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3-MANIFOLDS WITH SUBGROUPS $Z \oplus Z \oplus Z$ IN THEIR FUNDAMENTAL GROUPS

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In this paper we characterize those 3-manifolds M^3 satisfying $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \subseteq \pi_1(M)$. All such manifolds M arise in one of the following ways: (I) $M = M_0 \# R$, (II) $M = M_0 \# R^*$, (III) $M = M_0 \cup_\partial R^*$. Here M_0 is any 3-manifold in (I), (II) and any 3-manifold having P^2 components in its boundary in (III). R is a flat space form and R^* is obtained from R and some involution $\iota\colon R \to R$ with fixed points, but only finitely many, as follows: if $C_1, \dots, C_n$ are disjoint 3-cells around the fixed points then R^* is the 3-manifold obtained from $(R - \operatorname{int}(C_1 \cup \cdots \cup C_n))/\iota$ by identifying some pairs of projective planes in the boundary.

1. Introduction. In [1] it was shown that the only possible finitely generated abelian subgroups of the fundamental groups of 3-manifolds are Z_n, $Z \oplus Z_2$, Z, $Z \oplus Z$ and $Z \oplus Z \oplus Z$. The purpose of this paper is to characterize all M^3 satisfying $Z \oplus Z \oplus Z \subseteq \pi_1(M)$.

To explain this characterization recall that the Bieberbach theorem (see Chapter 3 of [**8**]) implies that if M is a closed 3-dimensional flat space form then $Z \oplus Z \oplus Z \subseteq \pi_1(M)$. We let $M_1, \dots, M_6$ denote the 6 compact connected orientable flat space forms in the order given on p. 117 of [**8**]. Similarly $N_1, \dots, N_4$ will denote the non-orientable ones. For explicit descriptions see §2. One of the main theorems from [**3**] is

(1.1) THEOREM. *The only space forms from the orientable ones $M_1, \dots, M_6$ which admit involutions having fixed points, but only finitely many, are M_1, M_2, M_6. Moreover these involutions are unique up to conjugacy and have* 8, 4, 2 *fixed points respectively.*

If $\iota\colon M_i \to M_i$, $i = 1$, 2 or 6, is such an involution and $x_1, \dots, x_n$ are the fixed points ($n = 8, 4, 2$) then there are disjoint 3-cells $C_1, \dots, C_n$ so that

$$x_i \in \operatorname{int} C_i \quad \text{and} \quad \iota(C_i) = C_i, \qquad 1 \le i \le n.$$

We let M_i^* denote the orbit manifold $M_i - \operatorname{int}(C_1 \cup \cdots \cup C_n)/\iota$. Thus ∂M_1^* consists of 8 projective planes, ∂M_2^* consists of 4 and ∂M_6^* has 2. Canonical presentations of M_i^*, $i = 1, 2, 6$, are given in [**3**]. By making identifications of pairs of such projective planes in ∂M_i^* we obtain new manifolds still containing $Z \oplus Z \oplus Z$ in their fundamental groups. We refer to any such manifold obtained this way as a projectively flat space form. In this identification procedure it is not necessary to identify all boundary components.

(1.2) EXAMPLE. M_1 is the torus $S^1 \times S^1 \times S^1$ and the involution can be taken to be $\iota(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$.

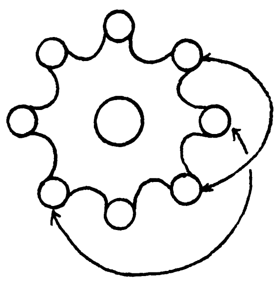

FIGURE 1

8 P^2 boundary components, 2 pairs of which have been identified.

If M_0 is a manifold having P^2 components in its boundary let $M_0 \cup_\partial R^*$ denote the manifold obtained from the disjoint union $M_0 \cup R^*$ of M_0 with a projectively flat space form R^* by identifying some P^2 components of ∂M_0 with some from ∂R^*. Then $\pi_1(M_0 \cup_\partial R^*)$ contains $Z \oplus Z \oplus Z$ as a subgroup.

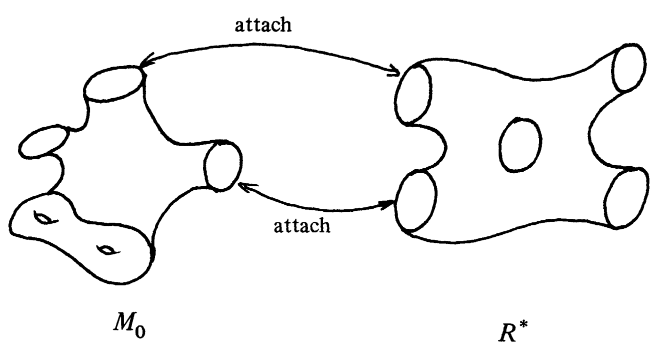

FIGURE 2

MAIN THEOREM. *Suppose M^3 is a 3-manifold. Then $\pi_1(M)$ admits $Z \oplus Z \oplus Z$ as a subgroup if and only if M has one of the following forms:*

(I) *$M = M_0 \# R$ for some flat space form R,*

(II) *$M = M_0 \# R^*$ for some projectively flat space form R^*,*

(III) *$M = M_0 \cup_\partial R^*$ for some projectively flat space form R^*, where, in case* (III), *M_0 is as above.*

Throughout we work in the PL category and use [**2**] for a standard reference. Section 2 contains the descriptions of the space forms, section 3 contains a topological characterization of them, and §4 has the proof of the main theorem.

