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FLOW UNDER A FUNCTION AND DISCRETE DECOMPOSITION OF PROPERLY INFINITE W^*-ALGEBRAS

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The purpose of this paper is to generalize the classical "flow under a function" construction to non-abelian W^*-algebras. That is, given an automorphism θ of a W^*-algebra N and a positive self-adjoint operator ϕ affiliated to the centre of N we show how to construct a continuous action α of the reals on a W^*-algebra M. The resulting covariant system $\{M, \alpha, \mathbf{R}\}$ is called the flow built on $\{N, \theta, \mathbf{Z}\}$ under the function ϕ.

1. Introduction. We obtain existence and uniqueness theorems for the representation of a given covariant system over the reals as a flow build under a function. As an application we generalize Connes' discrete decomposition theorems ([**3**] Théorème 5.3.1 and Théorème 5.4.2) using Takesaki's continuous decomposition theorems ([**8**], Theorem 8.1, Lemma 8.2 and Corollary 8.4).

In §2 we fix notation and state some results on covariant systems. In §3 we define flow built under a function and give necessary and sufficient conditions for a covariant system over the reals to be isomorphic to a flow built under a function. §4 deals with the uniqueness problem. That is, we show the relationship between $\{N_1, \theta_1, \phi_1\}$ and $\{N_2, \theta_2, \phi_2\}$ when the corresponding flows are isomorphic. In §5 we derive discrete decomposition theorems for properly infinite W^*-algebras using Takesaki's continuous decomposition theorem and our results on flow built under a function.

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2. Preliminaries. Let (Ω, μ) be a complete σ-finite measure space. An automorphism T of (Ω, μ) is a bijection $T: \Omega \to \Omega$ such that T and T^{-1} are measurable and $\mu \circ T^{-1}$ is equivalent to μ. A measurable action of a locally compact, σ-compact group G on (Ω, μ) is a homomorphism $t \to W_t$ of G into the group of automorphisms of (Ω, μ) such that the map

$(\omega, t) \to W_t(\omega)$ is measurable when $\Omega \times G$ is equipped with the completion of the product of μ with Haar measure. Measurable actions W and $\overline{W}$ of G on (Ω, μ) and $(\overline{\Omega}, \bar{\mu})$ respectively are called isomorphic iff there are G invariant conull sets $\Omega_0 \subset \Omega$ and $\overline{\Omega}_0 \subset \overline{\Omega}$ and a bijection $S: \Omega_0 \to \overline{\Omega}_0$ such that S and S^{-1} are measurable, $\bar{\mu} \circ S$ is equivalent to the restriction of μ_0 to Ω_0 and $S \circ W_t(\omega) = \overline{W}_t \circ S(\omega)$ for all $t \in G$ and all $\omega \in \Omega$.

If $t \to W_t$ is a measurable action of G on (Ω, μ) then we get a homomorphism $t \to \alpha_t$ of G into the group of automorphisms of $L^\infty(\Omega, \mu)$ be defining $\alpha_t f = f \circ W_t - 1$ for $f \in L^\infty(\Omega, \mu)$. The map $t \to \alpha_t$ is continuous in the sense that for every $\xi \in L^1(\Omega, \mu)$, $t \to \int_\Omega f \circ W_{t^{-1}}(\omega)\xi(\omega)\, d\mu(\omega)$ is continuous. More generally, a continuous action of a locally compact group G on a W^*-algebra M is a homomorphism $t \to \alpha_t$ of G into the group of automorphisms of M such that for each $x \in M$ the map $t \to \alpha_t(x)$ is ultraweakly continuous. In this case the triple $\{M, \alpha, G\}$ is called a covariant system. A homomorphism $\kappa: \{M, \alpha, G\} \to \{N, \beta, G\}$ of covariant systems is a continuous W^*-algebra homomorphism of M into N such that $\kappa\alpha_t = \beta_t\kappa$ for all $t \in G$.

As stated above, a measurable action of G on (Ω, μ) gives rise to a continuous action of G on $L^\infty(\Omega, \mu)$. The converse is also true:

PROPOSITION 2.1. *Let $\{M, \alpha, G\}$ be a covariant system where G is a locally compact σ-compact group and M is abelian and σ-finite. Then there is a measurable action $t \to W_t$ of G on a complete σ-finite measure space (Ω, μ) and an isomorphism κ of $L^\infty(\Omega, \mu)$ with M such that for all $t \in G$ and $f \in L^\infty(\Omega, \mu)$, $\kappa(f \circ W_{t^{-1}}) = \alpha_t(\kappa f)$.*

If G is a locally compact group $\{L^\infty(G), \sigma, G\}$ will denote the covariant system where $(\sigma_t f)(s) = f(t^{-1}s)$ for $t \in G$ and $f \in L^\infty(G)$. If G is abelian with dual group G then for $p \in \hat{G}$ define $\chi_p \in L^\infty(G)$ by $\chi_p(t) = \langle p, t\rangle$ ($\langle \cdot, \cdot \rangle$ is the pairing between G and $\hat{G}$).

If $\{M, \alpha, G\}$ is a covariant system, M^α will denote the fixed subalgebra $M^\alpha = \{x \in M: \alpha_t(x) = x$ for all $t \in G\}$. The following is a special case of [7] Theorem 2.

PROPOSITION 2.2. *Let $\{M, \alpha, G\}$ be a covariant system where G is abelian. Suppose that $p = U_p$ is a strongly continuous unitary representation of $\hat{G}$ in the centre of M such that $\alpha_t(U_p) = \overline{\langle p, t\rangle} U_p$ for all $t \in G$ and all $p \in \hat{G}$. Then there is an isomorphism κ of $\{M, \alpha, G\}$ with $\{M^\alpha \otimes L^\infty(G)$, $\mathrm{id} \otimes \sigma, G\}$ such that $\kappa(x) = x \otimes 1$ for $x \in M^\alpha$ and $\kappa(U_p) = 1 \otimes \chi_p$ for $p \in \hat{G}$.*

A consequence of this proposition is:

PROPOSITION 2.3. *Let $\{M, \alpha, G\}$ be a covariant system, H a locally compact abelian group and $(g, p) \to v(g, p)$ a strongly continuous mapping of $G \times \hat{H}$ into the unitaries in the centre of M such that $v(gk, p) = v(g, p)\alpha_g(v(k, p))$ and $v(g, p + q) = v(g, p)v(g, q)$ for all $g, k \in G$, $p, q \in \hat{H}$. Then there is a continuous action $\bar{\alpha}$ of G on $M \otimes L^\infty(H)$ commuting with* id $\otimes \sigma$ *such that $\bar{\alpha}_g(x \otimes 1) = \alpha_g(x) \otimes 1$ and $\bar{\alpha}_g(1 \otimes \chi_p) = v(g, p) \otimes \chi_p$ for all $x \in M$, $g \in G$ and $p \in \hat{H}$.*

