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WEAK COMPACTNESS OF REPRESENTING MEASURES FOR $R(K)$

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Let K be a compact subset of the complex plane, with connected interior K°. Suppose that $p \in K^\circ$ has a weakly compact set of representing measures on ∂K with respect to the algebra $R(K)$. Then every representing measure for p is mutually absolutely continuous with respect to harmonic measure, as is every nonzero orthogonal measure on ∂K. A class of champagne bubble sets with weakly compact sets of representing measures is constructed.

1. Introduction. Let K be a compact plane set, and let $R(K)$ be the algebra of continuous functions on K that can be approximated uniformly on K by rational functions with poles off K. A *representing measure* for a point $p \in K$ is a Borel probability measure λ on K such that $f(p) = \int f\,d\lambda$ for all $f \in R(K)$. The representing measures for p form a convex, weak-star compact set of measures on K. Our aim is to obtain information about $R(K)$ in the case that the set of representing measures on the topological boundary ∂K of K for some fixed $p \in K$ is a weakly compact subset of the Banach space of finite measures.

In the case that $p \in K$ has a finite dimensional set of representing measures on ∂K, reasonably complete information is available concerning the Gleason part of p and the corresponding orthogonal measures. If such a p belongs to ∂K, then in fact p is a peak point for $R(K)$, so that the point mass at p is the only representing measure for p. If such a p belongs to the interior K° of K, then the connected component U of K° containing p is finitely connected and forms a Gleason part for $R(K)$. Moreover, the boundary values of any conformal map from a canonical circle domain D into U determines a Borel isomorphism which transplants harmonic measure on ∂D to harmonic measure on ∂U, and which carries measures in $R(\overline{D})^\perp$ to those measures in $R(K)^\perp$ corresponding to the part U.

Following a conjecture of E. Bishop [**1**, p. 347, problem 8], S. Fisher [**3**] initiated the study of norm compact sets of representing measures, giving conditions for compactness and also for non-compactness in the norm topology of the set of representing measures. The study was continued by the author in [**4**], where it was shown that if $p \in K$ has a norm compact set of representing measures, then either $p \in K^\circ$, or else p is a peak point for $R(K)$. The proof in [**4**], which depends upon Iversen's theorem on cluster values together with a theorem of Hoffman and Rossi, is valid under the weaker hypothesis of a weakly compact set of representing measures.

Our first aim is to establish in §2 an abstract version of Iversen's theorem. This will be combined in §3 with a result from [**10**], which depends crucially on the Hoffman-Rossi theorem, to establish a theorem on the weak-star approximation of representing measures by absolutely continuous representing measures. The approximation theorem readily yields information on weakly compact sets of representing measures, including the following result.

THEOREM 1. *Suppose that K° is connected, and that $p \in K^\circ$ has a weakly compact set of representing measures on ∂K. Then any representing measure on ∂K for a point of K° is mutually absolutely continuous with respect to harmonic measure. If moreover K° is dense in K, then every nonzero measure on ∂K orthogonal to $R(K)$ is mutually absolutely continuous with respect to harmonic measure.*

Incidentally this shows that each point $q \in \partial K$ is a peak point for $R(K)$. Otherwise one constructs an orthogonal measure on ∂K that charges the singleton $\{q\}$, contradicting the fact that harmonic measure does not charge singletons. Thus the only nontrivial Gleason part for $R(K)$ is precisely K°. It would be interesting to determine in these circumstances whether $R(K)$ is pointwise boundedly dense in $H^\infty(K^\circ)$; this would imply in particular that $R(K) = A(K)$.

The second statement of Theorem 1 can be obtained from the first by a straightforward and standard argument. It can also be obtained, immediately, by appealing to a theorem of I. Glicksberg [**11**] to the effect that every nonzero orthogonal measure is mutually absolutely continuous with respect to a representing measure.

In §5, we observe that the basic results are also valid for certain algebras related to $R(K)$. In §§6 and 7 we discuss various classes of compact sets K, the roadrunner sets and the champagne bubble sets.

As a source for standard notation and terminology we refer to [**8**].

2. The Shilov boundary of $H^\infty(\lambda_Q)$. In this section, K is an arbitrary compact subset of the complex plane, and Q is a nontrivial Gleason part for $R(K)$. Let λ_Q denote the restriction of the area measure $dx\,dy$ to Q, and let $H^\infty(\lambda_Q)$ denote the weak-star closure of $R(K)$ in $L^\infty(\lambda_Q)$. By Davie's theorem, every function in $H^\infty(\lambda_Q)$ can be approximated pointwise a.e. $(d\lambda_Q)$ by a bounded sequence in $R(K)$. The point evaluation at a point $\zeta \in Q$ extends to be a weak-star continuous homomorphism of $H^\infty(\lambda_Q)$, which will be denoted by φ_ζ. Every weak-star continuous homomorphism (nonzero, complex-valued) arises in this manner, from a point of Q. For these results, and for further background material on $H^\infty(\lambda_Q)$, see [**9**] and [**10**].