2. Flat 3-dimensional space forms. In this section we will briefly summarize some of the basic facts about flat space forms — see [**8**] for details. Recall that a complete connected riemannian manifold is a flat space form if its sectional curvature is constantly zero, and that the classical Bieberbach theorem states that a 3-manifold M^3 is a flat space form if and only if its universal covering space is $\mathbf{R}^3$ and the deck transformation group $\pi_1(M)$ is acting on $\mathbf{R}^3$ by rigid motions. This is also equivalent to the existence of a regular covering by a flat torus $S^1 \times S^1 \times S^1 \to M$ in case M is closed.

For such a 3-manifold M there is therefore an extension

$$1 \to \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \to G \to \Psi \to 1,$$

where $G = \pi_1(M)$ is torsion free and Ψ is a finite group. Moreover the abelian subgroup $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ can be taken to be maximal abelian. Conversely we have the following result of Bieberbach: an abstract group G is the fundamental group of a 3-dimensional flat closed space form if G is torsion free and there exists an extension $1 \to \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \to G \to \Psi \to 1$ with Ψ finite and $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ maximal abelian in G. If ρ: $\Psi \to \mathrm{Gl}_3(\mathbf{Z})$ is the representation associated to an extension then $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ is maximal abelian if and only if ρ is faithful. Thus the affine classification of flat space forms in dimension 3 proceeds by first classifying the finite subgroups of $\mathrm{Gl}_3(\mathbf{Z})$ up to conjugacy and then by determining which ones correspond to torsion free extensions.

If ρ: $\Psi \to \mathrm{Gl}_3(\mathbf{Z})$ is a representation then the congruence classes of extensions $1 \to \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \to G \to \Psi \to 1$ associated to ρ are in 1-1 correspondence with $H^2(\Psi; R)$, where R is the Ψ module $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. Then it is easy to see that G is torsion free if and only if the cohomology class $\chi \in H^2(G; R)$ restricts to a non-zero class in $H^2(\mathbf{Z}_p; R)$ for each subgroup $\mathbf{Z}_p \subseteq G$, p a prime.

Proceeding in this way it is a routine matter to classify 3-dimensional flat space forms up to affine diffeomorphism. It turns out that the only possible holonomy groups Ψ are 1, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, $\mathbf{Z}_6$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ in the orientable case and $\mathbf{Z}_2$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ in the non-orientable case.

Let a_1, a_2, a_3 denote a fixed basis of $\mathbf{R}^3$ and let t_1, t_2, t_3 be the corresponding translations. If A is a 3×3 matrix with respect to this basis and $v \in \mathbf{R}^3$ then let (A, t_v) denote the affine map

$$(A, t_v)\colon u \to v + A(u)$$

Then the classification of space forms up to affine equivalence is given by the following two theorems.

THEOREM (2.1). *Up to affine equivalence there are* 6 *orientable closed flat* 3-*dimensional space forms. They are represented by the manifolds* $\mathbf{R}^3/G$, *where G is one of the groups below*:

1. $\Psi = \{1\}$ *and G is generated by* t_1, t_2, t_3.
2. $\Psi = \mathbf{Z}_2$ *and G is generated by* t_1, t_2, t_3 *and* $\alpha = (A, t_{a_1/2})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

3. $\Psi = \mathbf{Z}_3$ *and G is generated by* t_1, t_2, t_3 *and* $\alpha = (A, t_{a_1/3})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

4. $\Psi = \mathbf{Z}_4$ *and G is generated by* t_1, t_2, t_3 *and* $\alpha = (A, t_{a_1/4})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

5. $\Psi = \mathbf{Z}_6$ *and G is generated by* t_1, t_2, t_3 *and* $\alpha = (A, t_{a_1/6})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

6. $\Psi = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *and G is generated by* t_1, t_2, t_3 *and* $\alpha = (A, t_{a_1/2})$, $\beta = (B, t_{(a_2+a_3)/2})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

REMARKS. (1) The normal subgroup $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ is generated by the translations t_1, t_2, t_3 and the corresponding representations $\Psi \to \mathrm{Gl}_3(\mathbf{Z})$ are given by the matrices A for cases $2, \ldots, 5$ and by A, B for the last case.

(2) This theorem explicitly describes the way in which the groups act by affine motions on $\mathbf{R}^3$. In order to put a metric of constant curvature

zero on the space forms, that is, to make these motions rigid, we must impose certain metric conditions on the a_i. But this does not concern us here.