Proof. For $p \in \hat{H}$ set $U_p = v(g, p) \otimes \chi_p \in M \otimes L^\infty(H)$. By Proposition 2.2 there is an automorphism β_g of $M \otimes L^\infty(H)$ commuting with each id $\otimes \sigma_t$ for $t \in H$ such that $\beta_g(x \otimes 1) = x \otimes 1$ and $\beta_g(1 \otimes \chi_p) = v(g, p) \otimes \chi_p$ for $x \in M$ and $p \in \hat{H}$. Set $\bar{\alpha}_g = \beta_g \alpha_g \otimes \mathrm{id}$. Then $g \to \bar{\alpha}_g$ is the required action. □

If $\{M, \alpha, G\}$ is a covariant system with G abelian there is a unique (up to isomorphism) covariant system $\{\hat{M}, \hat{\alpha}, \hat{G}\}$ such that $\hat{M}$ is generated by an isomorphic image π: $M \to \hat{M}$ of M together with a strongly continuous unitary representation $g \to U_g$ of G satisfying: $U_g \pi(x) U_g^* = \pi(\alpha_g(x))$, $\hat{\alpha}_p(\pi(x)) = \pi(x)$ and $\hat{\alpha}_p(U_g) = \overline{\langle p, g\rangle} U_g$ for all $x \in M$, $g \in G$ and $p \in \hat{G}$ (see [**8**]). $\hat{M}$ is called the crossed product of the covariant system and is denoted by $M \times_\alpha G$. Takesaki ([**8**] Theorem 4.5) has shown that $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ is isomorphic to $M \otimes B(L^2(G))$.

PROPOSITION 2.4. *In the situation of Proposition* 2.3, *let β be the action of $G \times H$ on $M \otimes L^\infty(H)$ given by $\beta_{(g,h)} = \bar{\alpha}_g \mathrm{id} \otimes \sigma_h$ for $(g, h) \in G \times H$. Then $M \otimes L^\infty(H) \times_\beta (G \times H)$ is isomorphic to $(M \times_\alpha G) \otimes B(L^2(\hat{H}))$.*

Proof. $M \otimes L^\infty(H) \times_\beta (G \times H)$ is generated by a copy $\pi(M)$ of M and three strongly continuous unitary representations $g \to U_g$, $p \to V_p$, $h \to W_h$ of G, $\hat{H}$ and H respectively. ($\pi(M)$ is the image of $M \otimes 1$ and V_p, for $p \in \hat{H}$, is the image of $1 \otimes \chi_p$ in the crossed product). Let M_1 be the W^*-algebra generated by $\pi(M)$ and $\{U_g$: $g \in G\}$. Since $\hat{\beta}_{(q,0)}(\pi(x)) = \pi(x)$ and $\hat{\beta}_{(q,0)}(U_g) = \overline{\langle q, g\rangle} U_g$ for $x \in M$, $q \in \hat{G}$ and $g \in G$, it follows that M_1 is isomorphic to $M \times_\alpha G$. Let M_2 be the W^*-algebra generated by M_1 and $\{V_p$: $p \in \hat{H}\}$ and let θ be the action of $\hat{H}$ on M_1 given by $\theta_p(y) = V_p y V_p^*$ for $y \in M_1$, $p \in \hat{H}$. It follows that M_2 is $M_1 \times_\theta \hat{H}$ with $\hat{\theta}$ given by $\hat{\theta}_h(y) = W_h y W_h^*$ for $y \in M_2$. Finally, since $\hat{\beta}_{(0,p)}(y) = y$ and $\hat{\beta}_{(0,p)}(W_h) = \overline{\langle p, h\rangle} W_h$ for $y \in M_2$, $p \in \hat{H}$ and $h \in H$, we have that

$M \otimes L^\infty(H) \times_\beta (G \times H)$ is isomorphic to $M_2 \times_{\hat{\theta}} H$. But $M_2 \times_{\hat{\theta}} H$ is isomorphic to $((M \times_\alpha G) \times_\theta \hat{H}) \times_{\hat{\theta}} H$ which by Takesaki's result is isomorphic to $(M \times_\alpha G) \otimes B(L^2(\hat{H}))$. □

3. **Flow under a function.** The classical "flow under a function" construction produces a measurable flow from an automorphism of a measure space and a function on the measure space. The construction is as follows (see **[1]**, **[2]** and **[6]**). Let T be an automorphism of the complete σ-finite measure space (Ω, μ) and let ϕ: $\Omega \to (0, \infty)$ be a measurable function satisfying

$$\sum_{n \geq 0} \phi(T^n \omega) = \infty = \sum_{n \geq 0} \phi(T^{-n}\omega) \quad \text{for all } \omega \in \Omega. \tag{3.1}$$

Set $\bar{\Omega} = \Omega \times \mathbf{R}$ and let $\bar{\mu}$ be the completion of the product of μ with Lebesgue measure. Let $\bar{T}$ and S_t, for $t \in \mathbf{R}$, be the automorphisms given by $\bar{T}(\omega, s) = (T\omega, s - \phi(\omega))$, $S_t(\omega, s) = (\omega, s + t)$ for $(\omega, s) \in \bar{\Omega}$. Let Ω_0 be the space of orbits under $\bar{T}$ and let Ω_1 be the region under the graph of ϕ i.e. $\Omega_1 = \{(\omega, s)$: $0 \leq s < \phi(\omega)\}$. By (3.1) Ω_1 is a transversal of the orbits under $\bar{T}$ so we may identify Ω_0 and Ω_1. Let μ_0 be the measure on Ω_0 obtained by restricting $\bar{\mu}$ to Ω_1. Since S_t and $\bar{T}$ commute, S_t descends to a flow $t \to S_t^0$ on (Ω_0, μ_0). See **[1]** formula 1.1 for the definition of S^0 as a flow on Ω_1. S^0 is called the flow built on the automorphism T under the function ϕ. Due to the identification of Ω_0 and Ω_1, T is called the base automorphism and ϕ the ceiling function for S^0. A slight extension of **[2]** theorem 4 is:

THEOREM 3.2 (*Ambrose and Kakutani*). *A measurable flow $t \to W_t$ on a complete σ-finite measure space is isomorphic to a flow built under a function iff W is proper in the sense that for any measurable subset E of positive measure there is a measurable subset $F \subset E$ and a number t_0 so that $W_{t_0}(F)$ intersects the complement of F in a non-null set.*

In this section we shall generalize the flow under a function construction and Theorem 3.2 to non-abelian W^*-algebras. We first repeat the flow under a function construction in terms of covariant systems. Let $\{N, \theta, \mathbf{Z}\}$ be the covariant system where $N = L^\infty(\Omega, \mu)$ and $\theta(f) = f \circ T^{-1}$ for $f \in N_0$. Let $\{M, \alpha, \mathbf{R}\}$ be the covariant system where $M = L^\infty(\Omega_0, \mu_0)$ and $\alpha_t(f) = f \circ S_{-t}^0$ for $t \in \mathbf{R}$ and $f \in M$. Let $\bar{\theta}$ be the automorphism of $N \otimes L^\infty(\mathbf{R}) = L^\infty(\bar{\Omega}, \bar{\mu})$ given by $\bar{\theta}(f) = f \circ \bar{T}^{-1}$ for $f \in N \otimes L^\infty(\mathbf{R})$. There is a natural identification of M with the fixed algebra $[N \otimes L^\infty(\mathbf{R})]^{\bar{\theta}}$ and under this identification $\alpha_t(y) = \mathrm{id} \otimes \sigma_t(y)$ for $y \in M$ and $t \in \mathbf{R}$. We can see that the construction depends only on $\{N, \theta, \mathbf{Z}\}$ and ϕ as a self-adjoint operator affiliated to N by noting that $\bar{\theta}$ is characterized by the equations, $\bar{\theta}(x \otimes 1) = \theta(x) \otimes 1$ and $\bar{\theta}(1 \otimes \chi_s) = \theta(e^{is\phi}) \otimes \chi_s$ for $x \in N$ and $s \in \mathbf{R}$. To extend the construction to non-abelian W^*-algebras we need an analog of property 3.1.