In many respects, $H^\infty(\lambda_Q)$ is similar to the algebra $H^\infty(D)$ of bounded analytic functions on a domain D. In this section, we wish to prove a theorem for $H^\infty(\lambda_Q)$ which is known to be valid for $H^\infty(D)$ and which is an abstract version of Iversen's theorem.

Let $\mathfrak{M}$ denote the maximal ideal space of $H^\infty(\lambda_Q)$. We will regard the functions in $H^\infty(\lambda_Q)$ as continuous functions on $\mathfrak{M}$. The coordinate function $Z \in R(K)$, when regarded as a continuous function on $\mathfrak{M}$, maps $\mathfrak{M}$ to the closure $\overline{Q}$ of Q. For $\zeta \in \overline{Q}$, define the fiber $\mathfrak{M}_\zeta$ of $\mathfrak{M}$ over ζ by

$$\mathfrak{M}_\zeta = Z^{-1}(\{\zeta\}) = \{\varphi \in \mathfrak{M} : Z(\varphi) = \zeta\}.$$

The Shilov boundary of $H^\infty(\lambda_Q)$ will be denoted by Ш. The result we require is the equality (2.1) of the following theorem, which is proved for $H^\infty(D)$ in [**5**].

THEOREM 2. *Let Q be a nontrivial Gleason part for $R(K)$, and define $H^\infty(\lambda_Q)$ as above, with maximal ideal space $\mathfrak{M}$, fibers $\mathfrak{M}_\zeta$, and Shilov boundary* Ш. *Then for each $\zeta \in \overline{Q}$, the restriction of $H^\infty(\lambda_Q)$ to $\mathfrak{M}_\zeta$ is closed subalgebra of $C(\mathfrak{M}_\zeta)$ whose maximal ideal space is $\mathfrak{M}_\zeta$. Furthermore, if $\zeta \notin K^\circ$, the Shilov boundary of the restriction algebra is given by*

$$\text{Ш}_\zeta = \text{Ш} \cap \mathfrak{M}_\zeta, \qquad \zeta \in \overline{Q} \setminus K^\circ. \tag{2.1}$$

Proof. We will follow the proof of the corresponding fact for $H^\infty(D)$, as given for instance in [**7**]. We provide only an outline of those parts of the proof which carry over virtually verbatim from $H^\infty(D)$.

The algebra $H^\infty(\lambda_Q)$ is invariant under the T_g-operators used in rational approximation theory. Using these, one can establish as in [**7**, §1], that

$$\sup\{|f(\varphi)| : \varphi \in \mathfrak{M}_\zeta\} = \limsup_{Q \ni q \to \zeta} f(q), \qquad f \in H^\infty(\lambda_Q).$$

In particular, if $f(q)$ tends to zero as $q \to \zeta$, $q \in Q$, then $f = 0$ on $\mathfrak{M}_\zeta$. Using the T_g-operators, one can also establish (as in [**9**, Corollary 2.2]) that for each $f \in H^\infty(\lambda_Q)$, there is a bounded sequence $\{f_j\}$ in $H^\infty(\lambda_Q)$ that converges uniformly to f on each subset of Q at a positive distance from ζ, while each f_j is analytic at ζ. In particular, $\{f_j\}$ converges uniformly to f on each compact subset of $\mathfrak{M} \setminus \mathfrak{M}_\zeta$, and each f_j is constant on $\mathfrak{M}_\zeta$.

As in the proof of [7, Lemma 6.3], the approximating sequence $\{f_j\}$ can be used to show that if ν is any measure on $\mathfrak{M}$ such that $\nu \perp H^\infty(\mu_Q)$ and $\nu(\mathfrak{M}_\zeta) = 0$, then the restriction of ν to $\mathfrak{M}_\zeta$ is also orthogonal to $H^\infty(\lambda_Q)$. Thus the proof of [7, Theorem 6.1] (see also [9, §6]) shows that if $\mathfrak{M}_\zeta$ is not a peak set for $H^\infty(\lambda_Q)$, there is a weak-star continuous homomorphism in $\mathfrak{M}_\zeta$, namely φ_ζ, and the kernel I_ζ of φ_ζ has the property that if a measure ν on $\mathfrak{M}$ is orthogonal to I_ζ, then the restriction of ν to $\mathfrak{M}_\zeta$ is orthogonal to I_ζ. In this case, the restriction of I_ζ to $\mathfrak{M}_\zeta$ is a closed subspace of $C(\mathfrak{M}_\zeta)$, so that also the restriction of $H^\infty(\lambda_Q)$ to $\mathfrak{M}_\zeta$ is closed. In the other case, in which $\mathfrak{M}_\zeta$ is a peak set, it is a consequence of Glicksberg's peak set theorem [8, Theorem II.12.7] that the restriction of $H^\infty(\lambda_Q)$ to $\mathfrak{M}_\zeta$ is closed. Since $\mathfrak{M}_\zeta$ is the level set of the function $Z \in H^\infty(\lambda_Q)$, each $\mathfrak{M}_\zeta$ is $H^\infty(\lambda_Q)$-convex and hence coincides with the maximal ideal space of the restriction algebra. This proves the first assertion of Theorem 2.

