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TYPESETS AND COTYPESETS OF RANK-2 TORSION FREE ABELIAN GROUPS

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A sufficient condition is given for a set of types to be the typeset of a rank-2 group, strengthening all previous results on this subject. A correct version of a theorem of Schultz on types and cotypes is provided, along with a variety of other results on typesets and cotypesets of rank-2 groups. Numerous examples are included.

Beaumont-Pierce [**3**], in 1961, posed the problem of finding necessary and sufficient conditions for a set of types to be the typeset of a rank-2 torsion free abelian group. They also, among other things, solved the problem in case the given set of types is finite. Koehler [**10**], in 1964, extended some of these results. Dubois [**5**] and [**6**], 1965 and 1966, used basic analytic number theory techniques to give some necessary and some sufficient conditions for a set of types to be realized as the typeset of a rank-2 torsion free abelian group.

Ito [**9**], in 1975, gave a sufficient condition for the realization of a set of types, which is easily seen to be equivalent to a sufficient condition of Dubois [**5**], Theorem 1. Ito's construction, however, is somewhat easier to understand, the group being given as a homomorphic image of a completely decomposable group rather than as a subgroup of the direct sum of two copies of the Z-adic integers.

Schultz [**11**], in 1978, claimed to have given necessary and sufficient conditions on two sets of types S_1 and S_2 such that there is a rank-2 group A with typeset$(A) = S_1$ and cotypeset$(A) = S_2$. A counterexample to this theorem is given by Vinsonhaler-Wickless [**12**] (also see Example 1.6). Vinsonhaler-Wickless [**12**] also give some simple necessary and sufficient conditions for a set of types to be realized as the cotypeset of a rank-2 group.

The theme of this paper is to examine Dubois' results from the point of view of Ito's construction. This point of view leads to:

THEOREM. *Let* $S = \{\tau_1, \tau_2, \ldots\}$ *be a set of types with* $\inf(\tau_i, \tau_j) = \mathrm{type}(Z)$ *whenever* $\tau_i \neq \tau_j$. *Assume that* $\{\tau_i \mid \tau_i$ *is very large*$\}$ *has no snarls in* S.

(a) *There is a rank-2 group* A *with* typeset$(A) = S$ *iff either* $\mathrm{type}(Z) \in S$ *or else* S *has an infinite subset with no snarls in* S;

(b) *Let* $S' = \{\sigma_1, \sigma_2, \ldots\}$ *be another set of types. Then there is a rank-2 group A with* typeset$(A) = S$ *and* cotypeset$(A) = S'$ *if and only if*

(i) *There is a type* $\sigma_0 = \sup\{\sigma_i, \sigma_j\}$ *for* $i \neq j$;

(ii) $\tau_i \leq \sigma_0$ *for each i*;

(iii) $\sigma_i = \sigma_0 - \tau_i$ *for each i; and*

(iv) *Either* type$(Z) \in S$ *or else S has an infinite subset with no snarls in S.*

Included as special cases are Dubois' theorems (Theorem 2.9(b) and Corollary 2.10); Koehler's results (Theorem 2.9(a)); Ito's theorem (Corollary 2.14(b) which includes the Beaumont-Pierce results), and a corrected version of Schultz's assertions (Corollary 2.11). Also included is a simplification of the proof of the Vinsonhaler-Wickless theorem (Theorem 3.1).

Of particular interest are the locally completely decomposable groups, discussed in §4. If A is a finite rank torsion free group then there are locally completely decomposable groups B and C with $B \subseteq A \subseteq C$; C/B torsion; typeset(A) = typeset(B), and cotypeset(A) = cotypeset(C) (Theorem 4.1).

Section 5 is devoted to some open questions.

This paper is largely self-contained, and as a result partially expository, due to the complexity and the history of the problems considered. However, references for published results are given as well as numerous examples.

0. Notation and preliminaries. The basic properties of finite rank torsion free groups may be found in Fuchs [7]. Special notation used herein includes: if h is a height sequence (characteristic) then $h(p)$ is the pth entry for a prime p; $\tau = [h]$ denotes the type of h, an equivalence class of height sequences; and write $h \in \tau$. If A is torsion free and $a \in A$ then $h^A(a)$, $h_p^A(a)$, and type$_A(a)$ denote the height sequence, p-height, and type of a in A, respectively. If rank $A = 1$ then type(A) = type$_A(a)$ for $0 \neq a \in A$.

If $\tau = [t]$ and $\sigma = [s]$ then $\inf\{\tau, \sigma\}$ and $\sup\{\tau, \sigma\}$ are defined by $[\min\{t, s\}]$ and $[\max\{t, s\}]$, respectively. For two types τ and σ, $\tau \leq \sigma$ iff there is $t \in \tau$ and $s \in \sigma$ with $t \leq s$. In this case $\sigma - \tau = [s - t]$, agreeing that $\infty - \infty = 0$.

If S is a subset of A then $\langle S \rangle$ denotes the subgroup of A generated by S and $\langle S \rangle_*$ denotes the pure subgroup of A generated by S.

Also assumed are some basic analytic number theory results as found, for example, in Hardy-Wright [8]. For a positive integer n, let $\pi(n)$ be the number of primes $\leq n$. Let $U = \{(r, s) \mid r, s \in Z,\ r \geq 0,\ \gcd(r, s) = 1\}$ and $I_n = \{(r, s) \in U \mid \max\{r, |s|\} \leq n\}$. Denote the ith prime by p_i.

LEMMA 0.1.

(a) (*Chebyshev's Theorem.*) *There is* $c_1 > 0$ *and* $c_2 > 0$ *such that* $c_1 n/\log n < \pi(n) < c_2 n/\log n$.

(b) $\lim_{n\to\infty} n/p_n = 0$.

(c) $|I_n| = 4\Phi(n)$ *where* $\Phi(n) = \Sigma\{\phi(i) \mid 1 \leq i \leq n\}$ *and* $\phi(i)$ *is the Euler* ϕ-*function.*

(d) $\phi(n) = 3n^2/\pi^2 + O(n \log n)$

(e) $\lim_{n\to\infty}(|I_n| - \pi(2n^2)) = \infty$.

Note that (b) follows from (a) since $n = \pi(p_n) < c_2 p_n/\log p_n$ implies that $n/p_n < c_2/\log p_n$. Also (c) is a fairly routine counting argument, while (e) follows from (d) and (a).

As a consequence of (b), if c is any constant then for sufficiently large n, $p_n > cn$.

1. Type sequences. A *type sequence* is a countably infinite sequence of types (repetition of types is permitted). Two type sequences T and T' are *equivalent*, $T \approx T'$, if one is a permutation of the other.

Let A be a rank-2 group and let $A_1, A_2, \ldots$ be an indexing of the pure rank-1 subgroups of A. Define $\tau_i = \text{typeset}(A_i)$ and $\sigma_i = \text{type}(A/A_i)$ for each i. Then $T_A = (\tau_1, \tau_2, \ldots)$ and $C_A = (\sigma_1, \sigma_2, \ldots)$ are type sequences. Note that T_A and C_A are unique up to equivalence. Define $\text{typeset}(A) = \{\tau_i \mid i \geq 1\}$ and $\text{cotypeset}(A) = \{\sigma_i \mid i \geq 1\}$.