DEFINITION 3.3. Let θ be an automorphism of a W^*-algebra N and let ϕ be a positive self-adjoint operator affiliated to the centre of N. ϕ is called a θ ceiling function iff there is a partition of unity $\{e_i: i \in I\}$ in the centre of N and numbers $\varepsilon_i > 0$, for each $i \in I$, such that $\theta(e_i) = e_i$ and $\phi e_i \geq \varepsilon_i e_i$, for each $i \in I$.

DEFINITION 3.4. Let $\{N, \theta, \mathbf{Z}\}$ be a covariant system and ϕ a θ ceiling function. Let $\bar{\theta}$ be the automorphism of $N \otimes L^\infty(\mathbf{R})$ (given by Proposition 2.3) which satisfies $\bar{\theta}(x \otimes 1) = \theta(x) \otimes 1$ and $\bar{\theta}(1 \otimes \chi_s) = \theta(e^{is\phi}) \otimes \chi_s$ for $x \in N$ and $s \in \mathbf{R}$. Let $M = [N \otimes L^\infty(\mathbf{R})]^{\bar{\theta}}$ and for $x \in M$ and $t \in \mathbf{R}$ let $\alpha_t(x) = \mathrm{id} \otimes \sigma_t(x)$. The covariant system $\{M, \alpha, \mathbf{R}\}$ is called the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ.

We can characterize flow under a function abstractly.

PROPOSITION 3.5. *$\{M, \alpha, \mathbf{R}\}$ is isomorphic to the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ iff there is W^*-algebra P with commuting actions $\bar{\alpha}$ of $\mathbf{R}$ and $\bar{\theta}$ of $\mathbf{Z}$ such that $\{N, \theta, \mathbf{Z}\}$ is isomorphic to $\{P^{\bar{\alpha}}, \bar{\theta}, \mathbf{Z}\}$, $\{M, \alpha, \mathbf{R}\}$ is isomorphic to $\{P^{\bar{\theta}}, \bar{\alpha}, \mathbf{R}\}$ and there is a strongly continuous unitary representation $s \to v_s$ of $\mathbf{R}$ in the centre of P such that $\bar{\alpha}_t(v_s) = \bar{e}^{ist} v_s$ and $\bar{\theta}(v_s) = \bar{\theta}(e^{is\phi}) v_s$ for $s, t \in \mathbf{R}$ (in this last formula we identify N with $P^{\bar{\alpha}}$). Moreover, in this case there is a strongly continuous unitary representation $p \to u_p$ of (the group) $[0, 2\pi)$ in the centre of P such that $\bar{\theta}(u_p) = e^{-ip} u_p$ for $p \in [0, 2\pi)$.*

Proof. If $\{M, \alpha, \mathbf{R}\}$ is the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ, take $P = N \otimes L^\infty(\mathbf{R})$, $\bar{\alpha} = \mathrm{id} \otimes \sigma$, $\bar{\theta}$ as in Definition 3.4 and let $v_s = 1 \otimes \chi_s$ for $s \in \mathbf{R}$. The converse follows from Proposition 2.2. For the last part we use the following lemma which will be needed later.

LEMMA 3.6. *Let γ be an automorphism of the W^*-algebra Q and let $s \to v_s$ be a strongly continuous unitary representation of $\mathbf{R}$ in the centre of Q such that $\gamma(v_s) = \gamma(e^{is\phi}) v_s$, $s \in \mathbf{R}$ for some γ ceiling ϕ affiliated to the centre of Q. Then there is a central projection e in Q such that $\{\gamma^n(e): n \in \mathbf{Z}\}$ is an orthogonal family and $1 = \sum_{n=-\infty}^{\infty} \gamma^n(e)$.*

Proof. Let k be the self-adjoint operator affiliated to the centre of Q such that $v_s = e^{isk}$ for $s \in \mathbf{R}$. Then $\gamma(k) = \gamma(\phi) + k$. Since ϕ is a γ ceiling operator, there is a spectral projection p of k corresponding to an interval of the form $(-\infty, a]$ for some $a \in \mathbf{R}$ which satisfies $\gamma(p) \leq p$, $\gamma(p) \neq p$, $\gamma^n(p) \to 0$ as $n \to \infty$ and $\gamma^n(p) \to 1$ as $n \to -\infty$. $e = p - \gamma(p)$ is the required projection. □

Now, applying Lemma 3.6 to the situation of Proposition 3.5 we obtain a central projection e in P with $\{\bar{\theta}^n(e)\colon n \in \mathbf{Z}\}$ an orthogonal family and $1 = \sum_{n=-\infty}^{\infty} \bar{\theta}^n(e)$. Set $u_p = \sum_{n=-\infty}^{\infty} e^{inp}\bar{\theta}_n(e)$. □

To obtain the generalization of Theorem 3.2 we shall use

LEMMA 3.7. *Let $\{M, \alpha, \mathbf{R}\}$ be a covariant system. Let M_1 be a W^*-subalgebra of M such that $\alpha_t(M_1) = M_1$ for all $t \in \mathbf{R}$ and centre $M_1 \subset$ centre M. Set $\alpha_t^1(x) = \alpha_t(x)$ for $t \in \mathbf{R}$ and $x \in M_1$. If $\{M_1, \alpha^1, \mathbf{R}\}$ is isomorphic to the flow built on $\{N_1, \theta_1, \mathbf{Z}\}$ under ϕ_1 then there is an imbedding of $\{N_1, \theta_1, \mathbf{Z}\}$ into a covariant system $\{N, \theta, \mathbf{Z}\}$ such that centre $N_1 \subset$ centre N and $\{M, \alpha, \mathbf{R}\}$ is isomorphic to the flow built on $\{N, \theta, \mathbf{Z}\}$ under $\phi = \phi_1$.*

