates A_1^*, A_2^* from B_3^*, B_5^*, B_6^* and B_8^*. Finally, a slight rotation of aff 1^* about G yields the desired hyperplane H. Thus, S_{325} too is polytopal.

This ends the proof of the main result in the present work, namely:

THEOREM 6. *There are precisely* 661 *quasisimplicial* 3-*spheres with eight vertices, and precisely* 20 *of them are not polytopal.*

8. REMARKS.

1. The sphere S_{617}, discussed in detail in §6 and whose polytopality follows from our general arguments, is actually the 3-sphere discussed by Kleinschmidt in [18], and shown by him to be a minimal polytope having a facet of non-arbitrary shape. Kleinschmidt proves the polytopality of this sphere by using a theorem by Shephard.

2. As already mentioned, 10 of our 18 non-polytopal spheres appear in [20]. They are the following, where the number in parentheses denotes the notation in [20]: 125(15), 194(6), 348(16), 349(24), 401(7), 487(25), 488(17), 506(8), 519(26) and 520(18). Using our terminology and Schulz's labelling of the vertices, the proof for the non-polytopality of those spheres (and also of Barnette's and Grünbaum's spheres, in Schulz's labelling) is common and brief. If S is any of those spheres, then the 2-faces in bd ast(8, S) are the triangles 123, 125, 134, 145, 236, 256, 456. Since 16, 24 and 35 are either edges or missing edges of valence 1 in ast(8, S), and 7 is an interior vertex there, none of them should appear in a refill for S at 8. But then even the first triangle, 123, cannot be completed into a 3-simplex. Thus all those spheres are not contractible at the vertex 8, and therefore are not polytopal.

3. It is a fact for which we have no explanation, that each of the 20 non-polytopal quasisimplicial 3-spheres with eight vertices is not contractible at precisely *two* vertices. For our 18 non-simplicial cases these two vertices are given in Table 1. Barnette's sphere is not contractible at the vertices 7, 8 (in the original labelling of [10]), and Grünbaum's sphere is not contractible at the vertices 4, 6 (in the original labelling of [15]). Hopefully, in the second part of this work we will find if for every nonpolytopal 3-sphere S with eight vertices, S is not contractible at some vertex, and if so, that it is not contractible at precisely two vertices. There is however, a non-polytopal simplicial 3-sphere with nine vertices that is contractible at each vertex (the non-polytopal sphere in [9]), as well as many such 3-spheres with 10 vertices ([3]). On the other hand, there is a 3-sphere with ten vertices which is not contractible at all, that is, there is no vertex at which this sphere is contractible (see [2]).

4. Each of our 18 non-polytopal spheres has the same number of simplicial refills (1 or 2) at each of the six vertices at which it is contractible. This probably reflects some symmetry properties of those spheres. As for the entire list of 520 3-spheres whose 3-cells are just tetrahedra and double-tetrahedra—which are exactly the spheres for which we computed all the simplicial refills at all the vertices—the number of simplicial refills usually varies from one vertex to the other in the same sphere. Many spheres admit just one simplicial refill at each vertex, but there are also spheres that admit many simplicial refills at the same vertex. There are three spheres (S_{61}, S_{63}, S_{249}) that admit 16 simplicial refills at some of their vertices, two spheres (S_{144}, S_{145}) that admit 23 simplicial refills at some of their vertices, and there is a sphere (S_{141}, see Table 2) which admits 35 simplicial refills at vertices 1, 2, and 3.

5. In §6 we used Theorems 3, 4 and 5 to prove the polytopality of most of our spheres. Those theorems, however, are not totally "disjoint". The case $S \cup T \neq D$ in Theorem 5 is "covered" also by Theorems 3, 4.

6. The non-polytopal 3-spheres discussed in the present work tend, with exceptions, to have high determinants, often the highest determinants (that is, the determinants of the edge-valence matrices) among the 3-spheres which share the same number of vertices and the same number of 3-cells, particularly when the number of 3-cells is even. The same phenomenon has been observed in the simplicial 3-spheres with nine vertices ([8]) and in the neighborly 3-spheres with 10 vertices ([3])—that is, among all the 3-spheres of which we have complete catalogues.

REFERENCES

[1]   A. Altshuler, *Combinatorial 3-manifolds with few vertices*, J. Combinatorial Theory (A), **16** (1974), 165–173.

[2]   _____, *A peculiar triangulation of the 3-sphere*, Proc. Amer. Math. Soc., **54** (1976), 449–452.

[3]   _____, *Neighborly 4-polytopes and neighborly combinatorial 3-manifolds with* 10 *vertices*, Canad. J. Math., **29** (1977), 400–420.

[4]   _____, *3-Pseudomanifolds with preassigned links*, Trans. Amer. Math. Soc., **241** (1978), 213–237.

[5]   A. Altshuler and I. Shemer, *Construction theorems for convex polytopes*, to appear.

[6]   A. Altshuler and L. Steinberg, *Neighborly 4-polytopes with nine vertices*, J. Combinatorial Theory (A), **15** (1973), 270–287.

[7]   _____, *Neighborly combinatorial 3-manifolds with nine vertices*, Discrete Math., **8** (1974), 113–137.

[8]   _____, *An enumeration of combinatorial 3-manifolds with nine vertices*, Discrete Math., **16** (1976), 91–108.

[9]   A. Altshuler, L. Steinberg and J. Bokowski, *The classification of simplicial 3-spheres with nine vertices into polytopes and nonpolytopes*, Discrete Math., **31** (1980), 115–124.

[10] D. Barnette, *The triangulations of the 3-sphere with up to 8 vertices*, J. Combinatorial Theory (A), **14** (1973), 37–53.

[11] U. Betke, Ch. Schulz and J. M. Wills, *Bänder und Mäbiusbänder in konvexen Polytopen*, Abh. Math. Sem. Hamburg, **47** (1978), 113–127.

[12] G. Danaraj and V. Klee, *Shellings of spheres and polytopes*, Duke Math. J., **41** (1974), 443–451.

[13] P. J. Federico, *Enumeration of polyhedra: The number of 9-hedra*. J. Combinatorial Theory, **7** (1969), 155–161.

[14] B. Grünbaum, *Convex Polytopes*, Interscience, John Wilet and sons, 1967.

[15] B. Grünbaum and V. P. Sreedharan, *An enumeration of simplicial 4-polytopes with 8 vertices*, J. Combinatorial Theory, **2** (1967), 437–465.

[16] P. Kleinschmidt, *Sphären mit wenigen Ecken*, Geom. Dedicata, **5** (1976), 307–320.

[17] _____, *Nicht-dualisierbar sphären*, Geom. Dedicata, **6** (1977), 511–515.

[18] _____, *On facets with non-arbitrary shapes*, Pacific J. Math., **65** (1976), 97–101.

[19] M. Reif, *4-polytopes with 8 vertices*, M. Sc. thesis under the supervision of A. Altshuler, Ben-Gurion University of the Negev, 1979.

[20] Ch. Schulz, *Nicht-Polytopale 3-sphären mit 8 ecken*, Geom. Dedicata, **13** (1982), 325–329.

[21] _____, *Geometrische Realisierungen Zellzerlegter Mannigfaltigkeiten in Euklidischen Räumen*, Habilitationsschrift, (mimeographed) 1980.

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