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A SOLUTION TO A PROBLEM OF E. MICHAEL

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A product space $X \times Y$ is *rectangularly normal* if every continuous real-valued function defined on a closed rectangle $A \times B$ in $X \times Y$ can be continuously extended onto $X \times Y$. It is known that products of normal spaces with locally compact metric spaces are rectangularly normal. In this paper we prove the converse of this theorem by showing there exists a normal space X such that its product $X \times M$ with a metric space M is rectangularly normal if and only if M is locally compact, thus answering positively a question raised by E. Michael.

Other related results are obtained; in particular, we show there exists a normal space X and a countable metric space M with one non-isolated point such that the product space $X \times M$ is not *rectangular* (in the sense of Pasynkov).

1. Introduction. Let R, Q and I denote the reals, the rationals and the unit segment. We say that a product space $X \times Y$ is *rectangularly normal* if every continuous real-valued function $f: A \times B \to R$ defined on a closed rectangle $A \times B$ in $X \times Y$ can be continuously extended onto $X \times Y$. The concept of rectangular normality—being a natural weakening of normality—first appeared implicitly in papers of Morita [**M9**], Starbird [**S, S2**] and Miednikov [**Mi**] in connection with their successful attempt to generalize the Borsuk Homotopy Extension Theorem. It turned out that even though normality and countable paracompactness of X are necessary (and sufficient) for the normality of the product $X \times I$, only normality of X suffices to ensure rectangular normality of $X \times I$. More generally, the following theorem holds:

1.1. THEOREM [**Mo, S2, Mi**]. *Products of normal spaces with locally compact metric spaces are rectangularly normal.* □

In this paper we prove the converse of this Theorem by showing that there exists a normal space X whose product with a metric space M is rectangularly normal if and only if M is locally compact (Example 2.5). In particular, X is a normal space whose product, $X \times Q$, with the space of rationals Q is not rectangularly normal. This answers a question raised by E. Michael.

The existence of the above space X is a consequence of Theorem 2.4, which states that $X \times M$ is rectangularly normal for some non-locally compact metric space M if and only if X is countably functionally Katětov (see the definition in §2), and the existence of a normal space which is not countably functionally Katětov ([**PW**]; Example 3).

Other related results are proved; in particular, we give an example of a normal space X and a countable metric space M with one non-isolated point, whose product $X \times M$ is not *rectangular* in the sense of B. A. Pasynkov [**Pa, Pa2**].

For all the undefined notions the reader is referred to [**E**]. For a cardinal number κ, we denote by $J(\kappa)$ the *hedgehog with κ spikes*, i.e. $J(\kappa) = \{0\} \cup \{\langle \alpha, t\rangle: \alpha \in \kappa \text{ and } 0 < t \leq 1\}$, where points $\langle \alpha, t\rangle$ have basic neighborhoods of the form $\{\langle \alpha, t'\rangle: t - 1/n < t' < t + 1/n\}$, $n = 1, 2, \ldots$, and the point θ has basic neighborhoods of the form

$$B(n) = \{\theta\} \cup \{\langle \alpha, t\rangle: \alpha \in \kappa \text{ and } t \leq 1/n\}.$$

By $J_0(\kappa)$ we denote the closed subspace $\{\theta\} \cup \{\langle \alpha, 1/n\rangle: \alpha \in \kappa,\ n = 1, 2, \ldots\}$ of $J(\kappa)$. One can easily see that every non-locally compact space contains a closed copy of $J_0(\omega)$.

A subset of a space is an *F_κ subset* if it is a union of $\leq \kappa$ closed sets. A subset A of a space X is *C-embedded* (*C^*-embedded*) in X if every continuous function $f: A \to R$ ($f: A \to I$) can be continuously extended over X. We say that a covering $\{W_s\}_{s \in S}$ of X is an *extension* of a covering $\{G_s\}_{s \in S}$ of its subspace A if $W_s \cap A = G_s$ for all $s \in S$.

2 Rectangular normality of products. We will deduce our results from Proposition 2.2 below. In its proof we will use the following result due to E. Michael (see [**S2**]):

2.1. THEOREM (*Michael*). *If F is a closed subset of a metric space Z and X is any space, then $X \times F$ is C-embedded in $X \times Z$.* □

We remark that an analogous theorem holds for compact spaces Z [**S2**], but is false for paracompact p-spaces [**Wa**].