Proof. Using Proposition 3.5 we obtain P_1, $\bar{\alpha}^1$, $\bar{\theta}_1$, $s \to v_s^1$ and $p \to u_p^1$ satisfying the conditions of the proposition. We identify M_1 with $P_1^{\bar{\theta}_1}$ and N_1 with $P_1^{\bar{\alpha}_1}$. For $t \in \mathbf{R}, p \in [0, 2\pi)$, $\bar{\alpha}_t^1(u_p^1)u_p^{1*}$ is fixed by $\bar{\theta}_1$. Hence there is a unitary $v(t, p)$ in the centre of M_1 such that $\alpha_t^1(u_p^1) = v(t, p)u_p^1$ for $t \in \mathbf{R}$, $p \in [0, 2\pi)$. The map $(t, p) \to v(t, p)$ satisfies the conditions of Proposition 2.3 with respect to α^1 and hence with respect to α (since centre $M_1 \subset$ centre M). By Proposition 2.2 we can identify P_1 to $M_1 \otimes l^\infty(\mathbf{Z})$. Under this identification $\bar{\theta}$ is id $\otimes\, \sigma$, u_p is $1 \otimes \chi_p$ for $p \in [0, 2\pi]$ and $\bar{\alpha}^1$ satisfies $\bar{\alpha}_t^1(x \otimes 1) = \alpha_t^1(x) \otimes 1$ and $\bar{\alpha}_t^1(1 \otimes \chi_p) = v(t, p) \otimes \chi_p$ for $t \in \mathbf{R}$, $x \in M_1$ and $p \in [0, 2\pi)$. Let $P = M \otimes l^\infty(\mathbf{Z})$, then $P_1 \subset P$, centre $P_1 \subset$ centre P and we can extend $\bar{\theta}^1$ to $\bar{\theta}$ on P by $\bar{\theta} = \mathrm{id} \otimes \sigma$. We can also use Proposition 2.3 to extend $\bar{\alpha}^1$ to $\bar{\alpha}$ on P. Let $N = P^{\bar{\alpha}}$ and for $x \in N$ let $\theta(x) = \bar{\theta}(x)$. Since centre $N_1 \subset$ centre $P_1 \subset$ centre P we have that centre $N_1 \subset$ centre N. Finally, set $u_p = u_p^1$ for $p \in [0, 2\pi)$ and $v_s = v_s^1$ for $s \in \mathbf{R}$. Proposition 3.5 now shows that $\{M, \alpha, \mathbf{R}\}$ is isomorphic to the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. □

The generalization of Theorem 3.2 is

THEOREM 3.8. *A covariant system $\{M, \alpha, \mathbf{R}\}$ is isomorphic to a flow built under a function iff the restriction of α to the centre of M is proper in the sense that for every non-zero central projection e there is a central projection $f \leq e$ and a number t_0 such that $(1 - f)\alpha_{t_0}(f) \neq 0$.*

Proof. Without loss of generality we can assume that the centre of M is σ-finite. Assume that $\{M, \alpha, \mathbf{R}\}$ is isomorphic to the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. Proposition 3.5 shows that the restriction of α to the centre of M is isomorphic to the flow built on the restriction of θ to the centre of N under ϕ. By Theorem 3.2 α is proper. For the converse, Lemma 3.7 shows that it suffices to obtain the restriction of α to the centre of M as a flow built under a function. Theorem 3.2 shows that this is possible. □

4. Uniqueness of flow under a function. Let $\{M, \alpha, \mathbf{R}\}$ be the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. In this section we investigate the extent to which the isomorphism class of $\{M, \alpha, \mathbf{R}\}$ determines $\{N, \theta, \mathbf{Z}\}$ and ϕ. The results are well known in the abelian case (see **[5]**).

We first exhibit two ways of modifying $\{N, \theta, \mathbf{Z}\}$ and ϕ so that the resulting flows are isomorphic.

LEMMA 4.1. *Let θ be an automorphism of N and ϕ a θ ceiling function. Suppose ξ is a self-adjoint operator affiliated to the centre of N such that $\psi = \phi + \theta(\xi) - \xi$ is also a θ ceiling function. Then the flows built on $\{N, \theta, \mathbf{Z}\}$ under ϕ and ψ are isomorphic.*

Proof. In the notation of Proposition 3.5 we have $\bar{\theta}(v_s) = \theta(e^{is\phi})v_s$ for all $s \in \mathbf{R}$. Set $v'_s = \theta(e^{is\xi})v_s$ for $s \in \mathbf{R}$. Then $\bar{\theta}(v'_s) = \theta^2(e^{is\xi})\theta(e^{is\phi})v_s = \theta(e^{is\psi})v'_s$ for all $s \in \mathbf{R}$. Also $\bar{\alpha}_t(v'_s) = e^{-ist}v'_s$ for all s and $t \in \mathbf{R}$. Hence by Proposition 3.5, both flows are isomorphic to the restriction of $\bar{\alpha}$ to $\bar{P}^{\theta}$. □

The second modification deals with induced automorphisms in the sense of Kakutani **[5]**. For this we need the notion of recurrent projections.

DEFINITION 4.2. Let θ be an automorphism of N and let e be a projection in the centre of N. e is said to be recurrent under θ iff $e \leq \bigvee_{n<0} \theta^n(e)$ and $e \leq \bigvee_{n>0} \theta^n(e)$.

There is a canonical way to partition a recurrent projection as $e = \sum_{n=1}^{\infty} e_n$ where each e_n is central and satisfies $\theta(e_1) \leq e$, and for $n \geq 2$, $\theta^j(e_n)e = 0$ for $j = 1, 2, \ldots, n-1$ and $\theta^n(e_n) \leq e$. It follows that $e = \sum_{n=1}^{\infty} \theta^n(e_n)$ and $\{\theta^j(e_n): j = 0, 1, \ldots, n-1, n = 1, 2, \ldots\}$ is an orthogonal family with

$$\bigvee_{n \in \mathbf{Z}} \theta^n(e) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \theta^j(e_n) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \theta^j(e_n).$$

The induced automorphism θ_e of N_e is defined by $\theta_e(x) = \Sigma_{n=1}^{\infty} \theta^n(xe_n)$ for $x \in N_e$. If ϕ is a θ ceiling function then

$$\phi_e = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \theta^{-m}(\phi) e_n$$

is called the induced ceiling function.

LEMMA 4.3. *If e is recurrent under the automorphism θ of N with $\bigvee_{n \in \mathbf{Z}} \theta^n(e) = 1$ then the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ is isomorphic to the flow built on $\{N_e, \theta_e, \mathbf{Z}\}$ under ϕ_e.*

Proof. Let $\{M, \alpha, \mathbf{R}\}$ be the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. We use the notation of Proposition 3.5. The projection e is recurrent for $\bar{\theta}$ and $\bar{\alpha}_t(e) = e$ for all $t \in \mathbf{R}$. Let $P_1 = P_e$, $\bar{\theta}_1 = \bar{\theta}_e$, $\bar{\alpha}_t^1(x) = \bar{\alpha}_t(x)$ for $x \in P_e$, $t \in \mathbf{R}$ and let $v_s^1 = v_s e$. Then $\bar{\alpha}^1(v_s^1) = \bar{e}^{ist} v_s^1$ and $\bar{\theta}(v_s^1) = \bar{\theta}_1(e^{is\phi_e}) v_s^1$ for $s, t \in \mathbf{R}$. Hence, the restriction of $\bar{\alpha}^1$ to $P^{\bar{\theta}_1}$ is isomorphic to the flow built on $\{N_e, \theta_e, \mathbf{Z}\}$ under ϕ_e. Since $\bigvee_{n \in \mathbf{Z}} \theta^n(e) = 1$, the map $x \to xe$ is an isomorphism of $P^{\bar{\theta}}$ with $P_1^{\bar{\theta}_1}$. Hence the restriction of $\bar{\alpha}$ to $P^{\bar{\theta}}$ is isomorphic to the restriction of $\bar{\alpha}^1$ to $P^{\bar{\theta}_1}$. □