The following proposition is folklore.

PROPOSITION 1.1. *Let A be a rank-2 group with* $T_A = (\tau_1, \tau_2, \ldots)$ *and* $C_A = (\sigma_1, \sigma_2, \ldots)$.

(a) *There is a type* τ_0 *such that* $\tau_0 = \inf\{\tau_i, \tau_j\}$ *for each* $i \neq j$ *and if* typeset(A) *is finite then* $\tau_0 = \tau_i$ *for some* $i \geq 1$.

(b) *There is a type* σ_0 *such that* $\sigma_0 = \sup\{\sigma_i, \sigma_j\}$ *for each* $i \neq j$ *and if* cotypeset(A) *is finite then* $\sigma_0 = \sigma_i$ *for some* $i \geq 1$.

(c) $\tau_i \leq \sigma_j$ *for each* $i \neq j$ *and* $\tau_0 \leq \sigma_0$.

(d) $\sigma_i - \tau_j = \sigma_j - \tau_i$ *for each* $i \neq j$ *with* $i \geq 0$ *and* $j \geq 0$.

(e) *If* $\tau_0 = \text{type}(Z)$ *then* $\sigma_i = \sigma_0 - \tau_i$ *for each* i.

Proof. (a) If $i \neq j$ then $A/(A_i \oplus A_j)$ is torsion. Thus, for each k, $\tau_k \geq \inf\{\tau_i, \tau_j\}$. Consequently, if $k \neq l$ then $\inf\{\tau_k, \tau_l\} \geq \inf\{\tau_i, \tau_j\} \geq \inf\{\tau_k, \tau_l\}$. Now assume that $\text{typeset}(A) = \{\tau_1, \tau_2, \ldots, \tau_n\}$. If $j > n$ then

$\tau_0 = \inf\{\tau_j, \tau_i\}$ for each $1 \leq i \leq n$. But $\tau_j = \tau_i$ for some $1 \leq i \leq n$ so that $\tau_0 = \tau_i \in \text{typeset}(A)$.

The proof of (b) is similar.

(c) follows from the fact that if $i \neq j$ then there is a monomorphism $A_i \to A/A_j$.

(d) First assume that $i \neq j$ are both non-zero. There is an exact sequence $0 \to A_i \to A/A_j \to A/(A_i \oplus A_j) \to 0$. Choose $0 \neq a_i \in A_i$ and $x_j \in A/A_j$ with $a_i \to x_j$. Then

$$0 \to A_i/Za_i \to \left(A/A_j\right)/Zx_j \to A/\left(A_i \oplus A_j\right) \to 0$$

is exact. Write $A/(A_i \oplus A_j) = \oplus_p Z(p^{k_p})$ so that

$$k_p = \left(p\text{-height of } x_j \text{ in } A/A_j\right) - \left(p\text{-height of } a_i \text{ in } A_i\right).$$

Then $[(k_p)] = \sigma_j - \tau_i$. Similarly, $[(k_p)] = \sigma_i - \tau_j$.

Given three distinct positive integers i, j, k,

$$\sigma_0 - \tau_j = \sup\{\sigma_i - \tau_j, \sigma_k - \tau_j\} = \sup\{\sigma_j - \tau_i, \sigma_j - \tau_k\} = \sigma_j - \tau_0.$$

(e) follows from (d).

Following Warfield [**14**], τ_0 is called the *inner type of A*, denoted by $\text{IT}(A)$, and σ_0 is called the *outer type of A*, denoted by $\text{OT}(A)$.

LEMMA 1.2. *Assume that $Zx \oplus Zy \subseteq A$, a rank-2 torsion free group and that $T_A = (\tau_1, \tau_2, \ldots)$. Define*

$$U_A = \{rx + sy \mid r, s \in Z, r \geq 0, \gcd(r, s) = 1\}.$$

(a) *For each $i \geq 1$ there is a unique $a_i \in U_A \cap A_i$. Moreover, $A_i \cap (Zx \oplus Zy) = Za_i$ and* $\text{type}_A(a_i) = \tau_i$.

(b) $\text{OT}(A) = [\max\{h^A(a) \mid a \in U_A\}]$.

Proof. (a) is routine.

(b) Write $A/(Zx \oplus Zy) = \oplus_p[Z(p^{i_p}) \oplus Z(p^{j_p})]$ with $0 \leq i_p \leq j_p \leq \infty$ for each p. Then $\text{IT}(A) = [(i_p)]$ and $\text{OT}(A) = [(j_p)]$ (Warfield [**14**] or Arnold [**2**]). If $a + (Zx \oplus Zy) \in (A/(Zx \oplus Zy))_p$ then order $(a + (Zx \oplus Zy)) =$ least j with $p^j a = mu$ for some $u \in U_A$ and some $m \in Z$ with $\gcd(p, m) = 1$. Since $j \leq h_p^A(u)$, it follows that $j_p \leq \max\{h_p^A(a) \mid a \in U\}$. But

$$A/\left(Zx \oplus Zy\right) \supseteq \left(A_i + \left(Zx \oplus Zy\right)\right)/\left(Zx \oplus Zy\right) \simeq A_i/Za_i$$
$$= \bigoplus_p Z(p^{l_p}),$$

where $l_p = h_p^A(a_i)$ (by (a)) so that $\max\{h_p^A(a) \mid a \in U_A\} \leq j_p$. Thus, $\mathrm{OT}(A) = [(j_p)] = [\max\{h^A(a) \mid a \in U_A\}]$.

The following lemma reduces the problem of realizing a type sequence $T = (\tau_1, \tau_2, \ldots)$ to the case that type $(Z) = \inf\{\tau_i, \tau_j\}$ whenever $i \neq j$.

LEMMA 1.3 (*Schultz* [**11**]). *Let* $T = (\tau_1, \tau_2, \ldots)$ *and* $C = (\sigma_1, \sigma_2, \ldots)$ *be type sequences with* $\tau_0 = \inf\{\tau_i, \tau_j\}$ *and* $\sigma_0 = \sup\{\sigma_i, \sigma_j\}$ *whenever* $i \neq j$. *There is a rank-*2 *group* A *with* $T_A = T$ *and* $C_A = C$ *iff there is a rank-*2 *group* B *with* $T_B = (\tau_1 - \tau_0, \tau_2 - \tau_0, \ldots)$, $C_B = (\sigma_1 - \tau_0, \sigma_2 - \tau_0, \ldots)$, $\mathrm{IT}(B) = \mathrm{type}(Z)$, *and* $\mathrm{OT}(B) = \sigma_0 - \tau_0$.