Now let $\zeta \in \overline{Q}$, and fix $\varphi \in Ш \cap \mathfrak{M}_\zeta$. Let N be any open neighborhood of φ. Since the generalized peak points are dense in Ш, there is a sequence $\{\varphi_j\}$ of generalized peak points for $H^\infty(\lambda_Q)$ such that $\varphi_j \in N$ for all j, and $Z(\varphi_j) \to \zeta$. The "independence of fibers" argument of J.-P. Rosay [14], as utilized in [5, §1], then shows that every point in $\mathfrak{M}$ adherent to the sequence $\{\varphi_j\}$ is a generalized peak point for $H^\infty(\lambda_Q)$. In particular, there exists $\psi \in N \cap \mathfrak{M}_\zeta$ that is a generalized peak point for $H^\infty(\lambda_Q)$, hence for the restriction of $H^\infty(\lambda_Q)$ to $\mathfrak{M}_\zeta$, so that $\psi \in Ш_\zeta$ and $Ш_\zeta$ meets N. Since N is arbitrary, $\varphi \in Ш_\zeta$. We conclude that

$$Ш_\zeta \supseteq Ш \cap \mathfrak{M}_\zeta, \qquad \zeta \in \overline{Q}. \tag{2.2}$$

If $\mathfrak{M}_\zeta$ is a peak set, then the reverse inclusion follows from an abstract fact about uniform algebras. Thus (2.1) holds for all $\zeta \in \overline{Q}\backslash Q$.

Fix $\zeta \in Q$. The splitting property of measures orthogonal to I_ζ, cited earlier in the proof, implies the following extension theorem (cf. [8, Theorem II.12.5] or [9, Lemma 6.1]): If h is any (strictly) positive continuous function on $\mathfrak{M}$, if $g \in C(\mathfrak{M}_\zeta)$ is the restriction of a function in I_ζ, and if $|g| \leq h$ on $\mathfrak{M}_\zeta$, then there exists $f \in I_\zeta$ such that $f = g$ on $\mathfrak{M}_\zeta$ while $|f| \leq h$ on $\mathfrak{M}$. Now suppose $\varphi \in Ш_\zeta$ is distinct from φ_ζ. Let N be any closed neighborhood of φ in $\mathfrak{M}$, and choose $h \in C(\mathfrak{M})$ such that $0 < h \leq 1$, $h < 1$ on $\mathfrak{M}\backslash N$, and $h = 1$ in a small neighborhood of φ. Since $\varphi \in Ш_\zeta$, there is a function g in the restriction algebra of $H^\infty(\lambda_Q)$ to $\mathfrak{M}_\zeta$ such that $g = 1$ somewhere on the small neighborhood of φ, $|g| \leq h$ on $\mathfrak{M}_\zeta$, and $g(\varphi_\zeta) = 0$. The latter condition guarantees that g is the restriction to $\mathfrak{M}_\zeta$ of a function in I_ζ. By the extension result, there is $f \in I_\zeta$ such that $|f| \leq h$, while $\|f\| = 1$. It follows that the Shilov boundary of $H^\infty(\lambda_Q)$ meets N. Since N is an arbitrary neighborhood of φ, $\varphi \in \mathrm{III}$, and we conclude that

$$\mathrm{III}_\zeta \subseteq (\mathrm{III} \cap \mathfrak{M}_\zeta) \cup \{\varphi_\zeta\}, \qquad \zeta \in Q. \tag{2.3}$$

Now suppose that $\zeta \in \bar{Q}$ is such that $\mathrm{III}_\zeta \neq \mathrm{III} \cap \mathfrak{M}_\zeta$. In view of (2.2) and (2.3), III_ζ is then obtained by adjoining φ_ζ to $\mathrm{III} \cap \mathfrak{M}_\zeta$. Since φ_ζ is an isolated point of III_ζ, it is also an isolated point of $\mathfrak{M}_\zeta$. Hence there exists $f \in H^\infty(\lambda_Q)$ such that $f(\varphi_\zeta) = 1$, while $|f| < 1/4$ on $\mathfrak{M}_\zeta \backslash \{\varphi_\zeta\}$. In particular, $|f| < 1/4$ on $\mathrm{III} \cap \mathfrak{M}_\zeta$. Choose $\delta > 0$ such that if Δ_δ is the open disc $\{|z - \zeta| < \delta\}$, then $|f| < 1/4$ on $\mathrm{III} \cap Z^{-1}(\bar{\Delta}_\delta)$. By shrinking δ, we may suppose also that the range of f on $Z^{-1}(\bar{\Delta}_\delta)$ is included in the union of the two discs $\{|z| < 1/4\}$ and $\{|z - 1| < 1/4\}$.