The uniqueness question splits naturally into the dissipative and conservative cases. Recall that an action α of a locally compact abelian group G on a W^*-algebra M is called dissipative iff there is a strongly continuous unitary representation $p \to U_p$ of $\hat{G}$ in the centre of M such that $\alpha_t(U_p) = \overline{\langle p, t \rangle} U_p$ for all $t \in G, p \in \hat{G}$. (In which case by Proposition 2.2, $\{M, \alpha, G\}$ is isomorphic to $\{M^\alpha \otimes L^\infty(G), \mathrm{id} \otimes \sigma, G\}$.) α is called conservative iff there are no non-zero, central, α invariant projections e such that α restricted to M_e is dissipative. A maximality argument shows that there is a largest central α-invariant projection e such that α restricted to M_e is dissipative and α is restricted to M_{1-e} is conservative. We denote this projection by $e(\alpha)$.

LEMMA 4.4. *Let $\{M, \alpha, R\}$ be the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. Then $\{N, \theta, \mathbf{Z}\}$ is dissipative iff $\{M, \alpha, R\}$ is dissipative. More generally, $\{M_{e(\alpha)}, \alpha, \mathbf{R}\}$ is isomorphic to the flow built on $\{N_{e(\theta)}, \theta, \mathbf{Z}\}$ under $\phi e(\theta)$ and $\{M_{1-e(\alpha)}\alpha, \mathbf{R}\}$ is isomorphic to the flow built on $\{N_{1-e(\theta)}, \theta, \mathbf{Z}\}$ under $\phi(1 - e(\theta))$.*

Proof. Suppose $\{N, \theta, \mathbf{Z}\}$ is dissipative. Let $Q = N^\theta$ then $\{N, \theta, \mathbf{Z}\}$ is isomorphic to $\{Q \otimes l^\infty(\mathbf{Z}), \mathrm{id} \otimes \sigma, \mathbf{Z}\}$. Hence we can find a self-adjoint operator η affiliated to the centre of N such that $\phi = \theta\eta - \eta$. In the notation of Proposition 3.5 let $U_s = \theta(e^{-is\eta})v_s$ for $s \in R$. Then $\bar{\theta}(U_s) = U_s$ $\bar{\alpha}_t(U_s) = e^{-ist}U_s$ for s, $t \in R$. Hence α is dissipative.

Conversely, if α is dissipative let $s \to U_s$ be a strongly continuous unitary representation in the centre of M such that $\alpha_t(U_s) = e^{-ist}U_s$ for $s, t \in R$. In the notation of Proposition 3.5 let $w_s = U_s^* v_s$ for $s \in R$. Then w_s is in the centre of P, $\bar{\alpha}_t(w_s) = w_s$ for $s, t \in R$ and $\bar{\theta}(w_s) = \theta(e^{is\phi})w_s$ for $s \in R$. By Lemma 3.6 there is a central projection e in N with $\{\theta^n(e)$: $n \in \mathbf{Z}\}$ an orthogonal family such that $\sum_{n=-\infty}^{\infty} \theta^n(e) = 1$. For $p \in [0, 2\pi)$ set $u_p = \sum_{n=-\infty}^{\infty} e^{inp}\bar{\theta}^n(e)$. Then u_p is in the centre of N and $\theta(u_p) = e^{-ip}u_p$ for $p \in [0, 2\pi)$. Hence $\{N, \theta, \mathbf{Z}\}$ is dissipative.

For the last part of the lemma, using the notation of Proposition 3.5 we have $e(\theta) \in N^\theta = (P^{\bar{\alpha}})^{\bar{\theta}} = (P^{\bar{\theta}})^{\bar{\alpha}} = M^\alpha$ and the flow built on $\{N_{e(\theta)}, \theta, \mathbf{Z}\}$ under $\phi e(\theta)$ is isomorphic to $\{M_{e(\theta)}, \alpha, R\}$. By the first part of the proof $e(\theta) = e(\alpha)$. □

The main result of this section is:

THEOREM 4.5. *The flow built on* $\{N_1, \theta_1, \mathbf{Z}\}$ *under* ϕ_1 *is isomorphic to the flow built on* $\{N_2, \theta_2, \mathbf{Z}\}$ *under* ϕ_2 *iff there are recurrent projections* e_j *in the centre of* N_j *with* $\bigvee_{n\in\mathbf{Z}} \theta_j^n(e_j) = 1$ *for* $j = 1, 2$ *and an isomorphism* κ *of* $\{(N_1)e_1, (\theta_1)e_1, \mathbf{Z}\}$ *with* $\{(N_2)e_2, (\theta_2)e_2, \mathbf{Z}\}$ *such that* $(\phi_2)e_2 = \kappa(\phi_1)e_1 + (\theta_2)e_2(\xi) - \xi$ *for some self-adjoint operator* ξ *affiliated to the centre of* $(N_2)e_2$.

Proof. Lemmas 4.1 and 4.3 show that the stated conditions imply that the flows are isomorphic. For the converse, Lemma 4.4 shows that we may deal with the dissipative and conservative cases separately.

Assume that $\{N_j, \theta_j, \mathbf{Z}\}$ is dissipative for $j = 1, 2$ and let $\{M, \alpha, R\}$ be the common flow. Proposition 3.5 and Lemma 4.4 show that $\{N_j, \theta_j, \mathbf{Z}\}$ is isomorphic to $\{N_j^{\theta_j} \otimes l^\infty(\mathbf{Z}), \mathrm{id} \otimes \sigma, R\}$ and $N_1^{\theta_1}$, $N_2^{\theta_2}$ and M^α are all isomorphic. It follows that $\{N_1, \theta_1, \mathbf{Z}\}$ is isomorphic to $\{N_2, \theta_2, \mathbf{Z}\}$. If $\{N, \theta, \mathbf{Z}\}$ denotes this common covariant system then we can find ξ_1, ξ_2 such that $\phi_j = \theta(\xi_j) - \xi_j$. Hence $\phi_1 = \phi_2 + \theta(\xi) - \xi$ where $\xi = \xi_1 - \xi_2$. This proves the theorem in the dissipative case. For the rest of the proof we assume that θ_1 and θ_2 are conservative. The property of conservative automorphisms which is needed is that all central projections are recurrent.