2.2. PROPOSITION (*Main*). *For a cardinal number κ and a closed subset F of a normal space X the following conditions are equivalent*:

(i) *$F \times J(\kappa)$ is C-embedded in $X \times J(\kappa)$*;

(ii) *$F \times J_0(\kappa)$ is C-embedded in $X \times J_0(\kappa)$*

(iii) *every countable locally finite covering of F by open F_κ subsets can be extended to a locally finite open covering of X.*

Proof. The implication (i) ⇒ (ii) is an obvious consequence of Theorem 2.1 and the fact that $J_0(\kappa)$ is a closed subspace of $J(\kappa)$.

(ii) ⇒ (iii). Suppose that $\{G_n\}_{n=1}^{\infty}$ is a countable, locally finite covering of F by open F_κ subsets. Hence there exist zero sets $F_{n,\alpha}$ in F such that

$$G_n = \bigcup_{\alpha<\kappa} F_{n,\alpha},$$

and continuous functions $f_{n,\alpha}\colon F \to I$ such that $f_{n,\alpha}(F_{n,\alpha}) \subset \{1\}$ and $f_{n,\alpha}(F\setminus G_n) \subset \{0\}$. Define a function $f\colon F \times J_0(\kappa) \to I$ as follows:

$$f(x, z) = \begin{cases} 0, & \text{if } z = \theta \\ f_{n,\alpha}(x), & \text{if } z = \langle \alpha, 1/n \rangle. \end{cases}$$

The function f is continuous. Indeed, if $x_0 \in F$ then there exists an $n_0 < \omega$ and a neighborhood V_0 of x_0 such that $V_0 \cap \bigcup_{i\geq n_0} G_i = \varnothing$, and therefore $f(x, z) = 0$ for all $\langle x, z\rangle$ in $V_0 \times B(n_0)$. By (ii) there exists a continuous extension $\tilde{f}\colon X \times J_0(\kappa) \to I$ of f onto $X \times J_0(\kappa)$. Define

$$\begin{aligned} G_n^* &= \{x \in X\colon |\tilde{f}(x, z) - \tilde{f}(x, \theta)| > 3/4 \text{ for some } z \in B(n)\} \\ &= \bigcup_{z\in B(n)} \{x \in X\colon |\tilde{f}(x, z) - \tilde{f}(x, \theta)| > 3/4\}. \end{aligned}$$

Clearly, the sets G_n^* are open in X. We will prove that the family $\{G_n^*\}$ is locally finite in X and that $G_n \subset G_n^*$.

Suppose that $x_0 \in X$. By the continuity of $\tilde{f}$, there exists a neighborhood V_0 of x_0 and an n_0 such that

$$\tilde{f}(V_0 \times B(n_0)) \subset \left(\tilde{f}(x_0, \theta) - \tfrac{1}{4}, \tilde{f}(x_0, \theta) + \tfrac{1}{4}\right)$$

and therefore $V_0 \cap \bigcup_{i\geq n_0} G_i^* = \varnothing$, which implies that the family $\{G_n^*\}_{n=1}^{\infty}$ is locally finite. Suppose now that $x_0 \in G_n$. There exists an $\alpha < \kappa$ such that $x_0 \in F_{n,\alpha}$ and, consequently, $f(x_0, \langle\alpha, 1/n\rangle) = f_{n,\alpha}(x_0) = 1$, but $f(x_0, \theta) = 0$, which implies $x_0 \in G_n^*$.

The covering $\{G_n^{**}\}_{n<\omega}$, where $G_n^{**} = G_n \cup (G_n^*\setminus F)$ for $n > 1$ and $G_1^{**} = G_1 \cup (X\setminus F)$, is obviously a locally finite open extension of $\{G_n\}_{n=1}^{\infty}$.

(iii) ⇒ (i). Suppose that $f\colon F \times J(\kappa) \to I$ is continuous. (For the sake of simplicity we assume that f is bounded; the proof for an unbounded f differs only inessentially, but is technically more complicated.)