Let $N = \{\varphi \in Z^{-1}(\Delta_\delta): |f(\varphi) - 1| < 1/4\}$. Then N is an open subset of $\mathfrak{M}$ which is disjoint from III, and furthermore the boundary of N is included in $Z^{-1}(\Delta_\delta)$. By the local maximum modulus principle, N is included in the $H^\infty(\lambda_Q)$-convex hull $\widehat{\partial N}$ of ∂N. Since $|f - 1| \leq 1/4$ on N, this estimate persists on $\widehat{\partial N}$, and consequently $\widehat{\partial N} \cap Z^{-1}(\Delta_\delta) = N$. Now by an abstract fact, $Z(\partial N)$ includes the topological boundary of $Z(\widehat{\partial N})$. Since $Z(\partial N)$ is included in $\partial\Delta_\delta$, we conclude that $Z(N)$ covers Δ_δ.

Fix $\xi \in \Delta_\delta$. Since $|f| \leq 1/4$ on $\mathrm{III} \cap \mathfrak{M}_\xi$, while $|f - 1| < 1$ somewhere on $\mathfrak{M}_\xi$, we see that $\mathrm{III}_\xi \neq \mathrm{III} \cap \mathfrak{M}_\xi$, and consequently $\xi \in Q$. Thus $\Delta_\delta \subset Q$, and ζ is an interior point of K. This shows that (2.1) is valid, and the proof is complete. □

3. Weak-star density of representing measures. Let Q be a non-trivial Gleason part for $R(K)$, as before. Let σ be a positive measure on K, and let $H^\infty(\sigma)$ denote the weak-star closure of $R(K)$ in $L^\infty(\sigma)$. We will write

$$H^\infty(\sigma) \cong H^\infty(\lambda_Q)$$

to mean that the identity map of $R(K)$ extends to an isometric isomorphism and weak-star homeomorphism of $H^\infty(\sigma)$ and $H^\infty(\lambda_Q)$. This occurs if and only if (i) σ is absolutely continuous with respect to some representing measure for some point of Q, and (ii) every point of Q has a representing measure that is absolutely continuous with respect to σ. The condition (i) means that σ lies in the minimal reducing band of measures corresponding to the Gleason part Q. (See [**2**, §20].) In particular, σ is supported on $\bar{Q}$. The condition (ii) is equivalent to asserting that the point evaluations at points of Q extend to be weak-star continuous homomorphisms of $H^\infty(Q)$, again denoted by φ_ζ, $\zeta \in Q$.

Our aim in this section is to establish the following theorem.

THEOREM 3. *Let σ be a positive measure on K such that $H^\infty(\sigma) \cong H^\infty(\lambda_Q)$, and let $\zeta \in Q$. If τ is any probability measure on the closed support of σ such that τ represents ζ on $R(K)$, then τ belongs to the weak-star closure of the set of representing measures for ζ that are absolutely continuous with respect to σ.*

Proof. This was proved in several special cases, including the case $\sigma = \lambda_Q$, in [**10**]. To prove the result in the case at hand it suffices to check the hypotheses of the abstract version, Theorem 7.3 of [**10**]. This amounts to checking that the property (♯) on [**10**, p. 137] holds, and for this we proceed as follows.

The restriction map $H^\infty(\sigma + \tau) \to H^\infty(\sigma)$ is an isometric isomorphism and a weak-star homeomorphism, so that

$$H^\infty(\sigma + \tau) \cong H^\infty(\sigma) \cong H^\infty(\lambda_Q).$$

Moreover, these isomorphisms respect the fibering of $\mathfrak{M}$ by the $\mathfrak{M}_\zeta$'s.

Let Σ denote the maximal ideal space of $L^\infty(\sigma)$. Then $H^\infty(\sigma)$ can be regarded as an algebra of continuous functions on Σ. In fact, $H^\infty(\sigma)$ becomes a uniform algebra on the quotient space $\Sigma/\sim$ obtained by identifying points of Σ that are identified by $H^\infty(\sigma)$. Thus $\Sigma/\sim$ can be regarded as a closed subset of $\mathfrak{M}$, and $\Sigma/\sim$ includes the Shilov boundary of $H^\infty(\lambda_Q)$.