LEMMA 4.6. *If the flow built on* $\{N_1, \theta_1, \mathbf{Z}\}$ *under* ϕ_1 *is isomorphic to the flow built on* $\{N_2, \theta_2, \mathbf{Z}\}$ *under* ϕ_2 *then there is a* W^**-algebra* Q *with* *commuting automorphisms* γ_1 *and* γ_2 *and a strongly continuous unitary representation* $s \to w_s$ *of* R *in the centre of* Q *such that* $\{N_1, \theta_1, \mathbf{Z}\}$ *is identified to* $\{Q^{\gamma_1}, \gamma_2, \mathbf{Z}\}$, $\{N_2, \theta_2, \mathbf{Z}\}$ *is identified to* $\{Q^{\gamma_1}, \gamma_2, \mathbf{Z}\}$, $\gamma_1(w_s) = \gamma_1(e^{is\phi_1})w_s$ *and* $\gamma_2(w_s) = \gamma_2(e^{-is\phi_2})w_s$ *for all* $s \in R$.

Proof. Let $\{M, \alpha, R\}$ be the common flow. We apply Proposition 3.5 obtaining P_j, $\bar{\theta}_j$, $\bar{\alpha}^j$ and v^j for $j = 1, 2$ satisfying the properties of Proposition 3.5. In particular, we identify M to both $P_1^{\bar{\theta}_1}$ and $P_2^{\bar{\theta}_2}$. Under this identification α is the restriction of $\bar{\alpha}^j$ for $j = 1, 2$. Let $p \to U_p$ be a strongly continuous unitary representation of $[0, 2\pi)$ in the centre of P_2 such that $\bar{\theta}_2(U_p) = e^{-ip}U_p$ for $p \in [0, 2\pi)$. Then $\bar{\alpha}_t^2(U_p) = v(t, p)U_p$ for a strongly continuous unitary map $(t, p) \to v(t, p)$ of $R \times [0, 2\pi)$ into the centre of $P_2^{\bar{\theta}_2}$. Note that centre $P_j^{\bar{\theta}_j} = (\text{centre}\, P_j)^{\bar{\theta}_j} = \text{centre}\, M$ for $j = 1, 2$ and $(t, p) \to v(t, p)$ satisfies the conditions of Proposition 2.3. By Proposition 2.2 we can identify P_2 with $P_1^{\bar{\theta}_1} \otimes l^\infty(\mathbf{Z})$ and under this identification $\bar{\theta}_2$ is id $\otimes\, \sigma$, U_p is $1 \otimes \chi_p$ for $p \in [0, 2\pi)$ and $\bar{\alpha}^2$ satisfies $\bar{\alpha}_t^2(x \otimes 1) = \bar{\alpha}_t^1(x) \otimes 1$ and $\bar{\alpha}_t^2(1 \otimes \chi_p) = v(t, p) \otimes \chi_p$ for $t \in R$, $x \in P_1^{\bar{\theta}_1}$ and $p \in [0, 2\pi)$. Now, set $P = P_1 \otimes l^\infty(\mathbf{Z})$, $\bar{\gamma}_1 = \bar{\theta}_1 \otimes \text{id}$, $\bar{\gamma}_2 = \text{id} \otimes \sigma$ and let β be the action of R on P (given by Proposition 2.3) which satisfies $\beta_t(x \otimes 1) = \bar{\alpha}_t^1(x) \otimes 1$ and $\beta_t(1 \otimes \chi_p) = v(t, p) \otimes \chi_p$ for $x \in P_1$, $t \in R$, $p \in [0, 2\pi)$. The automorphisms $\bar{\gamma}_1$, $\bar{\gamma}_2$ and β_t commute for all $t \in R$, $\{P_1, \bar{\theta}_1 \times \bar{\alpha}^1, \mathbf{Z} \times R\}$ is identified to $\{P^{\bar{\gamma}_2}, \bar{\gamma}_1 \times \beta, \mathbf{Z} \times R\}$ and $\{P_2, \bar{\theta}_2 \times \bar{\alpha}^2, \mathbf{Z} \times R\}$ is identified to $\{P^{\bar{\gamma}_1}, \bar{\gamma}_2 \times \beta, \mathbf{Z} \times R\}$. Let $Q = P^\beta$, $\gamma_j = \bar{\gamma}_j|_Q$ for $j = 1, 2$ and let $w_s = v_{-s}^2 v_s^1$ for $s \in R$. Now, $\{N_1, \theta_1, \mathbf{Z}\}$ is identified to $\{P_1^{\bar{\alpha}_1}, \bar{\theta}_1, \mathbf{Z}\}$ which is identified to $\{(P^{\bar{\gamma}_2})^\beta, \bar{\gamma}_1, \mathbf{Z}\} = \{(P^\beta)^{\bar{\gamma}_2}, \bar{\gamma}_1, \mathbf{Z}\} = \{Q^{\gamma_2}, \gamma_1, \mathbf{Z}\}$. Similarly $\{N_2, \theta_2, \mathbf{Z}\}$ is identified to $\{Q^{\gamma_1}, \gamma_2, \mathbf{Z}\}$. Finally for $s \in R$ we have

$$\begin{aligned}\gamma_1(w_s) = \bar{\gamma}_1(v_{-s}^2 v_s^1) &= v_{-s}^2 \bar{\gamma}_1(e^{is\phi_1}) v_s^1 \\ &= v_{-s}^2 \gamma_1(e^{is\phi_1}) v_s^1 = \gamma_1(e^{is\phi_1}) w_s.\end{aligned}$$

Similarly $\gamma_2(w_s) = \gamma_2(e^{-is\phi_2})w_s$ for all $s \in R$. □

In the notation of Lemma 4.6, let $w_s = e^{isk}$, for $s \in R$, where k is a self-adjoint operator affiliated to the centre of Q. Let f be the spectral projection of k corresponding to the set $(-\infty, 0]$. Since $\gamma_1(k) = k + \gamma_1(\phi_1)$ we have that $\gamma_1(f) \leq f$. Since ϕ_1 is a γ_1 ceiling function it follows that $\gamma_1^n(f) \to 0$ and $\gamma_1^{-n}(f) \to 1$ ultraweakly as $n \to \infty$. Now set $e = f - \gamma_1(f)$. Then $\{\gamma_1^n(e) \colon n \in \mathbf{Z}\}$ is an orthogonal family with $\sum_{n=-\infty}^{\infty} \gamma_1^n(e) = 1$. Since $\gamma_2(k) = k - \gamma_2(\phi_2)$ we have that $\gamma_2^{-1}(e)\gamma_1^{-n}(e) = 0$ for $n \geq 1$. It follows that $\gamma_2^{-1}(e) = \sum_{n=0}^{\infty} g_n \gamma_1^n(e)$ where $\{g_n \colon n = 0, 1, 2, \ldots\}$ is a partition of unity consisting of γ_1 invariant projections. Set $e_2 = 1 - g_0 \in Q^{\gamma_1} = N_2$. We shall show that $\bigvee_{n\in\mathbf{Z}} \theta_2^n(e_2) = 1$. Suppose there is a non-zero projection p in the centre of Q fixed by both γ_1 and γ_2 with $p \leq g_0$. Then $\gamma_2^{-1}(ep) = ep$ and $\{\gamma_1^n(ep)\colon n \in \mathbf{Z}\}$ is an orthogonal family. This contradicts the assumption that θ_1 is conservative.