Proof. ($\Leftarrow$) Let X be a rank-1 group with type $(X) = \tau_0$ and define $A = X \otimes_Z B$. Then Y is a pure rank-1 subgroup of A iff $Y \simeq X \otimes_Z D$ for some pure rank-1 subgroup D of B. Moreover, $A/Y \simeq X \otimes_Z (B/D)$; $\mathrm{type}(Y) = \tau_0 + \mathrm{type}(D)$; and $\mathrm{type}(A/Y) = \tau_0 + \mathrm{type}(B/D)$. Thus, $T_A = T$ and $C_A = C$.

($\Rightarrow$) Choose $h_i \in \tau_i$ for $i \geq 0$ such that $h_0 \leq h_i$ for each i. First of all, it suffices to assume that τ_0 is idempotent: Let X be a rank-1 group with $\mathrm{type}(X) = \tau_0$ and $x \in X$ with $h^X(x) = h_0$. Define $A' = \mathrm{Hom}(X, A)$. Then $\phi\colon A' \to \{a \in A \mid h^A(a) \geq h_0\}$ is an isomorphism, where $\phi(f) = f(x)$. Thus $T_{A'} = (\tau_1 - \tau_0', \tau_2 - \tau_0', \ldots)$, $C_{A'} = (\sigma_1 - \tau_0', \sigma_2 - \tau_0', \ldots)$, and $\mathrm{IT}(A') = \tau_0 - \tau_0'$ where $\tau_0' = [h_0']$, $h_0'(p) = 0$ if $h_0(p) = \infty$, and $h_0(p) = h_0'(p)$ if $h_0(p) < \infty$. Therefore, $\tau_0 - \tau_0'$ is idempotent.

Now assume that τ_0 is idempotent, say $\tau_0 = [h_0]$ with $h_0(p) = 0$ or ∞ for each p. Let F be a free subgroup of A with A/F torsion and define $B = A \cap (\bigcap\{F_p \mid h_0(p) = \infty\})$. Then $B_p = F_p$ if $h_0(p) = \infty$ and $B_p = A_p$ if $h_0(p) = 0$. Let R be the subring of Q generated by $\{1/p \mid h_0(p) = \infty\}$ and define $\theta\colon R \otimes_Z B \to A$ by $\theta(r \otimes b) = rb$, noting that $RA = A$. Then θ is an epimorphism, hence an isomorphism, since $\mathrm{rank}(R \otimes_Z B) = \mathrm{rank}(A) = 2$. Finally, if A_i is a pure rank-1 subgroup of A let

$$B_i = A_i \cap B = A_i \cap \left(\bigcap \{F_p \mid h_0(p) = \infty\}\right).$$

Then B_i is a pure rank-1 subgroup of B and $\mathrm{type}_B(B_i) = \tau_i - \tau_0$, since $(B_i)_p = (A_i)_p$ if $h_0(p) = 0$ and $\bigcap\{(B_i)_p \mid h_0(p) = \infty\}$ is pure in $\bigcap\{F_p \mid h_0(p) = \infty\}$. Consequently, $T_B = (\tau_1 - \tau_0, \tau_2 - \tau_0, \ldots)$, $C_B = (\sigma_1 - \tau_0, \sigma_2 - \tau_0, \ldots)$, and $\mathrm{IT}(B) = \mathrm{type}(Z)$, as desired.

Let $T = (\tau_1, \tau_2, \ldots)$ be a type sequence with $\mathrm{type}(Z) = \inf\{\tau_i, \tau_j\}$ whenever $i \neq j$, let $h_i \in \tau_i$ for each i, and let τ be a type with $h \in \tau$. Then

τ is a *snarl of* T if $\{p \mid 0 < h(p) < h_j(p) = \infty$ for some $j\}$ is infinite. Note that this definition depends only upon τ and T and not upon the choice of $h \in \tau$ and $h_i \in \tau_i$. Snarls of sets of types are defined analogously.

Suppose that A is a rank-2 group with $T_A = (\tau_1, \tau_2, \ldots)$. By Proposition 1.1.a, $\mathrm{IT}(A)$ is the only type in T_A that may be repeated. Following Beaumont-Pierce [3], A is *completely anisotropic* if $\tau_i \neq \tau_j$ for each $i \neq j$. In this case, $\mathrm{IT}(A)$ appears at most one time in T_A.

THEOREM 1.4 (Dubois [1]). *Let A be a rank-2 group with $T_A = (\tau_1, \tau_2, \ldots)$ and $\mathrm{IT}(A) = \mathrm{type}(Z)$. Then T_A has an infinite subsequence T' such that no snarls of T' are in T_A.*

Proof. Choose $Zx \oplus Zy \subseteq A$ with $\inf\{h^A(x), h^A(y)\} = 0$. There is an indexing $u_1, u_2, \ldots$ of U_A such that $u_1 = x$, $u_2 = y$, $u_i = r_i x + s_i y$ and $\max\{r_i, |s_i|\} \leq \max\{r_j, |s_j|\}$ if $i < j$. Relabel T_A so that $\tau_i = \mathrm{type}_A(u_i)$. Let $h_i = h^A(u_i) \in \tau_i$ for each $i \geq 1$.

Define $K = \{j \mid$ for each p, $h_j(p) < \infty$ or $h_j(p) = \infty$ and there is no $i < j$ with $0 < h_i(p) < h_j(p) = \infty\}$. Let T' be the subsequence of T_A determined by K. Then for each i, τ_i is not a snarl of T' since

$$\left\{p \mid 0 < h_i(p) < h_j(p) = \infty, \text{ for some } j \in K\right\} \subseteq \left\{p \mid 0 < h_i(p) < h_j(p) = \infty, i > j \in K\right\}$$

is finite (recalling that $\inf\{h_i(p), h_j(p)\} = 0$ for almost all p, since $\mathrm{IT}(A) = \mathrm{type}(Z)$, and that there are only finitely many $j < i$).

It now suffices to prove that K is infinite. Let $I_n = \{i \mid \max\{r_i, |s_i|\} \leq n\}$. If $j \in I_n \setminus K$ then there is some p and some $i < j$ with $0 < h_i(p) < h_j(p) = \infty$ and $\max\{r_j, |s_j|\} \leq n$. Now $r_i u_j = r_j u_i - (r_j s_i - r_i s_j) y$ and $s_i u_j = s_j u_i + (r_j s_i - r_i s_j) x$. Since $\inf\{h_p^A(x), h_p^A(y)\} = 0$, p divides $r_j s_i - r_i s_j$. Furthermore, $|r_j s_i - r_i s_j| \leq 2n^2$. Thus, $|I_n \setminus K| \leq \pi(2n^2)$, the number of primes $\leq 2n^2$, since for each p there is at most one j with $h_j(p) = \infty$. It follows that

$$|I_n \cap K| = |I_n| - |I_n \setminus K| \geq |I_n| - \pi(2n^2).$$

Now apply Lemma 0.1.e to see that K is infinite.

EXAMPLE 1.5. Let $T = (\tau_1, \tau_2, \ldots)$ be given by $\tau_i = [h_i]$ where $h_1 = (1, 1, 1, \ldots)$; $h_2 = (\infty, 0, 0, \ldots)$; $h_3 = (0, \infty, 0, \ldots)$, $h_4 = (0, 0, \infty, \ldots), \ldots$.