Let Σ_ζ denote the fiber of $\Sigma/\sim$ over $\zeta \in \bar{Q}$:

$$\Sigma_\zeta = \{\varphi \in \Sigma/\sim \, : Z(\varphi) = \zeta\}, \qquad \zeta \in \bar{Q}.$$

The range of $f \in H^\infty(\sigma)$ on Σ_ζ is precisely the σ-essential cluster set of f at ζ. By Theorem 2, Σ_ζ includes the Shilov boundary Ш_ζ of the restriction of $H^\infty(\sigma) \cong H^\infty(\lambda_Q)$ to $\mathfrak{M}_\zeta$. Thus if $f \in H^\infty(\sigma)$, then

$$\sup\{|f(\varphi)| : \varphi \in \mathfrak{M}_\zeta\} \le \sigma\text{-ess} \limsup_{z \to \zeta} |f(z)|.$$

Now the τ-essential cluster set of f at ζ is included in $f(\mathfrak{M}_\zeta)$, as can be seen by regarding $H^\infty(\sigma + \tau) \cong H^\infty(\sigma)$ as an algebra of continuous functions on the maximal ideal space of $L^\infty(\sigma + \tau)$. It follows that

$$\tau\text{-ess} \limsup_{z \to \zeta} |f(z)| \le \sigma\text{-ess} \limsup_{z \to \zeta} |f(z)|$$

for all $\zeta \in \bar{Q}$ that belong to supp τ. This is property (♯) of [**10**, p. 137], so we may now appeal to Theorem 7.3 of [**10**] to deduce Theorem 3. □

4. Weakly compact sets of representing measures. Let Q be a nontrivial Gleason part for $R(K)$, and suppose $q \in Q$ has a weakly compact set of representing measures on ∂K. Then q has a dominant representing measure on ∂K, that is, a representing measure η that dominates every representing measure for q on ∂K. Furthermore, the Radon-Nikodym derivatives of the representing measures for q with respect to η form a weakly compact subset of $L^1(\eta)$. This is equivalent to asserting that they form a bounded, weakly closed subset of $L^1(\eta)$ which is uniformly integrable.

Each point of Q has a weakly compact set of representing measures just as soon as one point does (cf. [**8**, VI.1]). Thus η dominates any representing measure on ∂K for any point of Q. In particular, η dominates the point mass at any point of $Q \cap \partial K$, so that there are at most countably many points of $Q \cap \partial K$. The main theorem of [**4**] asserts that $Q \subseteq K^\circ$, and we will give a proof of this fact presently. First we give the following corollary to Theorem 3.

THEOREM 4. *Let Q be a Gleason part for $R(K)$, and let σ be a positive measure in ∂K such that $H^\infty(\sigma) \cong H^\infty(\lambda_Q)$. Suppose that the set of representing measures on ∂K for some (hence for all) $q \in Q$ is weakly compact. Then every representing measure for any $q \in Q$ is absolutely continuous with respect to σ.*

Proof. Since point evaluations at points of Q are weak-star continuous on $H^\infty(\sigma)$, σ has mass in any neighborhood of each point of $\overline{Q} \cap \partial K$, and $\operatorname{supp} \sigma = \overline{Q} \cap \partial K$. Let ν be supported on $\overline{Q}$, so that $\operatorname{supp} \nu \subseteq \operatorname{supp} \sigma$. By Theorem 3, there is a net $\{\nu_\alpha\}$ of representing measures for q that converges weak-star to ν, such that $\nu_\alpha \ll \sigma$. Passing to a subnet, we may assume that $\{\nu_\alpha\}$ converges weakly, evidently to ν. Since the representing measures dominated by σ form a weakly closed set, we obtain $\nu \ll \sigma$, as required. □

Now we indicate how to prove the main result of [**4**], that $Q \subseteq K^\circ$ whenever points of Q have weakly compact sets of representing measures. Let η be a dominant representing measure on ∂K for some point of Q, and let $p \in \partial K$. Write $\eta = \sigma + a\delta_p$, where $a \geq 0$ and $\sigma(\{p\}) = 0$. It is easy to check that every point in Q has a representing measure dominated by σ, so that $H^\infty(\sigma) \cong H^\infty(\lambda_Q)$. Theorem 4 shows that $\eta \ll \sigma$. This implies that $\eta(\{p\}) = 0$. Hence η has no point measures on ∂K, and Q is disjoint from ∂K. So $Q \subseteq K^\circ$, and in fact Q is a union of connected components of K°.

It is not known (assuming weak compactness of the set of representing measures for $q \in Q$) whether the Gleason part Q is connected. However, in the case that K° is connected, the part Q coincides with K°. Moreover, if σ is any representing measure on ∂K for a point $q \in K^\circ$, then

$H^\infty(\sigma) \cong H^\infty(\lambda_Q)$, so that Theorem 4 shows that any other representing measure is absolutely continuous with respect to σ. Thus all representing measures on ∂K are mutually absolutely continuous. This proves Theorem 1.

5. The algebra $A(D)$. The line of argument above applies not only to $R(K)$ but to any T-invariant algebra [**2**, §7]. Here we mention the analogue of Theorem 1 for the T-invariant algebra $A(D)$.