Hence, by replacing θ_2 by $(\theta_2)e_2$ we may assume that $\gamma_2^{-1}(e) = \sum_{n=1}^{\infty} g_n\gamma_1^n(e)$ where $\{g_n\colon n = 1,2,\ldots\}$ is a partition of unity consisting of central γ_1 invariant projections. It follows that $\{\gamma_2^n(e)\colon n \in \mathbf{Z}\}$ is an orthogonal family. Set $e_1 = \sum_{n=-\infty}^{\infty} \gamma_2^n(e) \in Q^{\gamma_2} = N_1$. Clearly $\bigvee_{n\in\mathbf{Z}} \theta_1^n(e_1) = 1$ and the canonical partition of e_1 is $(e_1)_n = \sum_{n=-\infty}^{\infty} \gamma_2^k(g_n e)$ for $n = 1,2,\ldots$. This means $(\gamma_1)_{e_1}(x) = \sum_{n=1}^{\infty} \gamma_1^n((e_1)_n x)$ for $x \in Q$. Hence $(\gamma_1)e_1(e) = \sum_{n=1}^{\infty} g_n\gamma_1^n(e) = \gamma_2^{-1}(e)$. So by replacing θ_1 by $(\theta_1)e_1$ we may assume that in the situation of Lemma 4.6 there is a central projection e with $\{\gamma_1^n(e)\colon n \in \mathbf{Z}\}$ a partition of unity and with $\gamma_1(e) = \gamma_2^{-1}(e)$. By Proposition 2.2 there is an isomorphism of Q with $N_2 \otimes l^\infty(\mathbf{Z})$ which carries γ_1 to $\mathrm{id} \otimes \sigma$ and γ_2 to $\theta_2 \otimes \sigma^{-1}$. If we regard $N_2 \otimes l^\infty(\mathbf{Z})$ as bounded functions $x\colon n \to x_n$ from $\mathbf{Z}$ to N_2 then we see that Q^{γ_2} consists of those operators x satisfying $x_n = \theta_2^{-n}(x_0)$. Hence $\kappa\colon x \to x_0$ is an isomorphism of N_1 onto N_2 such that $\theta_2\kappa = \kappa\theta_1$. Now, let k be the self-adjoint operator affiliated to the centre of Q such that $w_s = e^{isk}$ for $s \in R$. Then $\gamma_1(k) = k + \gamma_1(\phi_1)$ and $\gamma_2^{-1}(k) = k + \phi_2$. Substituting one equation into the other yields $\phi_2 = \gamma_2^{-1}(k) - \gamma_1(k) + \gamma_1(\phi_1)$. That is, for each $n \in \mathbf{Z}$, $\phi_2 = \theta_2^{-1}(k_{n-1}) - k_{n-1} + (\phi_1)_{n-1}$. Take $n = 1$ to obtain $(\phi_1)_0 = \phi_2 + \phi_2(\xi) - \xi$ where $\xi = \theta_2^{-1}(k_0)$. Hence $\kappa(\phi_1) = \phi_2 + \theta_2(\xi) - \xi$. □

5. Crossed products and traces.

THEOREM 5.1. *Let $\{M, \alpha, \mathbf{R}\}$ be the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. Then M is properly infinite iff N is properly infinite and in this case $M \times_\alpha \mathbf{R}$ is isomorphic to $N \times_\theta \mathbf{Z}$. More generally, the tensor product of the crossed products with the factor of type I_∞ are isomorphic.*

Proof. By Lemma 3.5 and Proposition 2.2 there is an isomorphism of $N \otimes L^\infty(\mathbf{R})$ with $M \otimes l^\infty(\mathbf{Z})$ and so M is properly infinite iff N is.

In the notation of Lemma 3.5 let β be the action of $\mathbf{Z} \times \mathbf{R}$ on P given by $\beta_{(n,t)} = \bar{\theta}^n\bar{\alpha}_t$. By Propositions 2.2 and 2.4, $P \times_\beta (\mathbf{Z} \times \mathbf{R})$ is isomorphic to $M \times_\alpha \mathbf{R} \otimes B(l^2(\mathbf{Z}))$ as well as $(N \times_\theta \mathbf{Z}) \otimes B(L^2(\mathbf{R}))$. Finally, recall that a W^*-algebra is properly infinite iff it is isomorphic to its tensor product with the type I_∞ factor. □

We can connect traces on M with traces on N.

THEOREM 5.2. *Let* $\{M, \alpha, \mathbf{R}\}$ *be the flow built on* $\{N, \theta, \mathbf{Z}\}$ *under* ϕ. *Then M is semi-finite iff N is semi-finite. Moreover, M has a faithful, normal, semi-finite* (*abbreviated f.n.s-f.*) *trace* τ_1 *with* $\tau_1 \circ \alpha_t = e^{-t}\tau_1$ *for all* $t \in \mathbf{R}$ *iff N has a f.n.s.-f. trace* τ_2 *with* $\tau_2 \circ \theta = \tau_2(e^{-\phi})$.

Proof. In the situation of Lemma 3.5, Proposition 2.2 shows that P is isomorphic to both $N \otimes L^\infty(\mathbf{R})$ and $M \otimes l^\infty(\mathbf{Z})$. Hence M is semifinite iff N is. Let m and n denote the usual traces on $L^\infty(\mathbf{R})$ and $l^\infty(\mathbf{Z})$. Let τ_1 be a f.n.s.-f. trace on M with $\tau_1 \circ \alpha_t = e^{-t}\tau_1$ for $t \in \mathbf{R}$. Using the isomorphism of $N \otimes L^\infty(\mathbf{R})$ with $M \otimes l^\infty(\mathbf{Z})$ we can transfer $\tau_1 \otimes n$ to a f.n.s.-f. trace $\bar{\tau}_1$ on $N \otimes L^\infty(\mathbf{R})$ satisfying $\bar{\tau}_1 \circ \bar{\theta} = \tau_1$ and $\bar{\tau}_1 \circ \mathrm{id} \otimes \sigma_t = \bar{e}^t\bar{\tau}_1$ for $t \in \mathbf{R}$. Let h be a self-adjoint operator affiliated to the centre of $N \otimes L^\infty(\mathbf{R})$ such that $e^{ish} = 1 \otimes \chi_s$ for $s \in \mathbf{R}$. Then $\mathrm{id} \otimes \sigma_t(e^h) = e^t e^h$ and $\bar{\theta}^{-1}(e^h) = e^{-\phi} \otimes 1e^h$ for $t \in \mathbf{R}$. Set $\tau = \bar{\tau}_1(e^h \cdot)$. It then follows that $\tau \circ \bar{\theta} = \tau(e^{-\phi} \otimes 1 \cdot)$ and $\tau \circ \mathrm{id} \otimes \sigma_t = \tau$. Hence there is a f.n.s.-f. trace τ_2 on N such that $\tau_2 \otimes m = \tau$ and so $\tau_2 \circ \theta = \tau_2(e^{-\phi} \cdot)$.