THEOREM (2.2). *Up to affine equivalence there are* 4 *non-orientable closed flat* 3-*dimensional space forms. They are represented by the manifolds* $\mathbf{R}^3/G$, *where G is one of the groups below*:

1. $\Psi = \mathbf{Z}_2$ *and G is generated by* t_1, t_2, t_3 *and* $\varepsilon = (A, t_{a_1/2})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

2. $\Psi = \mathbf{Z}_2$ *and G is generated by* t_1, t_2, t_3 *and* $\varepsilon = (A, t_{a_1/2})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

3. $\Psi = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *and G is generated by* t_1, t_2, t_3 *and* $\alpha = (A, t_{a_1/2})$, $\varepsilon = (B, t_{a_2/2})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

4. $\Psi = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *and G is generated by* t_1, t_2, t_3 *and* $\alpha = (A, t_{a_1/2})$, $\varepsilon = (B, t_{(a_2+a_3)/2})$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We let $M_1, \ldots, M_6$ denote the orientable space forms and $N_1, \ldots, N_4$ the non-orientable ones. Perusing the list of groups in the orientable case reveals that the subgroup generated by t_2, t_3 is normal in G. In the first 5 cases $M_1, \ldots, M_5$ we have an extension

$$1 \to \mathbf{Z} \oplus \mathbf{Z} \to G \xrightarrow{\theta} \mathbf{Z} \to 1$$

and for M_6 we have an extension

$$1 \to \mathbf{Z} \oplus \mathbf{Z} \to G \xrightarrow{\theta} \mathbf{Z}_2 * \mathbf{Z}_2 \to 1$$

where $\mathbf{Z} \oplus \mathbf{Z}$ is the subgroup generated by t_2, t_3. The matrices associated to the extension in the first 5 cases are given by conjugation by t_1 for M_1

and by α for $M_2, \ldots, M_5$. They are, respectively,

$$(2.3) \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Notice that these matrices are in $\mathrm{Sl}_2(Z)$ and have orders 1, 2, 3, 4 and 6.

Geometrically this means that $M_1, \ldots, M_5$ are orientable torus bundles over S^1 resulting from orientation preserving homeomorphisms

$$\phi: S^1 \times S^1 \to S^1 \times S^1$$

having finite orders, and with $\phi_*: H_1(S^1 \times S^1) \to H_1(S^1 \times S^1)$ given by the matrices in (2.3) respectively. In terms of complex coordinates these homeomorphisms can be described as follows:

$$\phi_i(x, y) = \begin{cases} (x, y) & \text{if } i = 1, \\ (\bar{x}, \bar{y}) & \text{if } i = 2, \\ (y, \bar{x}\bar{y}) & \text{if } i = 3, \\ (y, \bar{x}) & \text{if } i = 4, \\ (y, \bar{x}y) & \text{if } i = 5. \end{cases}$$

Then M_i, $1 \le i \le 5$, is the mapping torus construction

$$M_i = S^1 \times S^1 \times [0,1]/(x, y, 0) \sim (\phi_i(x, y), 1).$$

The torus bundle structure over S^1 is given by Figure 3.

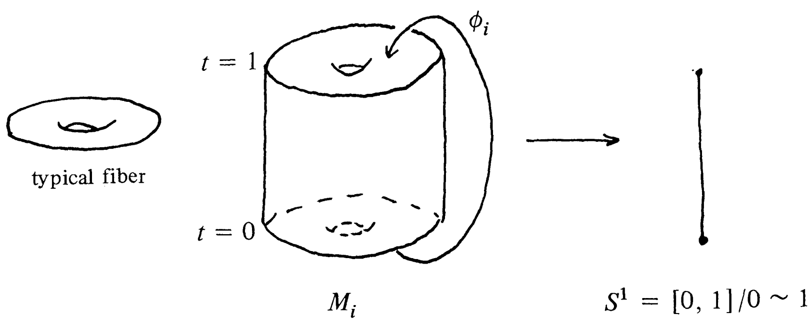

FIGURE 3

The last orientable flat space form M_6, the so-called Hantzsche-Wendt manifold, is not a torus bundle over the 1-sphere. However, M_6 is the union of 2 copies of the orientable twisted I-bundle over the Klein bottle. A particular model W for this twisted I-bundle is $W = S^1 \times S^1 \times [0,1]/(x, y, 0) \sim (\tau(x, y), 0)$ where $\tau: S^1 \times S^1 \to S^1 \times S^1$ is any fixed point free orientation reversing homeomorphism, e.g., $\tau(x, y) = (-x, -\bar{y})$

or $\tau(x, y) = (-\bar{x}, -y)$. The I-bundle structure is given by sliding down the t-axis. Notice that $S^1 \times S^1/(x, y) \sim \tau(x, y)$ is the Klein bottle K and ∂W is a torus. (See Figure 4.)