THEOREM 5. *Let D be a bounded domain in the complex plane whose boundary ∂D has no isolated points, and let $A(D)$ denote the algebra of continuous functions on $\bar{D}$ that are analytic on D. If $p \in D$ has a weakly compact set of representing measures, then all representing measures on ∂D for points of D are mutually absolutely continuous with respect to harmonic measure.*

Proof. The analogue of Theorem 3 is valid for $A(D)$. To apply it as in the preceding section, we need only to establish one minor point, namely, that the closed support of any representing measure on ∂D for $q \in D$ coincides with ∂D. For this, let S denote the Shilov boundary of $A(D)$. It is easy to see that any representing measure on S for $q \in D$ has closed support S. So it suffices to show that $S = \partial D$.

Now S is a closed subset of ∂D, and all functions in $A(D)$ extend analytically to $\bar{D} \backslash S$ [**2**, Lemma 17.3]. In particular, the point masses at points of $(\partial D)\backslash S$ must be absolutely continuous with respect to a dominant measure for p on ∂D. Thus there are at most countably many points in $(\partial D)\backslash S$, and since ∂D has no isolated points, in fact $S = \partial D$, as required. □

6. Roadrunner sets. A roadrunner set is a compact set K obtained from the closed unit disc $\bar{\Delta}$ by excising a sequence of open subdiscs Δ_j with pairwise disjoint closures, such that the Δ_j's accumulates only at zero. The question of which roadrunner sets correspond to weakly compact sets of representing measures is settled by the work in [**3**] and [**4**]. In fact, Fisher [**3**] proved that if 0 is a peak point for $R(K)$, then points of K° have norm compact sets of representing measures. On the other hand, if points of K° have weakly compact sets of representing measures, then 0 is necessarily a peak point for $R(K)$, by the proof in [**4**]. In particular, weak compactness and norm compactness are equivalent. For a more interesting class of examples, we turn to champagne bubble sets.

7. Champagne bubble sets. A champagne bubble set is a compact set K obtained from the closed unit disc $\bar{\Delta}$ by excising a sequence of open subdiscs Δ_j with pairwise disjoint closures, such that the Δ_j accumulate only on $\partial\Delta$.

THEOREM 6. *Let $K = \bar{\Delta}\setminus(\bigcup \Delta_j)$ be a champagne bubble set, fix $p \in K^\circ$, and let M_p be the set of representing measures on ∂K for p. Then the following are equivalent.*

(i) *M_p is norm compact.*

(ii) *M_p is weakly compact.*

(iii) *If Δ_ε is an open annulus $\{1 - \varepsilon < |z| < 1\}$, then $\nu(A_\varepsilon) \to 0$, uniformly in $\nu \in M_p$, as $\varepsilon \to 0$.*

(iv) *The restriction measures $\{\nu|_\Delta : \nu \in M_p\}$ form a weak-star closed set.*

Proof. Evidently (iii) and (iv) are equivalent, while (i) implies (ii). If (ii) is valid, then the Radon-Nikodym derivatives of the representing measures with respect to some fixed dominant measure are uniformly integrable, so that (iii) is valid. Finally, Fisher's work in **[3]** shows that (iii) implies (i). □

Note in particular that if $\partial\Delta$ is a null set for M_p, then condition (iv) holds, so that the set of representing measures is weakly compact. B. Øksendal **[13]** has given necessary and sufficient conditions, in terms of analytic capacity, for $\partial\Delta$ to be a null set. His condition is that the holes Δ_j be large in the sense of analytic capacity γ, specifically, that

$$\liminf_{\delta\to 0} \frac{\gamma\big(\Delta(\zeta;\delta) \cap \big(\bigcup \Delta_j\big)\big)}{\delta} > 0 \tag{7.1}$$

for a set of $\zeta \in \partial\Delta$ of full linear measure. It is easy to construct the Δ_j's large enough so that the condition (7.1) is valid for all $\zeta \in \partial\Delta$. In this way we obtain a champagne bubble set such that points of K° have compact sets of representing measures, all carried by $\bigcup \partial\Delta_j$.

It may occur that the set of representing measures M_p is norm compact, even though the discs Δ_j accumulate on all of $\partial\Delta$ and the restriction of harmonic measure to $\partial\Delta$ is mutually absolutely continuous with respect to arc length there. Such an example is constructed, using Fisher's argument, as follows. Choose a sequence $\{\zeta_n\}$ of complex numbers of unit modulus which has each point of $\partial\Delta$ as a limit point. Suppose discs $\Delta_1, \ldots, \Delta_{n-1}$ have been chosen appropriately. Let Δ_n be a disc in Δ with center so near to ζ_n and radius so small that, for

$$K_n = \bar{\Delta}\setminus\left(\bigcup_{j=1}^{n} \Delta_j\right),$$

the supremum of $\nu(\partial\Delta_n)$ for all representing measures ν on ∂K_n for p with respect to $R(K_n)$ is less than $1/2^n$. Let $\eta \in M_p$, and let $S_n\eta$ denote the sweep of η to ∂K_n via harmonic measure. Then $S_n\eta$ is a representing measure for p on $R(K_n)$, and $S_n\eta$ converges weak-star to η as $n \to \infty$. The sweep of $S_{m+1}\eta$ to ∂K_m is $S_m\eta$, and it is obtained by sweeping the mass of $S_{m+1}\eta$ on $\partial\Delta_{m+1}$ to ∂K_m. Since the mass of $S_{m+1}\eta$ actually swept is less than $1/2^{m+1}$, we obtain

$$\|S_{m+1}\eta - S_m\eta\| \leq 1/2^{m+1}.$$

This shows that $S_m\eta$ converges in norm as $m \to \infty$, and the limit must be η. Moreover, the estimate shows that

$$\|\eta - S_m\eta\| \leq \sum_{j=m+1}^{\infty} \|S_{j+1}\eta - S_j\eta\| \leq \frac{1}{2^m}.$$