It suffices to show that there exists a continuous function $h\colon X \times J(\kappa) \to I$ such that $f^{-1}(0) \subset h^{-1}(0)$ and $f^{-1}(1) \subset h^{-1}(1)$. Since the space $J(\kappa)\setminus\{\theta\}$ is locally compact, by Theorem 1.1 there exists a continuous function $g\colon X \times (J(\kappa)\setminus\{\theta\}) \to I$ extending $f\restriction F \times (J(\kappa)\setminus\{\theta\})$. There also exists a continuous function $g_0\colon X \to I$ such that $\{x \in F\colon f(x, \theta) \leq \frac{1}{3}\} \subset g_0^{-1}(0)$ and $\{x \in F\colon f(x, \theta) \geq \frac{2}{3}\} \subset g_0^{-1}(1)$.

For every $n = 1, 2, \dots,$ let

$$G_n = \{x \in F\colon |f(x, z) - f(x, \theta)| > \tfrac{1}{6}, \text{ for some } z \in B(n)\}.$$

As above, one easily shows that the family $\{G_n\}$ is locally finite, decreasing and consists of open F_κ sets. By (iii) we can find a locally finite family $\{G_n^*\}$ of open subsets of X such that $G_n^* \cap F = G_n$ for all $n = 1, 2, \dots$.

For every $\alpha < \kappa$ and $n = 1, 2, \dots,$ define

$$K_{n,\alpha} = \{x \in F\colon |f(x, z) - f(x, \theta)| \geq \tfrac{1}{3},$$
$$\text{for some } z \in \{\alpha\} \times [1/(n+1), 1/n]\}.$$

Clearly, $K_{n,\alpha} \subset G_n$ and since the set $\{\alpha\} \times [1/(n+1), 1/n]$ is compact in $J(\kappa)$, one easily checks that the sets $K_{n,\alpha}$ are also closed. Let $F_{n,\alpha} = \bigcup_{i \geq n} K_{i,\alpha}$. The sets $F_{n,\alpha}$ are decreasing, closed and $F_{n,\alpha} \subset G_n$. The closedness of $F_{n,\alpha}$ follows from the local finiteness of $\{G_n\}$ and the inclusion $K_{n,\alpha} \subset G_n$.

We define:

$$F_\alpha = \bigcup_{n=1}^{\infty} \left(F_{n,\alpha} \times \{\alpha\} \times \left[\frac{1}{n+1}, \frac{1}{n}\right]\right),$$

$$G_\alpha = \bigcup_{n=1}^{\infty} \left(G_n^* \times \{\alpha\} \times \left(\frac{1}{n+2}, 0\right]\right),$$

$$F = \bigcup_{\alpha < \kappa} F_\alpha,$$

$$G = \bigcup_{\alpha < \kappa} G_\alpha.$$

One easily sees that the sets F and G are, respectively, closed and open in $X \times (J(\kappa) \backslash \{\theta\})$, and $F \subset G$. Using Theorem 10 in [**Mi**] it is not difficult to construct a continuous function $\phi\colon X \times (J(\kappa)\backslash\{\theta\}) \to I$ such that $\psi(F) \subset \{1\}$ and $\phi^{-1}((0, 1]) \subset G$.

Let $h\colon X \times J(\kappa) \to I$ be defined as follows:

$$h(x, z) = \begin{cases} g_0(x), & \text{if } z = \theta \\ \phi(x, z) \cdot g(x, z) + (1 - \phi(x, z)) \cdot g_0(x), & \text{if } z \neq \varnothing. \end{cases}$$

Clearly h is continuous at all points of $X \times (J(\kappa)\backslash\{\theta\})$. We shall show that h is continuous at all points $\langle x, \theta\rangle$, $x \in X$. Let $x_0 \in X$. There exists an n_0 and a neighborhood U_0 of x_0 such that $U_0 \cap \bigcup_{n \geq n_0} G_n^* = \varnothing$. Therefore, if $x \in U_0$ and $x \in B(n_0 + 2)\backslash\{\theta\}$, then $\langle x, z\rangle \notin G$ and $\phi(x, z) = 0$. Consequently, $h(x, z) = g_0(x)$, which implies continuity at $\langle x_0, \theta\rangle$.

It remains to show that $f^{-1}(0) \subset h^{-1}(0)$ and $f^{-1}(1) \subset h^{-1}(1)$.