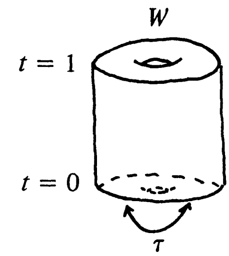

FIGURE 4

To see how M_6 is the union of 2 copies of W we first analyze the structure of $G = \pi_1(M_6)$. From the above description of G it follows that there is a decomposition of G as an amalgamated free product $G_1 *_{\mathbf{Z}\oplus\mathbf{Z}} G_2$, where

$$G_1 = \text{the subgroup generated by } t_1, t_2, \alpha,$$
$$G_2 = \text{the subgroup generated by } t_1, t_2, \beta.$$

In fact G_1, G_2 are Klein bottle groups and M_6 decomposes as follows:

$$M_6 = S^1 \times S^1 \times [0, 1]/(x, y, 0) \sim (-x, -\bar{y}, 0) \text{ and}$$
$$(x, y, 1) \sim (-\bar{x}, -y, 1)$$

W_0, W_1 are copies of W and $W_0 \cap W_1$ is an incompressible torus.

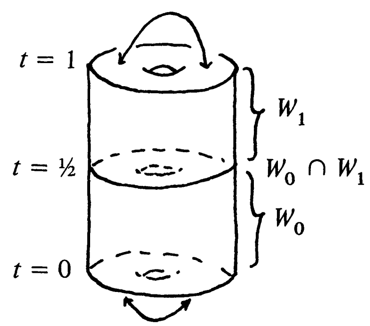

FIGURE 5

We can summarize this construction as follows. Let W be an orientable twisted I-bundle over a Klein bottle. Then $H_1(W) \cong \mathbf{Z} \oplus \mathbf{Z}_2$, and $H_1(\partial W) \cong \mathbf{Z} \oplus \mathbf{Z}$ has a natural basis b_0, b_1 such that, with respect to the inclusion $\iota\colon \partial W \to W$, $\iota_*(b_0)$ is a generator of $2\mathbf{Z}$ and $\iota_*(b_1)$ is the non-zero element of $\mathbf{Z}_2$. Now let W_0, W_1 be two copies of W and let ϕ: $\partial W_0 \to \partial W_1$ be an orientation preserving homeomorphism with ϕ_*: $H_1(\partial W_0) \to H_1(\partial W_1)$ given by the matrix $[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]$ with respect to the natural basis. Then $M_6 = W_0 \cup W_1/x \sim \phi(x)$.

Up to conjugacy a complete list of matrices of finite order in $\mathrm{Sl}_2(\mathbf{Z})$ is given by (2.3). The corresponding list in $\mathrm{Gl}_2(\mathbf{Z})$ has 2 more representatives, namely

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Note that although these matrices are similar over the ring $\mathbf{Z}[\frac{1}{2}]$, they are not similar over $\mathbf{Z}$. The corresponding torus bundles over S^1 turn out to be N_1, N_2 respectively. To see this for N_1 we note that $G = \pi_1(N_1)$ satisfies an extension

$$1 \to \mathbf{Z} \oplus \mathbf{Z} \to G \overset{\theta}{\to} \mathbf{Z} \to 1, \quad \text{where } \mathbf{Z} \oplus \mathbf{Z}$$

is generated by t_2, t_3 and $\theta(\varepsilon)$ is a generator of $\mathbf{Z}$. Since $\varepsilon t_2 \varepsilon^{-1} = t_2$ and $\varepsilon t_3 \varepsilon^{-1} = t_3^{-1}$ it follows that the matrix of this extension is $[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]$. Accordingly N_1 is the mapping torus construction

$$N_1 = S^1 \times S^1 \times [0,1]/(x, y, 0) \sim (x, \bar{y}, 1).$$

In the case of N_2 we also have an extension

$$1 \to \mathbf{Z} \oplus \mathbf{Z} \to G \overset{\theta}{\to} \mathbf{Z} \to 1, \quad \text{where } G = \pi_1(N_2),$$

$\mathbf{Z} \oplus \mathbf{Z}$ is generated by $t_1 t_2$, t_3, and $\theta(\varepsilon) =$ a generator of $\mathbf{Z}$. Now $\varepsilon t_1 t_2 \varepsilon^{-1} = t_1 t_2$, $\varepsilon t_3 \varepsilon^{-1} = t_1 t_2 t_3^{-1}$ and so the matrix is $[\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}]$. Therefore

$$N_2 = S^1 \times S^1 \times [0,1]/(x, y, 0) \sim (xy, \bar{y}, 1).$$

We now have a complete list of conjugacy classes of matrices of finite order in $\mathrm{Gl}_2(\mathbf{Z})$. There are five orientable matrices leading to the space forms $M_1, \ldots, M_5$ and two non-orientable matrices corresponding to N_1, N_2. The other space forms are not torus bundles over a circle. As noted above M_6 is the union of two twisted I-bundles over the Klein bottle.