This estimate is uniform in η. Since the range of each $S_m\mathfrak{M}_p$ is finite dimensional, the limit $\mathfrak{M}_p$ is norm compact. We may assume that the radii of the discs Δ_j are summable. Then the measure dz on ∂K is a finite measure that is orthogonal to $R(K)$. By Theorem 1, it is mutually absolutely continuous with respect to harmonic measure. Thus the restriction of harmonic measure to $\partial\Delta$ is comparable to arc length.

By combining Øksendal's condition with Fisher's approximation technique, it is possible to concoct a wide variety of champagne bubble sets with norm compact sets of representing measures.

THEOREM 7. *Let E be a Borel subset of $\partial\Delta$. Then there is a champagne bubble set K such that points $p \in K^\circ$ have norm compact sets of representing measures on ∂K, and such that each representing measure is comparable to the arc length measure on $E \cup (\bigcup_{j=1}^{\infty} \partial\Delta_j)$.*

Proof. If E is a subset of $\partial\Delta$ of full arc length measure, then the construction discussed above provides such an example. Thus we may assume that $F = (\partial\Delta)\setminus E$ has positive arc length. Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of F such that the F_i's are pairwise disjoint, $F\setminus(\bigcup F_i)$ has zero arc length, each F_i has positive arc length, and for $i \geq 2$ the length of F_i is bounded by $1/2^i$. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of E such that $E\setminus(\bigcup E_i)$ has zero arc length.

Construct by induction open subsets U_k and W_k of Δ, $k \geq 1$, such that (i) U_k is the union of open discs $\{\Delta_{kj}\}_{j=1}^{\infty}$ with pairwise disjoint closures which accumulate on F_k; (ii) U_k is contained in the $(1/k)$-neighborhood of F_k; (iii) $\overline{U}_k$ is disjoint from $\overline{U}_j$ for $j < k$; (iv) if

$$K_k = \overline{\Delta}\setminus(U_1 \cup \cdots \cup U_k),$$

then F_k is a null-set for $R(K_k)^\perp$; (v) if u_k is the harmonic function on K_{k-1}° with boundary values 1 on F_k and 0 on $(\partial K_{k-1})\backslash F_k$, then $u_k > 1/2$ on U_k; (vi) W_k is a domain in Δ whose boundary is a simple closed rectifiable Jordan curve such that $(\partial W_k) \cap (\partial\Delta) = E_k$; and (vii) $\overline{W}_k$ is disjoint from $\overline{U}_j$ for all k and j. To see that this can be done, assume that $U_1, \ldots, U_{k-1}$ and $W_1, \ldots, W_{k-1}$ have been chosen. Let $0 < \varepsilon < 1/k$, and let V_k be the intersection of the ε-neighborhood of F_k with the set of points in $\Delta\backslash K_{k-1}$ at which $u_k > 1/2$. For $\varepsilon > 0$ small enough, $\overline{V}_k$ is disjoint from $\overline{U}_j$ and from $\overline{W}_j$ for $1 \le j < k$. For $i \ge 1$, let L_i be the intersection of V_k with the circle $\{|z| = 1 - 1/2^i\}$. Fix $c > 0$ small, say $c = 1/100$, and choose points $z_{i1}, z_{i2}, \ldots$ on L_i so that $\Delta(z_{ij}; c/2^i) \subset V_k$, such that the discs $\Delta(z_{ij}; c/2^i)$ have pairwise disjoint closures, and such that any subarc of L_i of length greater than $c/(8 \cdot 2^i)$ includes one of the points z_{ij}. Let U_k be the union of the discs $\Delta(z_{ij}; c/2^i)$ for $j \ge 1$ and $1 \le i < \infty$. Let $\zeta \in F_k$ be a point of full linear density of F_k. Then $u_k(z)$ tends to 1 through any cone in Δ with vertex at ζ, so that all points in any such curve and sufficiently close to ζ lie eventually in V_k. If the aperture of the cone is chosen to be sufficiently large, say $\pi/2$, then the construction is devised so that there is a sequence $w_i = z_{ij(i)}$, $i \ge i_0$, such that $|w_i - \zeta| \le 2/2^i$. Fix $\delta > 0$ small. Choose the smallest integer i such that $4/2^i < \delta$. For δ sufficiently small, $i \ge i_0$. Note also that $\delta < 8/2^i$, and furthermore that $\Delta(\zeta; \delta)$ includes $\Delta(w_i; c/2^i)$. Since the analytic capacity of a disc is equal to its radius,

$$\gamma(\Delta(\zeta; \delta) \cap U_k) \ge \gamma\big(\Delta(w_i; c/2^i)\big) = c/2^i \ge c\delta/8.$$