(a) There is no rank-2 group A with $T_A = T$.

(b) There is no rank-2 group A with $\mathrm{typeset}(A) = \{\tau_i \mid i \geq 1\}$.

(c) There is no rank-2 completely anisotropic group A with $\mathrm{typeset}(A) = \{\tau_i \mid i \geq 0\}$ where $\tau_0 = \mathrm{type}(Z) = \inf\{\tau_i, \tau_j\}$ for $i \neq j$.

Proof. (a) Note that τ_1 is a snarl of every infinite subsequence of T and apply Theorem 1.4.

(b) If there is a rank-2 group with typeset$(A) = \{\tau_i \mid i \geq 1\}$ then $\mathrm{IT}(A) = \mathrm{type}(Z) \notin \mathrm{typeset}(A)$. Thus $T_A = (\tau_1, \tau_2, \ldots)$ since IT(A) is the only type that can be repeated and IT(A) does not appear in T_A. But this contradicts (a).

(c) For such an A, $T_A = (\tau_0, \tau_1, \tau_2, \ldots)$ since A is assumed to be completely anisotropic. Once again, τ_1 is a snarl of every infinite subsequence of T_A contradicting Theorem 1.4.

EXAMPLE 1.6. Let $S_1 = \{\tau_i \mid i \geq 1\}$ be as defined in Example 1.5, $\sigma_0 = \mathrm{type}(Q)$, and $S_2 = \{\sigma_0 - \tau_i \mid i \geq 1\}$. Then there is no rank-2 group A with typeset$(A) = S_1$ and cotypeset$(A) = S_2$ by Example 1.5. On the other hand, S_1 and S_2 satisfy the hypotheses of Theorem 1, Schultz [**11**]. Thus Schultz's main theorem is incorrect as stated.

2. Realization of type sequences and typesets. In this section the following notation is consistently employed: $T = (\tau_1, \tau_2, \ldots)$ is a type sequence with $\inf\{\tau_i, \tau_j\} = \mathrm{type}(Z)$ if $i \neq j$; $h_i \in \tau_i$ for all i; $a_i = r_i x + s_i y$ is a denumeration of $U = \{rx + sy \in Zx \oplus Zy \mid r, s \in Z,\ r \geq 0$, and $\gcd(r, s) = 1\}$; $\det(i, j) = r_i s_j - s_i r_j$; and $\det_p(i, j) = h_p^Z(r_i s_j - s_i r_j)$.

The type sequence T is *admissible* if there is an indexing of U so that for each i there is an $N > 0$ such that $h_n(p) = \infty$ for some $n > N$ implies $\det_p(i, n) = h_i(p)$.

Since for each p there is at most one n with $h_n(p) = \infty$, the admissibility of T does not depend on the choice of $h_i \in \tau_i$ or the ordering of T.

LEMMA 2.1. *Given $h_i' \in \tau_i$ for each i, there exists $h_i \in \tau_i$ for each i such that $h_i \leq h_i'$ and* (a) *If $j < k$ and $h_k(p) < \infty$ then $\min\{h_j(p), h_k(p)\} = 0$; and* (b) *If $h_k(p) < \infty$ and $\det_p(i, k) > 0$ for some $i < k$ then $h_k(p) = 0$.*

Proof. Assume $h_1, \ldots, h_{n-1}$ have been chosen such that $h_i \in \tau_i$, $h_i \leq h_i'$ and (a) and (b) are satisfied for $i, j, k < n$. Define $h_n(p) = 0$ if $h_n'(p) < \infty$ and either $0 < h_i(p)$ for some $i < n$ or $\det_p(i, n) > 0$ for some $i < n$; and define $h_n(p) = h_n'(p)$ otherwise.

Note that there are only finitely many $i < n$ and only a finite number of primes can divide $\det(i, n)$ if $i \neq n$. Furthermore, $h_n'(p) > 0$ and $h_i(p) > 0$ for some $i < n$ can happen for only finitely many p since $\inf\{\tau_n, \tau_i\} = \mathrm{type}(Z)$. Thus $h_n(p) = h_n'(p)$ in case $h_n'(p) = \infty$ and for almost all p, and $h_n(p) \leq h_n'(p)$. Therefore $h_n \in \tau_n$ and $h_1, \ldots, h_n$ satisfy (1) and (2). The proof is now complete by induction on n.

LEMMA 2.2. *Suppose that* $h_i \in \tau_i$ *for each* i. *Define* A *to be the subgroup of* $Qx \oplus Qy$ *generated by* $\{a_i/p^j \mid p$ *is a prime*, $0 \leq j \leq h_i(p)$, $i = 1, 2, \ldots\}$. *Then* $\mathrm{OT}(A) = [h]$ *where* $h = \max\{h_i \mid i \geq 1\}$.

Proof. By Proposition 1.1(a),

$$\mathrm{OT}(A) = \sup\{\mathrm{type}(A/A_1), \mathrm{type}(A/A_2)\},$$

where A_1 and A_2 are the pure rank-1 subgroups of A generated by x and y, respectively. Since

$$\mathrm{type}(A/A_1) = \mathrm{type}\langle s_i/p^j \in Q \mid p \text{ is a prime}, 0 \leq j \leq h_i(p), i = 1, 2, \ldots\rangle,$$

$$\mathrm{type}(A/A_2) = \mathrm{type}\langle r_i/p^j \in Q \mid p \text{ is a prime}, 0 \leq j \leq h_i(p)\rangle,$$

and $\gcd(r_i, s_i) = 1$ for each i, the result follows.

LEMMA 2.3. *Let* p *be a prime, and* $Zx \oplus Zy \subseteq A \subseteq Qx \oplus Qy$ *with* $h_p^A(a_i) = 0$ *for each* i, $B = \langle A \cup \{a_k/p^j \mid j \leq e\}\rangle$ *for some* $k > 0$, *and* $0 < e \leq \infty$. *Then for each* i, $h_p^B(a_i) = \min\{e, \det_p(i, k)\}$.

Proof. Fix $i > 0$ and assume that $\gcd(p, s_k) = 1$. We first show that $h_p^B(x) = 0$. Suppose that $x/p = a + ca_k/p^l$ for some $a \in A$, $c \in Z$, $0 \leq l \leq e$, $l < \infty$. Then $x = pa + ca_k/p^{l-1}$ so $ca_k/p^{l-1} \in A$. Thus we may assume that $c/p^{l-1} = c' \in Z$ with $\gcd(c', p) = 1$. By Lemma 1.2, $a = (m/n)a_i$ for some $i \neq k$, $\gcd(m, n) = 1$. Equating coefficients of y gives $0 = (pm/n)s_i + c's_k$. Since $\gcd(c's_k, p) = 1$, p divides n. This contradicts $h_p^A(a_i) = 0$.