The non-orientable space forms $N_1, \ldots, N_4$ admit Klein bottle bundle structures (for a different approach see **[4]**). In the following we derive explicit descriptions. First of all N_1 is homeomorphic to $K \times S^1$, where K is the Klein bottle. To see this note that

$$\begin{aligned} N_1 &= S^1 \times S^1 \times [0,1]/(x, y, 0) \sim (x, \bar{y}, 1) \\ &= S^1 \times \{S^1 \times [0,1]/(y, 0) \sim (\bar{y}, 1)\} = S^1 \times K. \end{aligned}$$

Now let σ: $S^1 \times S^1 \to S^1 \times S^1$ be the homeomorphism $\sigma(x, y) = (xy, \bar{y})$. Thus N_2 is the torus bundle over S^1 associated to σ. Then consider the circle $S_\theta = \{(x, y) \in S^1 \times S^1 \mid x^2 y = e^{2\pi i\theta}\}$, $0 \leq \theta \leq 1$. It is easy to verify that S_θ is invariant under σ and that the following diagram commutes:

$$\begin{array}{ccc} S_\theta & \stackrel{\cong}{\to} & S^1 \\ \downarrow \sigma & & \downarrow \rho_\theta \\ S_\theta & \stackrel{\cong}{\to} & S^1 \end{array}$$

where $S_\theta \stackrel{\cong}{\to} S^1$ is the homeomorphism $(x, y) \to x$ and ρ_θ: $S^1 \to S^1$ is $x \to e^{2\pi i\theta}\bar{x}$

Put $K_\theta = S_\theta \times [0, 1]/(x, y, 0) \sim (\sigma(x, y), 1) \subseteq N_2$. Then K_θ is a Klein bottle and $N_2 = \bigcup_{0 \leq \theta \leq 1} K_\theta$. In other words, N_2 is a twisted product of $K = S^1 \times [0, 1]/(x, 0) \sim (\bar{x}, 1)$ and S^1. To see how this twisting works consider the map

$$f: S^1 \times [0, 1] \times [0, 1] \to S^1 \times S^1 \times [0, 1],$$
$$f(x, t, \theta) = (xe^{2\pi i\theta t}, x^{-2}e^{2\pi i\theta(1-2t)}, t).$$

The following are easy to verify:

(i) $f(S^1 \times [0, 1] \times \theta) = S_\theta \times [0, 1]$, in fact f induces a homeomorphism $S^1 \times [0, 1] \times \theta \to S_\theta \times [0, 1]$.

(ii) $f(x, 0, \theta) \sim f(\bar{x}, 1, \theta)$, as points in N_2, for $0 \leq \theta \leq 1$.

Therefore f induces: F: $S^1 \times [0, 1] \times [0, 1]/(x, 0, \theta) \sim (\bar{x}, 1, \theta) \to N_2$, $F[x, t, \theta] = [xe^{2\pi i\theta t}, x^{-2}e^{2\pi i\theta(1-2t)}, t]$. In fact F induces homeomorphisms $K \times \theta \stackrel{\cong}{\to} K_\theta$, $0 \leq \theta \leq 1$. For $\theta = 0, 1$ we have

$$F[x, t, 0] = [x, x^{-2}, t], F[x, t, 1] = [xe^{2\pi it}, x^{-2}e^{-4\pi it}, t]$$

and hence

$$N_2 = K \times [0, 1]/[x, t, 0] \sim [xe^{-2\pi it}, t, 1].$$

To determine the Klein bottle bundle structure on N_3 we first show that there exists an extension $1 \to Q \to G \stackrel{\theta}{\to} \mathbf{Z} \to 1$, where $G = \pi_1(N_3)$ and Q is the fundamental group of the Klein bottle. From the description of G as a group of rigid motions on $\mathbf{R}^3$ we can derive the following relations:

$$\alpha^2 = t_1, \quad \varepsilon^2 = t_2, \quad \varepsilon\alpha\varepsilon^{-1} = t_2\alpha, \quad \alpha t_2\alpha^{-1} = t_2^{-1},$$
$$\alpha t_3\alpha^{-1} = t_3^{-1}, \quad \varepsilon t_1\varepsilon^{-1} = t_1, \quad \varepsilon t_3\varepsilon^{-1} = t_3^{-1}.$$

Now it follows that the subgroup generated by ε, t_3 is a choice for Q and $\theta(\alpha)$ is a generator for $\mathbf{Z}$. Thus $N_3 = K \times [0, 1]/(p, 0) \sim (\sigma(p), 1)$, where σ: $K \to K$ is the homeomorphism inducing conjugation by α on $\pi_1(K) = Q$.

But

$$\alpha\varepsilon\alpha^{-1} = \varepsilon^{-1} \quad \text{and} \quad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

and therefore we may choose σ to be $\sigma[x, t] = [x, 1 - t]$. To see this consider the corresponding homeomorphism on $S^1 \times [0, 1]$. Therefore $N_3 = K \times [0, 1]/[x, t, 0] \sim [x, 1 - t, 1]$.