Suppose that $f(x, z) = 0$. If $z = \theta$, then clearly $g(x, z) = 0$. Assume $z \neq 0$. There are two cases. Either, $|f(x, z) - f(x, \theta)| < \frac{1}{3}$ in which case $f(x, \theta) < \frac{1}{3}$, $g_0(x) = 0$ and, consequently $h(x, z) = 0$, or $|f(x, z) - f(x, \theta)| \geq \frac{1}{3}$, in which case $\langle x, z \rangle \in F$, $\phi(x, z) = 1$ and consequently $h(x, z) = g(x, z) = f(x, z) = 0$.

The proof for $f(x, z) = 1$ is similar. This completes the proof of the Proposition. □

REMARK. It follows from the above proof that in statements (i) and (ii) C-embedding can be replaced by C^*-embedding. □

The notions of a countably Katětov and countably functionally Katětov space were defined in [**PW**] in answer to questions raised by M. Katětov in 1958. It turns out that these notions are closely related to rectangular normality.

DEFINITION. A normal space X is *countably (functionally) Katětov* if every countable locally finite open (cozero) of any closed subspace can be extended to a locally finite open covering of X. □

Normality and countable paracompactness implies countably Katětov; countably Katětov implies countably functionally Katětov, which implies normal, but none of these implications can be reversed [**PW**].

2.3. THEOREM. *The following conditions are equivalent for a topological space* X:

(i) $X \times J(\kappa)$ *is rectangularly normal for every* $\kappa \in$ Card;

(ii) X *is countably Katětov.*

Proof. Implication (i) ⇒ (ii) follows immediately from Proposition 2.2. The implication (ii) ⇒ (i) follows from Proposition 2.2 and Theorem 2.1. □

2.4. THEOREM. *The following conditions are equivalent for a topological space* X:

(i) $X \times J(\omega)$ *is rectangularly normal;*

(ii) $X \times M$ *is rectangularly normal for some non-locally compact metric space* M;

(iii) X *is countably functionally Katětov.*

Proof. The implications (i) $\Rightarrow$ (ii) is obvious. If (ii) holds then M contains a closed copy of $J_0(\omega)$ and therefore also $X \times J_0(\omega)$ is rectangularly normal, which in view of Proposition 2.2 implies (iii). The implication (iii) $\Rightarrow$ (i) follows from Proposition 2.2 and Theorem 2.1. □

The following example answers positively a question raised by E. Michael.

2.5 EXAMPLE. There exists a (collectionwise) normal space X such that the product space $X \times M$ with a metric space M is rectangularly normal if and only if M is locally compact.

In particular, $X \times Q$ is not rectangularly normal.

Proof. By **[PW]** there exists a collectionwise normal space which is not countably functionally Katětov, hence it suffices to apply Theorems 2.4 and 1.1. □

2.6. EXAMPLE. ($V = L$). There exists a Dowker space X such that $X \times J(\kappa)$ is rectangularly normal for every $\kappa \in$ Card.

Proof. By **[PW]**, under $V = L$ there exists a Dowker countably Katětov space. □

It would be interesting to characterize the class of spaces X such that $X \times M$ is rectangularly normal, for every metric space M (see **[P]**). The author believes that there exists a Dowker space in this class. The existence of such a space would even more strongly underscore the difference between normality and rectangular normality of products. For information on this and related matters the reader is referred to **[P]** and **[Wa]**.

3. Rectangular products. In **[Pa, Pa2]** B. A. Pasynkov introduced the notion of a rectangular product and proved that for rectangular products $\dim(X \times Y) \leq \dim X + \dim Y$. A product space $X \times Y$ is *rectangular* if every two-element cozero covering of $X \times Y$ has a σ-locally finite refinement consisting of cozero rectangles, i.e. sets of the form $U \times V$, where U and V are cozero subsets of X and Y, respectively. Pasynkov proved that every *normal* product $X \times M$, with M metric, is rectangular. On the other hand, the following example shows that even the product of a countable metric space and a normal space need not be rectangular. For related examples, see **[HM]** and **[Wg]**.

3.1. EXAMPLE. There exists a (collectionwise) normal space X and a countable metric space M with one non-isolated point such that $X \times M$ is not rectangular.