The lemma now follows, in this case, from the equation $s_k a_i - s_i a_k = \det(i, k)x$. Indeed, since $h_p^B(x) = 0$ and $\gcd(s_k, p) = 1$,

$$h_p^B(a_i) \geq \min\{e, \det_p(i, k)\}.$$

On the other hand, if p^l divides a_i in B, then $l \leq e$ by the construction of B, so p^l divides a_k in B. Since $h_p^B(x) = 0$, the equation implies p^l divides $\det(i, k)$. Hence, $h_p^B(a_i) \leq \min\{e, \det_p(i, k)\}$ as desired.

A similar argument shows that if $\gcd(p, r_k) = 1$, then $h_p^B(y) = 0$ and again $h_p^B(a_i) = \min\{e, \det_p(i, k)\}$. Since $\gcd(r_k, s_k) = 1$, the proof is complete.

LEMMA 2.4. *Let* p *be a prime*, $Zx \oplus Zy \subseteq A \subseteq Qx \oplus Qy$ *with* $h_p^A(a_i) = 0$ *for each* i, α *an irrational* p*-adic integer and* $0 < t \leq \infty$. *Define*

$$B = \left\langle A \cup \left\{a_i/p^j \mid p^j \text{ divides } r_i - \alpha s_i \text{ and } j \leq t, 1 \leq i < \infty\right\}\right\rangle.$$

Then

$$h_p^B(a_i) = \max\{j|p^j \text{ divides } r_i - \alpha s_i \text{ and } j \leq t\} < \infty$$

for each i.

Proof. Given i, let $m = \max\{j|p^j$ divides $r_i - \alpha s_i$ and $j \leq t\}$. Note that if p divides $r_i - \alpha s_i$ then $\gcd(s_i, p) = 1$ since $\gcd(s_i, r_i) = 1$. Clearly, $h_p^B(a_i) \geq m$ and $m < \infty$. It therefore suffices to show $h_p^B(a_i)$ is not greater than m. Suppose $(1/p^{m+1})a_i \in B$. Then $1/p^{m+1}a_i = a + \sum_{k=1}^{l} c_k a_k/p^{e(k)}$, where $a \in A$, $c_k \in Z$, $p^{e(k)}$ divides $r_k - \alpha s_k$ and $e(k) \leq t$. Choose $1 \leq j \leq l$ so that $e(j)$ is maximal among the $e(k)$. Note that

$$s_j a_k = s_k a_j + s_k(r_j - \alpha s_j)x + s_j(\alpha s_k - r_k)x.$$

Since $e(j)$ is maximal and $\gcd(s_j, p) = 1$, this equation implies that each $(c_k/p^{e(k)})a_k$ may be replaced by an expression of the form $b_k + (c_k'/p^{e(j)})a_j$ where $b_k \in A$ and $c_k' \in Z$. Thus we may write $(1/p^{m+1})a_i = a + (c/p^e)a_j$ where $a \in A$, $c \in Z$, p^e divides $r_j - \alpha s_j$ and $e \leq t$. This shows that $(1/p^{m+1})a_i$ is in fact an element of the group $A' = A \cup \{a_j/p^i \,|\, i \leq e\}$. By Lemma 2.3, $m + 1 \leq \min\{e, \det_p(i, j)\}$. In particular, $\det_p(i, j) \geq m + 1$. Therefore, p^{m+1} divides $(r_i s_j - s_i r_j)$. Since $e \geq m + 1$, p^{m+1} divides $(r_j - \alpha s_j)s_i$; thus p^{m+1} divides $r_i s_j - \alpha s_j s_i$ and p divides s_j. However p divides $r_j - \alpha s_j$, so p divides r_j contradicting $\gcd(r_j, s_j) = 1$. Thus $h_p^B(a_i)$ is not greater than m.

The next theorem is stated in Dubois [**6**].

THEOREM 2.5. *There is a rank-2 group A with $T_A \approx T$ if and only if T is admissible.*

Proof. ($\Rightarrow$) Assume that $Zx \oplus Zy \subseteq A \subseteq Qx \oplus Qy$ and let $a_1 = x$, $a_2 = y$, $a_3, \ldots$ be an indexing of U such that $\text{type}_A(a_i) = \tau_i$ for each i. Define $h_i = h^A(a_i)$ for each i, let

$$N = \max\{j|h_j(p) = \infty, \inf\{h_1(p), h_2(p)\} > 0\},$$

and let $N = 1$ if no such j exists. Given i and $n > N$ with $h_n(p) = \infty$, then $h_i(p) = \det_p(n, i)$ since $s_i a_n - s_n a_i = \det(n, i)x$ and $r_n a_i - r_i a_n = \det(n, i)y$. Note that for this choice of $h_i \in \tau_i$, N does not depend on i.

($\Leftarrow$) If T is admissible, choose $h_i \in \tau_i$ satisfying (a) and (b) of Lemma 2.1. Next define $h_i' \leq h_i$ by setting $h_i'(p) = 0$ if $h_k(p) = \infty$ for some $k \neq i$ and $h_i'(p) = h_i(p)$ otherwise. Note that h_i' need not be in τ_i. Given p, this implies (along with (a)) that $h'_i(p) > 0$ for at most one i. Let

$$A(p) = \left\langle (Zx \oplus Zy) \cup \left\{ a_i/p^j \middle| 0 \le j \le e(i) \right\} \right\rangle$$

if $e(i) = h'_i(p) > 0$ for some i, and let $A(p) = Zx \oplus Zy$ otherwise. Define $A = \Sigma_p A(p)$. Note that $h_p^A(a_i) = h_p^{A(p)}(a_i)$ for all i and p, so we can apply Lemma 2.3 to show $\text{type}_A(a_i) = \tau_i$ for all i as follows:

Let $P_1 = \{p \mid h'_i(p) = 0 \text{ for each } i\}$. Then for each $p \in P_1$, $h_p^A(a_i) = 0 = h_i(p)$ for each i.

Let $P_2 = \{p \mid h'_k(p) = \infty \text{ for some } k = k(p)\}$. If $p \in P_2$ then $h_p^A(a_i) = \det_p(i, k(p))$ by Lemma 2.3. By the admissibility condition, $h_p^A(a_i) = h_i(p)$ for almost all $p \in P_2$.

Let $P_3 = \{p \mid 0 < h'_k(p) = h_k(p) < \infty \text{ for some (unique) } k = k(p)\}$. If $p \in P_3$, then

$$h_p^A(a_i) = \min\{h_{k(p)}(p), \det_p(i, k(p))\}.$$

If $i = k(p)$ then $h_p^A(a_i) = h_{k(p)}(p)$. On the other hand, if $i \ne k(p)$ and $\det_p(i, k(p)) > 0$ then $k(p) < i$ by condition (b) on the h_i's. This implies $h_p^A(a_i) = 0 = h_i(p)$ except for a finite number of p. Thus $h_p^A(a_i) = h_i(p)$ for almost all $p \in P_3$. Consequently, $\text{type}_A(a_i) = \tau_i$ for each i as desired.