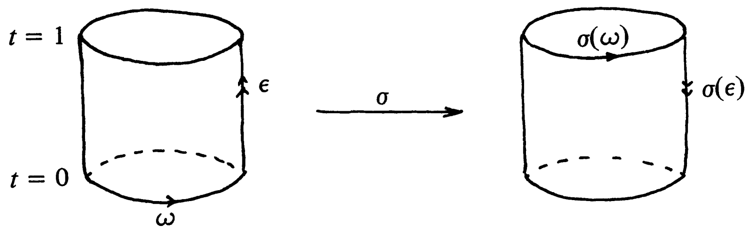

FIGURE 6

Finally it remains to describe the K-bundle structure on N_4. The fundamental group $G = \pi_1(N_4)$ has the relations

$$\alpha^2 = t_1, \quad \varepsilon^2 = t_2, \quad \varepsilon\alpha\varepsilon^{-1} = t_2 t_3 \alpha, \quad \alpha t_2 \alpha^{-1} = t_2^{-1},$$
$$\alpha t_3 \alpha^{-1} = t_3^{-1}, \quad \varepsilon t_1 \varepsilon^{-1} = t_1, \quad \varepsilon t_3 \varepsilon^{-1} = t_3^{-1}.$$

Again we choose $Q =$ the subgroup generated by ε, t_3. Then Q is normal and is isomorphic to $\pi_1(K)$. There is an extension $1 \to Q \to G \xrightarrow{\theta} \mathbf{Z} \to 1$, where $G = \pi_1(N_4)$ and $\theta(\alpha)$ is a generator for $\mathbf{Z}$. One can easily check that

$$\alpha\varepsilon\alpha^{-1} = t^{-1}\varepsilon^{-1}, \qquad \alpha t_3 \alpha^{-1} = t_3^{-1}.$$

The homeomorphism $\sigma\colon K \to K$ inducing conjugation by α is $\alpha[x, t] = [xe^{-2\pi i t}, 1 - t]$. Thus

$$N_4 = K \times [0, 1]/[x, t, 0] \sim [xe^{-2\pi i t}, 1 - t, 1].$$

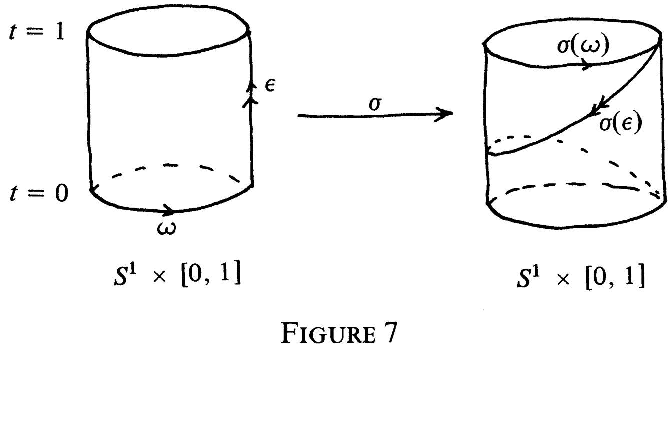

FIGURE 7

We can summarize the preceding results as follows. Let $\psi_i \colon K \to K$ be homeomorphisms, $i = 1, 2, 3, 4$, so that the induced isomorphisms ψ_{i*} on $H_1(K) \cong \mathbf{Z} \oplus \mathbf{Z}_2$ are given by the following "matrices":

$$(2.4) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then N_i is the mapping torus construction

$$N_i = K \times [0,1]/(x, y, 0) \sim (\psi_i(x, y), 1).$$

Note $N_1, \ldots, N_4$ comprise all possible K-bundles over S^1.

Abelianizing the fundamental groups, the first homology groups of the space forms are easily computed to be as follows:

(2.5)

$$\begin{aligned} H_1(M_1) &= H_1(S^1 \times S^1 \times S^1) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \\ H_1(M_2) &= \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\ H_1(M_3) &= \mathbf{Z} \oplus \mathbf{Z}_3 \\ H_1(M_4) &= \mathbf{Z} \oplus \mathbf{Z}_2 \\ H_1(M_5) &= \mathbf{Z} \\ H_1(M_6) &= \mathbf{Z}_4 \oplus \mathbf{Z}_4 \end{aligned} \qquad \begin{aligned} H_1(N_1) &= \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2 \\ H_1(N_2) &= \mathbf{Z} \oplus \mathbf{Z} \\ H_1(N_3) &= \mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\ H_1(N_4) &= \mathbf{Z} \oplus \mathbf{Z}_4. \end{aligned}$$

Since the identification maps in the bundle structures of all ten space forms $M_1, \ldots, M_6$, $N_1, \ldots, N_4$ have finite order it follows that all ten space forms admit Seifert fibrations. The exceptional fibers correspond to fixed points of the group actions generated by the identification maps. See also [**4**].

The canonical involutions on M_1, M_2, M_6 can now be easily described in terms of bundle coordinates:

(1) M_1, $\iota_8[x, y, t] = [\bar{x}, \bar{y}, 1 - t]$ has 8 fixed points $[\pm 1, \pm 1, 0]$, $[\pm 1, \pm 1, \frac{1}{2}]$.