For the converse, if $\tau_2 \circ \theta = \tau_2(e^{-\phi} \cdot)$ we transfer $\tau_2 \otimes M(e^{-h} \cdot)$ to $M \otimes l^\infty(\mathbf{Z})$ obtaining a f.n.s.-f. trace τ^1 which satisfies $\tau^1 \circ \mathrm{id} \otimes \sigma = \tau^1$ and $\tau^1 \circ \bar{\alpha}_t = e^{-t}\tau^1$ for $t \in \mathbf{R}$. Hence there is a f.n.s.-f. trace τ_1 on M such that $\tau_1 \otimes n = \tau^1$. This implies that $\tau_1 \circ \alpha_t = e^{-t}\tau_1$ for $t \in \mathbf{R}$. □

6. Discrete and continuous decompositions.

DEFINITION 6.1 ([**4**]). Let P be a properly infinite W^*-algebra. A continuous decomposition of P is a covariant system $\{M, \alpha, \mathbf{R}\}$ with the properties:

(i) $M \times_\alpha \mathbf{R}$ is isomorphic to P.

(ii) M is properly infinite and semi-finite.

(iii) There is a f.n.s.-f. trace τ on M such that $\tau \circ \alpha_t = e^{-t}\tau$ for all $t \in R$.

Combining the results of [**4**] and [**8**], Connes and Takesaki showed that continuous decompositions exist and are unique up to isomorphism. Let $P = M \times_\alpha R$ be a continuous decomposition of P and let C be the centre of M. Then the covariant system $\{C, \alpha|_C, \mathbf{R}\}$ is an invariant of the isomorphism type of P. $\{C, \alpha|_C, \mathbf{R}\}$ is called the flow of weights for P.

DEFINITION 6.2 ([**4**]). A discrete decomposition of a properly infinite W^*-algebra P is a covariant system $\{N, \theta, \mathbf{Z}\}$ with the properties:

(i) $N \times_\theta \mathbf{Z}$ is isomorphic to P.

(ii) N if properly infinite and semi-finite.

(iii) N has a f.n.s.-f. trace τ with $\tau \circ \theta = \tau(e^{-\phi} \cdot\,)$ for some θ ceiling function ϕ.

Connes [**3**] showed that for factors of type III_λ, $\lambda \neq 1$, discrete decompositions exist and are unique up to induction (see §4). Our results on flow under a function yield the following connection between discrete and continuous decompositions.

THEOREM 6.3. *Let P be a properly infinite W^*-algebra, $\{N, \theta, \mathbf{Z}\}$ and $\{M, \alpha, \mathbf{R}\}$ covariant systems and let ϕ be a θ ceiling function. Then any two of the following imply the third*:

(i) *$\{M, \alpha, \mathbf{R}\}$ is a continuous decomposition of P.*

(ii) *$\{N, \theta, \mathbf{Z}\}$ is a discrete decomposition of P with $\tau \circ \theta = \tau(e^{-\phi} \cdot\,)$ for some f.n.s.-f. trace τ on N.*

(iii) *$\{M, \alpha, R\}$ is isomorphic to the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ.*

Proof. Assume (i) and (ii) and let $\{M_1, \alpha^1, \mathbf{R}\}$ be the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. Theorems 5.1 and 5.2 show that $\{M_1, \alpha^1, \mathbf{R}\}$ is a continuous decomposition of P. By the uniqueness of continuous decomposition $\{M, \alpha, R\}$ is isomorphic to $\{M_1, \alpha^1, \mathbf{R}\}$. Now assume (i) and (iii). By Theorem 5.2 there is a f.n.s.-f. trace τ on N with $\tau \circ \theta = \tau(e^{-\phi} \cdot\,)$. Theorem 5.1 shows that $N \times_\theta \mathbf{Z}$ is isomorphic to P. Hence $\{N, \theta, \mathbf{Z}\}$ is a discrete decomposition of P. Finally, assume (ii) and (iii). Theorems 5.1 and 5.2 show that $\{M, \alpha, R\}$ is a continuous decomposition of P. □

Theorems 6.3 and 3.8 give a generalization of [**3**] Théorème 5.3.1.

COROLLARY 6.4. *A properly infinite W^*-algebra P has a discrete decomposition iff the flow of weights is proper.*

Proof. Let $\{M, \alpha, \mathbf{R}\}$ be a continuous decomposition of P then the restriction of α to the centre of M is proper. By Theorem 3.8 $\{M, \alpha, \mathbf{R}\}$ can be expressed as the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ. By Theorem 6.3, $\{N, \theta, \mathbf{Z}\}$ is a discrete decomposition of P.

Conversely, if $P = N \times_\theta \mathbf{Z}$ is a discrete decomposition where $\tau \circ \theta = \tau(e^{-\phi} \cdot\,)$ for some f.n.s.-f. trace τ on N and θ ceiling function ϕ then by

Theorem 3.8 the flow built on $\{N, \theta, \mathbf{Z}\}$ under ϕ is proper when restricted to its centre. By Theorem 6.3 this flow is the flow of weights. □

Theorems 4.5 and 6.3 give a generalization of [3] Théorème 5.4.2.

COROLLARY 6.5. *For* $j = 1, 2$, *let* $\{N_j, \theta_j, \mathbf{Z}\}$ *be discrete decompositions of* P_j. *Then* P_1 *is isomorphic to* P_2 *iff for* $j = 1, 2$ *there are recurrent projections* e_j *in the centre of* N_j *with* $\bigvee_{n \in \mathbf{Z}} \theta_j^n(e_j) = 1$ *such that the induced covariant systems* $\{(N_j)e_j, (\theta_j)e_j, \mathbf{Z}\}$ *are isomorphic.*

Proof. For $j = 1, 2$ let τ_j be a f.n.s.-f. trace on N_j such that $\tau_j \circ \theta_j = \tau_j(e^{-\phi_j} \circ)$ for some θ_j ceiling function ϕ_j.

If P_1 is isomorphic to P_2 then by Theorem 6.3 and the uniqueness of continuous decomposition the flows built on $\{N_j, \theta_j, \mathbf{Z}\}$ under ϕ_j are isomorphic for $j = 1, 2$. Theorem 4.5 now gives the desired conclusion.

Conversely, let $\bar{\tau}_j$ be the restriction of τ_j to $(N_j)e_j$ for $j = 1, 2$. Then we have $\bar{\tau}_j \circ (\theta_j)e_j = \tau_j(e^{-(\phi_j)e_j} \cdot)$ for $j = 1, 2$. Let κ be the isomorphism of $\{(N_1)e_1, (\theta_1)e_1, \mathbf{Z}\}$ with $\{(N_2)e_2, (\theta_2)e_2, \mathbf{Z}\}$ and let h be the self-adjoint operator affiliated to the centre of $(N_2)e_2$ such that $\bar{\tau}_1 \circ \kappa^{-1} = \bar{\tau}_2(e^{-h} \cdot)$. It follows that $(\phi_2)e_2 = \kappa(\phi_1)e_1 + (\theta_2)e_2^{-1}(h) - h$. By Theorem 4.5 the flows built on $\{N_j, \theta_j, \mathbf{Z}\}$ under ϕ_j are isomorphic for $j = 1, 2$. From Theorem 6.3, P_1 and P_2 are isomorphic. □

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