THEOREM 2.6. *Given a rank-2 group A and a type $\sigma \ge \text{OT}(A)$, there is a rank-2 group B with $\text{OT}(B) = \sigma$ and $T_A = T_B$.*

Proof. Assume that $Zx \oplus Zy \subseteq A \subseteq Qx \oplus Qy$. Choose $h \in \sigma$ and $h_i \in \text{type}_A(a_i)$ such that $h \ge h_i$ for each i, and such that the h_i's satisfy (a) and (b) of Lemma 2.1. Note that $h \ge \max\{h_i \mid i \ge 1\}$ and that $h(p) = \infty$ for all p in $P_\infty = \{p \mid h_k(p) = \infty \text{ for some } k\}$. In view of Theorem 2.5, it suffices to assume that $A = \Sigma_p B(p)$, where

$$B(p) = \left\langle Zx \oplus Zy \cup \left\{ a_i/p^j \middle| 0 \le j \le h'_i(p), i = 1, 2, \dots \right\} \right\rangle,$$

$h'_i(p) = 0$ if $p \in P_\infty$ and $h_i(p) < \infty$, $h'_i(p) = h_i(p)$ otherwise. Thus, $\min\{h'_i(p), h'_j(p)\} = 0$ for each p and $i \ne j$. We will construct B using Lemma 2.4. This involves choosing an index $k = k(p)$ and a p-adic integer $\alpha = \alpha(p)$ for an appropriate collection of primes p.

First consider $P_1 = \{p \mid h(p) > 0 \text{ and } h_i(p) = 0 \text{ for all } i\}$. Write $P_1 = \{q(1), q(2), \dots\}$ with $q(i) < q(i+1)$. For each $p = q(t) \in P_1$, let $k(p) = t$, and let $\alpha(p)$ be an irrational p-adic integer such that p does not divide $r_i - \alpha(p)s_i$ for $i \le t$.

Next consider $P_2 = \{p \mid 0 < h'_k(p) < h(p) \le \infty \text{ for some (unique) } k = k(p)\}$. Assume $\gcd(s_k, p) = 1$ and choose an irrational p-adic integer

$\alpha = \alpha(p)$ with p-height$(r_k - \alpha s_k) = h'_k(p)$. (If p divides s_k, then $\gcd(r_k, p) = 1$ and the roles of s and r may be reversed in the proof. For example, α would be chosen so that p-height$(\alpha r_k - s_k) = h'_k(p)$.)

Denote $P_3 = P_1 \cup P_2$. For $p \in P_3$, let $k = k(p)$, $\alpha = \alpha(p)$, and let

$$A(p) = \left\langle Zx \oplus Zy \cup \left\{a_i/p^j \mid p^j \text{ divides } r_i - \alpha s_i \text{ and } j \leq h(p)\right\}\right\rangle.$$

By Lemma 2.4, $h_p^{A(p)}(a_i) < \infty$ for each i. Moreover, if $p \in P_2$ and $h_p^{A(p)}(a_i) > 0$, then p divides $r_i - \alpha s_i$, and hence

(*) p divides $s_k(r_i - \alpha s_i) - s_i(r_k - \alpha s_k) = s_k r_i - s_i r_k = \det(i, k)$.

Define $B = \Sigma\{A(p) \mid p \in P_3\} + \Sigma\{B(p) \mid p \notin P_3\}$. Then $h_p^B(a_i) = h_p^{A(p)}(a_i)$ for each i and $p \in P_3$. By Lemma 2.2, $\mathrm{OT}(B)$ and σ agree on P_3, and therefore $\mathrm{OT}(B) = \sigma$. To see that $\mathrm{type}_A(a_i) = \mathrm{type}_B(a_i)$, first note that if $p = q(t)$ is an element of P_1, then for $i \leq t$, $h_p^B(a_i) = 0$ by Lemma 2.4 and the choice of $\alpha(p)$. Thus $\mathrm{type}_A(a_i)$ and $\mathrm{type}_B(a_i)$ agree on P_1. Next let $p \in P_2$. If $i = k(p)$, then

$$\begin{aligned} h_p^B(a_i) &= \max\{j \mid p^j \text{ divides } r_i - \alpha(p)s_i \text{ and } j \leq h(p)\} \\ &= h_i(p) = h'_i(p) \end{aligned}$$

by Lemma 2.4 (since $h_{k(p)}(p) < h(p)$). Moreover, as in the proof of Theorem 2.5, $h_p^A(a_i) = h'_i(p)$, so that $h_p^B(a_i) = h_p^A(a_i)$ in this case. On the other hand, if $i \neq k(p)$ and $h_p^B(a_i) > 0$, then p divides $\det(i, k(p))$ as shown above (*). By condition (b), this happens only if $k(p) > i$, since $0 < h_{k(p)}(p) < \infty$. Thus, $i \neq k(p)$ and $h_p^B(a_i) > 0$ can happen for at most finitely many $p \in P_2$, and $h_p^A(a_i) = h_p^B(a_i)$ for almost all $p \in P_2$. It follows that $\mathrm{type}_A(a_i) = \mathrm{type}_B(a_i)$.

LEMMA 2.7 (Dubois [5]). *Let $T' = (\tau'_1, \tau'_2, \ldots)$ be a type sequence with* $\mathrm{type}(Z) = \inf\{\tau'_i, \tau'_j\}$ *whenever $i \neq j$. Assume that T' has an infinite subsequence T'_0 with no snarls of T'_0 in T'. Then there is a type sequence $T = (\tau_1, \tau_2, \ldots)$ and $h_i \in \tau_i$ for each i such that $T \approx T'$ and*

(a) *If p_i is the ith prime, then $h_j(p_i) = 0$ whenever $j \geq 2i$.*

(b) *If $K = \{k \mid$ for each p either $h_k(p) < \infty$ or else $h_k(p) = \infty$ and there is no $j < k$ with $0 < h_j(p) < h_k(p) = \infty\}$ then $\{k \mid \tau'_k \in T'_0\} \subseteq K$ so that K is infinite.*

Proof. (a) Let T_1 be the subsequence of types in T' with an infinity at some p and let T_2 be the complement of T_1 in T'. Order the elements of T_1

so that if $\tau_i' = [h_i']$, $\tau_j' = [h_j']$ are elements of T_1 and $\min\{p \mid h_i'(p) = \infty\} < \min\{p \mid h_j'(p) = \infty\}$ then $\tau_i' \leq \tau_j'$. Let T be the type sequence defined by:

$\tau_{2i} = i$th element in T_2 (if it exists)

$\tau_{2i-1} = i$th element in T_1 (if it exists)

If either T_1 or T_2 is finite, use the elements of the infinite sequence when the elements of the finite sequence are exhausted and if both are finite, use type(Z). By Lemma 2.1.a there is $h_i \in \tau_i$ for each i so that if $h_i(p) > 0$ and $j > i$, then $h_j(p) = 0$ or ∞. It follows that (a) is satisfied by $T \approx T'$.