(2)

$$M_2, \iota_4[x, y, t] = \begin{cases} [-x, -y, \frac{1}{2} - t] & \text{if } 0 \le t \le 1/2, \\ [-\bar{x}, -\bar{y}, \frac{3}{2} - t] & \text{if } 1/2 \le t \le 1. \end{cases}$$

The 4 fixed points are $[\pm i, \pm i, \frac{3}{4}]$.

$$M_2 = S^1 \times S^1 \times [0,1]/(x, y, 0) \sim (\bar{x}, \bar{y}, 1).$$

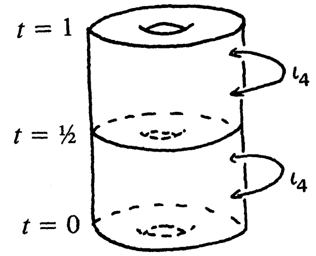

FIGURE 8

(3) M_6, $\iota_2[x, y, t] = [\bar{x}, -y, t]$ has two fixed points $[i, \pm 1, 0]$.

Finally, there are 2-fold coverings $M_1 \to M_2$ and $M_2 \to M_6$. By varying ι_2, ι_4, ι_8 within their conjugacy class we can make these involutions compatible with these coverings. To do this consider the three involutions of $M_1 = S^1 \times S^1 \times S^1$ defined in terms of complex coordinates as follows:

$$\sigma(x, y, z) = (-x, \bar{y}, \bar{z}), \qquad \rho(x, y, z) = (\bar{x}, -\bar{y}, z),$$
$$\iota(x, y, z) = (-\bar{x}, -\bar{y}, \bar{z}).$$

Then it is easily checked that σ, ρ, ι pairwise commute, σ: $M_1 \to M_1$ is the covering translation for $M_1 \to M_2$, and the induced involution $\bar{\rho}$: $M_2 \to M_2$ is the covering translation for $M_2 \to M_6$. Since ι commutes with σ, ρ it gives involutions on M_2 and M_6. These induced involutions are the canonical ones.

For a more explicit description of M_1^*, M_2^* and M_6^* see [**3**]. In conclusion we have the following hierarchy of coverings:

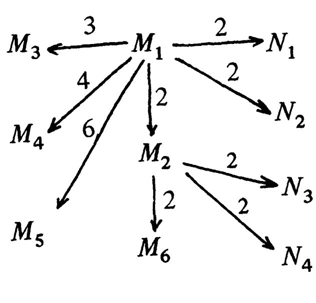

FIGURE 9

3. The topology of flat space forms. In this section we characterize the compact 3-dimensional flat space forms as those connected P^2-irreducible M^3 satisfying $Z \oplus Z \oplus Z \subseteq \pi_1(M)$. (A 3-manifold M is irreducible if each 2-sphere in M bounds a 3-cell; it is P^2-irreducible if it is irreducible and if it does not contain 2-sided projective planes.)

(3.1) LEMMA. *Suppose M^3 is a compact connected P^2-irreducible 3-manifold with $\pi_1(M) = \pi_1(R)$ for some compact flat space form R. Then M is sufficiently large.*

Proof. By (2.5) in §2 the first homology group $H_1(M)$ is infinite except when $R = M_6$. Thus M is sufficiently large in these cases. If $R = M_6$ then $\pi_1(M)$ splits as a free product with amalgamation $G_1 *_{Z \oplus Z} G_2$ (see §2) and therefore there is a 2-sided incompressible surface in M (see [7]). Hence M is sufficiently large. □

As a corollary it follows that the space forms $M_1, \ldots, M_6$, $N_1, \ldots, N_4$ are sufficiently large. They are P^2-irreducible because their universal coverings are $\mathbf{R}^3$.

(3.2) LEMMA. *Let M be a connected 3-manifold so that $\pi_2(M) = 0$ and $Z \oplus Z \oplus Z \subseteq \pi_1(M)$. Then M is closed and $Z \oplus Z \oplus Z$ has finite index in $\pi_1(M)$.*

Proof. We have coverings $M' \to M'' \to M$ where M' is the universal covering of M and M'' corresponds to $Z \oplus Z \oplus Z$. Now suppose that either M is not closed or the index is infinite. Then $H_3(M'') = 0$. Now M' is non-compact and hence $H_3(M') = 0$. But $\pi_1(M') = \pi_2(M') = 0$ and therefore $\pi_3(M') = 0$ by the Hurewicz isomorphism theorem. In other words M' is contractible, and this implies that M'' is a $K(Z \oplus Z \oplus Z, 1)$. Thus $H_3(M'') = H_3(Z \oplus Z \oplus Z) = Z$. Contradiction.. □

(3.3) THEOREM. *Let M be a P^2-irreducible connected 3-manifold such that $Z \oplus Z \oplus Z$ is a subgroup of $\pi_1(M)$. Then M is a compact flat space form.*

Proof. By the sphere theorem we have $\pi_2(M) = 0$, and since $\pi_1(M)$ is infinite the universal covering space is contractible. In other words M is a $K(G, 1)$. Lemma (3.2) now implies that M is closed and $Z \oplus Z \oplus Z$ has finite index in G. Replacing $Z \oplus Z \oplus Z$ by the intersection of its conjugates gives us an extension $1 \to Z \oplus Z \oplus Z \to G \to \Psi \to 1$ with Ψ a finite group. By **[1]** G is torsion free and according to **[6]** there is a compact flat space form R with $\pi_1(R) = G$. Hence there is a homotopy equivalence $M \simeq R$ because M, R are both spaces of type $K(G, 1)$. But M, R are sufficiently large by (3.1) and therefore we can deform the homotopy equivalence $M \simeq R$ into a homeomorphism. □