Hence

$$\liminf_{\delta \to 0} \frac{\gamma(\Delta(\zeta; \delta) \cap U_k)}{\delta} \ge \frac{c}{8}.$$

Since this is valid for almost-all points of F_k, Øksendal's theorem shows that F_k is a null set for all measures in $R(\overline{\Delta}\backslash U_k)^\perp$, and hence for all measures in $R(K_k)^\perp$. Thus U_k has all the desired properties, and since E_k is disjoint from $F_1 \cup \cdots \cup F_k$, it is easy to construct W_k with the properties asserted above.

Now let $K = \lim K_k = \overline{\Delta}\backslash(\bigcup U_k)$, which is a champagne bubble set. Since F_k is a null set for $R(K_k)^\perp$, F_k is also a null set for $R(K)^\perp$, and hence F is a null set for $R(K)^\perp$. Let λ_k be the measure dz on ∂W_k. Since $\overline{W}_k \subseteq K$, and $W_k \subset K^\circ$, λ_k is orhtogonal to $R(K)$. Furthermore, the restriction of λ_k to E_k is comparable to arc length on E_k. It follows that any null set of $R(K)^\perp$ lying inside $\bigcup E_k$ has zero length. Hence any dominant representing measure for a point of K° is equivalent to the arc length measure on $(\bigcup \partial\Delta_j) \cup E$. To complete the proof, it suffices to show that points of K° have compact sets of representing measures on ∂K.

Since $F_1 \cup \cdots \cup F_k$ is a null-set for measures orthogonal to $R(K_k)$, Theorem 1 of [3] shows that points of K_k° have norm compact sets of representing measures on ∂K_k. It suffices to show that representing measures on ∂K are uniformly limits in norm of representing measures on ∂K_k.

Assume for convenience that $0 \in K^\circ$, and let η be a representing measure on ∂K for 0. For each k, let η_k be the sweep of η to ∂K_k, via harmonic measure. Suppose $k \geq 2$. Then η_{k-1} is obtained by sweeping η_k to ∂K_{k-1}. Now ∂K_k is the disjoint union of ∂K_{k-1} and ∂U_k. Since the sweep of η_{k-1} to $\partial\Delta$ with respect to Δ is $d\theta/2\pi$, $\eta_{k-1} \leq d\theta/2\pi$ on F_k, and consequently the mass of η_k swept from ∂U_k to F_k does not exceed $2^{-k}/2\pi \leq 2^{-k-2}$. Since $u_k > 1/2$ on U_k, at least half the mass of η_k on ∂U_k is swept to F_k, and consequently the total mass of η_k on ∂U_k does not exceed 2^{-k-1}. Hence

$$\|\eta_k - \eta_{k-1}\| \leq 2^{-k-1}, \qquad k \geq 2.$$

This estimate shows that $\{\eta_k\}$ converges in norm, evidently to η, and moreover

$$\|\eta - \eta_k\| \leq 2^{-k}.$$

Thus the set of representing measures for 0 on ∂K is the limit of the set of representing measures for 0 on ∂K_k, with respect to $R(K_k)$. Since the limit of compact sets is compact, points of K° have compact sets of representing measures. □

By utilizing the technique in the example preceding Theorem 7, one may pluck out a further sparse sequence of discs so that the Δ_j's in Theorem 7 accumulate on all of $\partial\Delta$.

On the basis of Theorem 1, it is possible to conclude that several standard examples of champagne bubble sets fail to have norm compact sets of representing measures. In one example, given in [**6**, p. 102], one chooses the Δ_j's with very small radii and with centers forming a dominating sequence for $H^\infty(\Delta)$. In this case, the restriction of harmonic measure to $\partial\Delta$ is comparable to arc length, yet there are representing measures for points of K° with no mass on $\partial\Delta$. Theorem 1 shows that the set of representing measures associated with such a set is not weakly compact.

The second example, pointed out by Fisher and mentioned in [**6**, §7], depends on a function constructed by A. Beurling and used by R.

McKissick [12] as the principal ingredient of his celebrated Swiss cheese. In Beurling's construction, the Δ_j's are chosen so that their radii are summable, but so that there exists a nonzero function f in $R(K)$ that vanishes identically on $\partial\Delta$. This property of f guarantees that harmonic measure for a point of K° has no mass on $\partial\Delta$. On the other hand, the measure dz on ∂K is a finite measure orthogonal to $R(K)$ which has mass on $\partial\Delta$. Again Theorem 1 shows that the sets of representing measures for points of K° are not weakly compact.

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