(b) It suffices to assume that for each j, $\{p \mid 0 < h_j(p) < h_k(p) = \infty$ for some $\tau_k' \in T_0'\}$ is empty, since τ_j is not a snarl of T_0' and setting $h_j(p) = 0$ for only finitely many p with $h_j(p) < \infty$ does not change the type of h_j. Consequently, $\{k \mid \tau_k' \in T_0'\} \subseteq K$ and K is infinite. Note that (a) is still satisfied.

LEMMA 2.8. *Suppose that* $(r_1, s_1), (r_2, s_2), \ldots, (r_n, s_n)$ *are distinct elements of* U; $q_1, q_2, \ldots, q_m$ *distinct primes with each* $q_j > n$ *and* $\det_q(i, l) = 0$ *for* $q = q_j$, $1 \leq i \neq l \leq n$, *and* $1 \leq j \leq m$; $e_1, e_2, \ldots, e_m$ *non-negative integers*; *and* $\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$. *Then there are infinitely many* (r, s) *in* U *such that*

(∗) *if* $1 \leq j \leq m$ *and* $q = q_j$ *then* $h_q^Z(rs_i - r_is) = 0$ *for* $1 \leq i \neq i_j \leq n$ *and* $h_q^Z(rs_i - r_is) = e_j$ *for* $i = i_j$.

Proof. The proof is by induction on m. Suppose that $m = 1$. Let $i = i_1$, $q = q_1$ and $e = e_1$. First assume that $e = 0$. For each i with $1 \leq i \leq n$, there is at most one t with $1 \leq t \leq q$ such that q divides $ts_i - r_i$. Indeed, if q divides $ts_i - r_i$ and q divides $t's_i - r_i$, then q divides $(t - t')s_i$ so that $t = t'$ or q divides s_i. The latter case is impossible since $\gcd(r_i, s_i) = 1$. Since $n < q$, there must exist some $1 \leq t \leq q$, such that $h_q^Z(ts_i - r_i) = 0$ for each i. In this case $(r, s) = (t, 1) \in U$ satisfies (∗). Next assume that $e > 0$. Choose $(r, s) \in U$ with $rs_i - r_is = q^e$. Then $h_q^Z(rs_l - r_ls) = 0$ whenever $1 \leq i \neq l \leq n$, otherwise $\det_q(i, l) = h_q^Z(r_is_l - r_ls_i) > 0$, which is impossible. Hence (r, s) satisfies (∗). Given $(r, s) \in U$ satisfying (∗) let $x' = r + aq^{e+1}$, $y' = s + bq^{e+1}$, $d = \gcd(x', y')$, $x = x'/d$ and $y = y'/d$. Then $(x, y) \in U$ and there are infinitely many such (x, y) which satisfy (∗).

Now assume inductively that $(r', s') \in U$ satisfies (∗) for $1 \leq j < m$. Let $i = i_m$, $q = q_m$, $e = e_m$, and let π be the product of $\{q_j^{n(j)} \mid 1 \leq j \leq m - 1, n(j) = e_j + 1\}$. Assume that $e = 0$. Since $n < q$ there is $1 \leq t \leq q$ such that $h_q^Z((r' + t\pi)s_l - s'r_l) = 0$ for each $1 \leq l \leq n$ (as above). Let

$r = (r' + t\pi)/d$, $s = s'/d$, where $d = \gcd(r' + t\pi, s')$. Then $(r, s) \in U$ satisfies $(*)$ since if $p = q_j \neq q$ then

$$h_p^Z((r' + t\pi)s_l - s'r_l) = h_p^Z(r's_l - s'r_l) \quad \text{for each } 1 \leq l \leq n.$$

Next assume that $e \neq 0$. Choose $a, b \in Z$ with $ar_i + bs_i = 1$. By the Chinese Remainder Theorem there is $x \geq 0$, y in Z with $x \equiv r' \pmod{\pi}$, $x \equiv bqe + r_i \pmod{q^{e+1}}$ and $y \equiv s' \pmod{\pi}$, $y \equiv -aq^e + s_i \pmod{q^{e+1}}$. Then $h_q^Z(xs_i - r_iy) = e$, $h_q^Z(xs_l - r_ly) = 0$ if $l \neq i$, and if $p = q_j \neq q$ then $h_p^Z(xs_l - r_ly) = h_p^Z(r's_l - r_ls')$. Consequently, $(r, s) \in U$ satisfies $(*)$, where $r = x/d$, $s = y/d$ and $d = \gcd(x, y)$.

If $x' = r + u\pi q^{e+1}$, $y' = s + v\pi q^{e+1}$, $d = \gcd(x', y')$, $r' = x'/d$, and $s' = y'/d$ then $(r', s') \in U$ and there are infinitely many such (r', s') satisfying $(*)$. By induction on m, the proof is complete.

A type τ is *very large* if $h \in \tau$ implies that $\{p \mid h(p) = \infty\}$ is infinite.

THEOREM 2.9. *Suppose that* $T = (\tau_1, \tau_2, \ldots)$ *is a type sequence having an infinite subsequence* T_0 *with no snarls in* T *and that* type$(Z) = \inf\{\tau_i, \tau_j\}$ *whenever* $i \neq j$.

(a) *There is a rank-2 group* A *with* $\mathrm{IT}(A) = $ type(Z) *and* $T_A = (\tau_1', \tau_2', \ldots)$, *where* $\tau_i' \geq \tau_i$ *for each* i *and if* $h_i' \in \tau_i'$, $h_i \in \tau_i$ *then* $h_i'(p) = \infty$ *iff* $h_i(p) = \infty$.

(b) *If* $\{\tau_j \mid \tau_j$ *very large*$\}$ *has no snarls in* T *then* A *may be chosen with* $T_A \approx T$.

Proof. (a) By Lemmas 2.7 and 2.1 it suffices to assume that there is $h_i \in \tau_i$ for each i such that if $j < k$ then $\inf\{h_j(p), h_k(p)\} = 0$ unless $h_k(p) = \infty$; $\{k \mid \tau_k \in T_0\} \subseteq K = \{k \mid$ for each p either $h_k(p) < \infty$ or else $h_k(p) = \infty$ and there is no j with $0 < h_j(p) < h_k(p) = \infty\}$; and $h_j(p_i) = 0$ whenever $j \geq 2i$ and p_i is the ith prime.

To construct an indexing of U, via Lemma 2.8, let $u_1 = (1, 0)$ and $u_2 = (0, 1)$.

If $k \geq 3$ and $k \in K$ choose $u_k = (r_k, s_k) \in U$ with $\max\{r_k, |s_k|\}$ minimal among the elements of U not already chosen.

If $k \geq 3$, $k \notin K$, and τ_k is not very large let $\{q_1, q_2, \ldots, q_m\} = \{p \mid h_k(p) = \infty\}$; $e_j = h_l(q_j)$ and $i_j = l$ if $0 < h_l(q_j)$; $e_j = 0$ and $i_j = l$ for some arbitrary $l < k$ if $h_i(q_j) = 0$ for all $1 \leq i < k$; and let n_k be the largest integer less than k such that $q_j > n_k$ and $\det_{q_j}(i, l) = 0$ whenever $1 \leq i \neq l \leq n_k$ and $1 \leq j \leq m$. By Lemma 2.8, there is $u_k = (r_k, s_k) \in U$, not already chosen, such that $h_i(p) = \det_p(k, i)$ whenever $p \in \{q_1, q_2, \ldots, q_m\}$ and $1 \leq i \leq n_k$.