4. Proof of the Main Theorem.

(4.1) LEMMA. *Let M be a compact irreducible 3-manifold. Then there exists an integer $n(M)$ such that if $P_1,\ldots,P_k$ are pairwise disjoint 2-sided projective planes in M and $k > n(M)$ then some pair P_i, P_j must be parallel (i.e., cobound a product) in M.*

Proof. A 2-sided projective plane in a 3-manifold is incompressible [**2**, Lemma (5.1)]. Then (4.1) is a special case of Lemma (3.2) in [**2**].

(4.2) LEMMA. *Let M^3 be a connected 3-manifold with $Z \oplus Z \oplus Z \subseteq \pi_1(M)$ and let $S^2 \times [-1,1]$ be a bicollar of the 2-sphere $S^2 = S^2 \times 0 \subset$* int M.

(I) *If S^2 does not separate M then $Z \oplus Z \oplus Z \subseteq \pi_1(M_0)$, where $M_0 = M -$* int *$S^2 \times [-1,1]$.*

(II) *If S^2 separates M into M_1, M_2 then at least one of $\pi_1(M_1)$, $\pi_1(M_2)$ contains $Z \oplus Z \oplus Z$.*

Proof. (II) follows from the Kurosh subgroup theorem in a standard way. Thus consider (I).

Let D^2 be a 2-cell such that $D^2 \times [-1,1] \subseteq M_0$, $D^2 \times [-1,1] \cap \partial M_0 = D^2 \times \{-1\} \cup D^2 \times \{1\}$, $D^2 \times \{-1\} \subseteq S^2 \times \{-1\}$ and $D^2 \times \{1\} \subseteq S^2 \times \{1\}$. Thus $M = M_0' \cup T$ and $M_0' \cap T = \partial T$ is a 2-sphere. By the van Kampen theorem $\pi_1(M) = \pi_1(M_0') * Z$ and so $Z \oplus Z \oplus Z \subseteq \pi_1(M_0')$ by the Kurosh subgroup theorem. Since $\pi_1(M_0') = \pi_1(M_0)$ the lemma follows. □

Now we complete the proof of the main theorem. There is a compact submanifold in M whose fundamental group contains $Z \oplus Z \oplus Z$ as a subgroup. Therefore we may assume that M is compact. If the prime decomposition of M is $M = M_1 \# \cdots \# M_n$ then by the Kurosh subgroup theorem $Z \oplus Z \oplus Z$ is a subgroup of some $\pi_1(M_i)$. Also M_i must be irreducible since the fundamental group of a 2-sphere bundle over S^1 is Z. Thus, without loss of generality, assume M is already irreducible. If M does not contain 2-sided projective planes then M is P^2-irreducible, and hence case (I) of the theorem now follows from (3.3).

Now assume that $P_1,\ldots,P_k \subseteq$ int M is a maximal collection of pairwise disjoint 2-sided projective planes such that no two are parallel and none are boundary parallel. Let $P_i \times [-1,1] \subseteq$ int M be disjoint bicollars and let $Q_1, \ldots, Q_m$ be the components of $M - \cup_i P_i \times (-1, 1)$. If p: $\tilde{M} \to M$ is the orientable covering then the $\tilde{Q}_j = p^{-1}(Q_j)$ are the connected pieces left after cutting $\tilde{M}$ along the thickened 2-spheres $p^{-1}(P_j) \times (-1, 1)$. Since $\pi_1(\tilde{M})$ is a subgroup of index 2 in $\pi_1(M)$ we conclude that $\pi_1(\tilde{M})$ has $Z \oplus Z \oplus Z$ for a subgroup. By (4.2) at least one of the pieces $\tilde{Q}_j$, say $\tilde{Q} = p^{-1}(Q)$, must contain $Z \oplus Z \oplus Z$ in its fundamental group. Next set $\hat{Q} =$ the manifold obtained from Q by capping all the 2-sphere boundary components with 3-cells. Then $\hat{Q}$ is irreducible (see Theorem F of **[5]**) and so is a space form by (3.3).

If ι: $\tilde{M} \to \tilde{M}$ is the deck transformation of $\tilde{M} \to M$ then $\iota(\tilde{Q}) = \tilde{Q}$ and so by radial extension we obtain an involution $\hat{\iota}$: $\hat{Q} \to \hat{Q}$ having fixed points but only finitely many. By (1.1) it follows that $\hat{\iota}$ is canonical. Thus Q is one of M_1^*, M_2^* or M_6^*.

To conclude the proof we need only analyze the way in which the pieces are sewn back together. Thus let R^* be the projectively flat space form obtained from Q by reattaching those $P_i \times [-1, 1]$ which were removed to produce Q. If $R^* = M$ we are in case (II) and if $R^* \neq M$ this is case (III).

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