Proof. By **[P2]** there exists a collectionwise normal space X and a countable locally finite family $\{G_n\}_{n=1}^{\infty}$ of its cozero subsets such that there is no locally finite family $\{W_n\}_{n=1}^{\infty}$ of open subsets of X such that $\overline{G}_n \subset W_n$, for every $n = 1, 2, \ldots$. Let $M = J_0(\omega)$.

Define a continuous mapping f: $X \times M \to I$ as in the proof of the implication (ii) $\Rightarrow$ (iii) in Proposition 2.2, for $F = X$ and $\kappa = \omega$. If $X \times M$ were rectangular, then there would exist a σ-locally finite refinement $\{U_s \times V_s: s \in S\}$ of the cozero covering $\{f^{-1}([0, \frac{2}{3})), f^{-1}((\frac{1}{3}, 1])\}$, consisting of cozero rectangles. For every $n = 1, 2, \ldots$ define

$$H_n = \bigcup \{U_s: s \in S \text{ and } B(n) \subset V_s\}.$$

Since the family $\{U_s: s \in S \text{ and } B(n) \subset V_s\}$ is σ-locally finite for every n, the sets H_n are cozero subsets of X and obviously the family $\{H_n\}$ is increasing and $\bigcup_{n=1}^{\infty} H_n = X$.

We claim that for every n the sets H_n and G_n are disjoint. Indeed, if $x_0 \in H_n$ then for some $s \in S$ we have $x_0 \in U_s$, $V_s \supset B(n)$ and either $U_s \times V_s \subset f^{-1}([0, \frac{2}{3}))$ or $U_s \times V_s \subset f^{-1}((\frac{1}{3}, 1])$. But if $x_0 \in G_n$, then $x_0 \in G_n^* = \{x \in X: |f(x, z) - f(x, \theta)| > \frac{3}{4}$, for some $z \in B(n)\}$, which is impossible.

Let $A_{n,m}$, $m = 1, 2, \ldots$, be open sets such that $H_n = \bigcup_{m=1}^{\infty} A_{n,m}$ and $A_{n,m+1} \supset \overline{A}_{n,m}$. Clearly we can also assume that $A_{n+1,m} \supset A_{n,m}$ for all n, m. The sets

$$W_n = X \setminus \overline{A}_{n,n}$$

are clearly open, $\overline{G}_n \subset W_n$ and the family $\{W_n\}$ is locally finite. This contradiction completes the proof. □

REFERENCES

[E] R. Engelking, *General Topology*, Polish Scientific Publishers, Warsaw 1977.

[HM] T. Hoshina and K. Morita, *On rectangular products of topological spaces*, Topology and Appl., **11** (1980), 47–57.

[M1] L. E. Miednikov, *Embeddings of bicompacta into the Tychonoff cube and extensions of mappings from subsets of products*, Dokl. Akad. Nauk SSSR, **222** (1975), 1287–1290 (in Russina).

[Mo] K. Morita, *On generalizations of Borsuk's homotopy extension theorem*, Fund. Math., **88** (1975), 1–6.

[Pa] B. A. Pasynkov, *On the dimension of rectangular products*, Dokl. Akad. Nauk SSSR, **221** (1975), 291–294 (in Russian).

[Pa2] _____, *On the dimension of products of normal spaces*, Dokl. Akad. Nauk SSSR, **209** (1973), 792–794 (in Russian).

[P] T. C. Przymusiński, *Extending functions from products with a metric factor and absolutes*, Pacific J. Math., **101** (1982), 463–475.

[P2] _____, *On locally finite coverings*, Colloq. Math., **38** (1978), 187–192.

[PW] T. C. Przymusiński and M. Wage, *Collectionwise normality and extensions of locally finite coverings*, Fund. Math., **109** (1980), 175–187.

[S] M. Starbird, *The Borsuk homotopy extension theorem without the binormality condition*, Fund. Math., **87** (1975), 207–211.

[S2] _____, *Extending Maps from Products*, in: Studies in Topology, Editors: N. H. Stavrakas and K. R. Allen, Academic Press 1975, 559–565.

[Wg] M. Wage, *The dimension of product spaces*, Proc. Natl. Acad. Sci. USA, **75** (1978), 4671–4672.

[Wa] A. Waśko, *Extension of functions defined on product spaces*, Fund. Math., to appear.

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