If $k \geq 3$, $k \notin K$, and τ_k is very large let $\{q_1, q_2, \ldots, q_m\} = \{p \mid 0 < h_j(p) < h_k(p) = \infty$ for some $j < k\}$; $e_j = h_l(q_j)$ and $i_j = l$ if $0 < h_l(q_j)$; and let n_k be the largest integer less than k such that $q_j > n_k$ and $\det_{q_j}(i, l) = 0$ whenever $1 \leq i \neq l \leq n_k$ and $1 \leq j \leq m$. By Lemma 2.8 there is $u_k = (r_k, s_k) \in U$, not already chosen, such that $h_i(p) = \det_p(k, i)$ whenever $p \in \{q_1, q_2, \ldots, q_m\}$ and $1 \leq i \leq n_k$.

Since K is infinite, every element of U is chosen. Moreover, if $k \in K$ then $\max\{r_k, |s_k|\} \leq k$, since only $k - 1$ elements of U have previously been chosen.

For each j define $\tau_j' = [h_j']$, where $h_j'(p) = \det_p(k, j)$ whenever $0 = h_j(p) < \det_p(k, j) < h_k(p) = \infty$ for some $j < k \notin K$ with τ_k very large, and define $h_j'(p) = h_j(p)$ otherwise. Note that $\tau_j' \geq \tau_j$ and $h_j'(p) = \infty$ iff $h_j(p) = \infty$.

By Theorem 2.5, it is sufficient to prove that $T' = (\tau_1', \tau_2', \ldots)$ is admissible relative to the chosen ordering of U. Fix j and let $m = \max\{r_j, |s_j|\}$.

Let $P_1 = \{p \mid 0 = h_j'(p) < \det_p(k, j) < h_k'(p) = \infty$ for some $k \in K\}$. If $p = p_i \in P_1$ then p_i divides $\det(k, j)$ while $\det(k, j) \leq 2mk \leq 4mi$, since $k \in K$ and $h_k(p_i) = \infty$ implies that $k \leq 2i$. By Lemma 0.1.b, $p_i > 4mi$ for sufficiently large i, so that P_1 is finite.

Next let $P_2 = \{p \mid h_j'(p) \neq \det_p(j, k) < h_k'(p) = \infty$, τ_k not very large, $j < k \notin K\}$. By the choice of $u_k = (r_k, s_k) \in U$, $j > n_k$ for each such k. Assume that P_2 is infinite. Then there are infinitely many $j < k \notin K$ with $j > n_k$, $\inf\{p \mid h_k(p) = \infty\} > j$ (noting that for each p there is at most one i with $h_i(p) = \infty$), and $h_j(p) \neq \det_p(j, k) < h_k(p) = \infty$ for some $p \in P_2$. For each such k, there is $1 \leq i \neq l \leq j$ with $0 < \det_p(i, l) < h_k(p) = \infty$ for some p, otherwise $j \leq n_k$ by the definition of u_k. But

$$\{p \mid \det_p(i, l) > 0 \text{ for some } 1 \leq i \neq l \leq j\}$$

is finite, which is a contradiction.

Finally, $P_3 = \{p \mid h_j'(p) \neq \det_p(j, k) < h_k'(p) = \infty$ for some $j < k \notin K$, τ_k very large$\}$ is empty by the definition of h_j'. Thus $P_1 \cup P_2 \cup P_3$ is finite so that T' is admissible.

(b) Note that $T_0 \cup \{\tau_j \mid \tau_j$ very large$\}$ generates an infinite subsequence of T with no snarls in T. Now apply the constructions of (a), noting that if $k \notin K$ then τ_k is not very large so that $h_j' = h_j$ for each j.

COROLLARY 2.10. *Let S be a set of types with* $\inf\{\tau, \tau'\} = \text{type}(Z)$ *whenever $\tau, \tau' \in S$ with $\tau \neq \tau'$. Assume that $\{\tau \in S \mid \tau$ is very large$\}$ has no snarls in S.*

(a) *There is a rank-2 group A with* typeset$(A) = S$ *iff either* type$(Z) \in S$ *or else S has an infinite subset with no snarls in S.*

(b) *There is a completely anisotropic rank-2 A with* typeset$(A) = S$ *iff S has an infinite subset with no snarls in S.*

Proof. (a) ($\Rightarrow$) Let $T_A = (\tau_1, \tau_2, \ldots)$. By Theorem 1.4, T_A has an infinite subsequence with no snarls in T_A. If $S = \{\tau_i \mid i \geq 1\} =$ typeset(A) does not have an infinite subset with no snarls in S then some τ_i must be repeated in T_A. But IT$(A) =$ type(Z) is the only type in T_A that may be repeated so that type$(Z) \in S$.

($\Leftarrow$) If S has an infinite subset with no snarls in S define $T = (\tau_1, \tau_2, \ldots)$ where $S = \{\tau_i \mid i \geq 1\}$. Otherwise type$(Z) \in S$, and in this case define $T = (\tau_1', \tau_2', \ldots)$ where $\tau_{2i-1}' = \tau_i$, $\tau_{2i}' =$ type(Z) for $i \geq 1$ if S is infinite. If $S = \{\tau_1, \tau_2, \ldots, \tau_n\}$ is finite define $\tau_{2i}' =$ type(Z), $\tau_{2i-1}' = \tau_i$ for $1 \leq i \leq n$ and $\tau_i' = \tau_{2i-1}' =$ type(Z) for $i > n$. For each of the above cases, T has an infinite subset with no snarls in T. By Theorem 2.9 there is a rank-2 group A with $T_A \approx T$ so that typeset$(A) = S$.

(b) is a consequence of (a) and the fact that A is completely anisotropic iff T_A has no repetitions.

COROLLARY 2.11. *Let $S_1 = \{\tau_1, \tau_2, \ldots\}$ be a set of types with* $\inf\{\tau_i, \tau_j\} =$ type(Z) *whenever $i \neq j$, and assume that $\{\tau_j \mid \tau_j$ is very large$\}$ has no snarls in S_1. Let $S_2 = (\sigma_1, \sigma_2, \ldots)$ be another set of types. Then there is a rank-2 group A with* typeset$(A) = S_1$ *and* cotypeset$(A) = S_2$ *if and only if*

(a) *There is a type σ_0 such that $\sigma_0 = \sup\{\sigma_i, \sigma_j\}$ for $i \neq j$;*

(b) *$\tau_i \leq \sigma_0$ for each i;*

(c) *$\sigma_i = \sigma_0 - \tau_i$ for each i; and*

(d) *Either* type$(Z) \in S_1$ *or else S_1 has an infinite subset with no snarls in S_1.*

Proof. ($\Rightarrow$) Apply Proposition 1.1 and Corollary 2.10.

($\Leftarrow$) In view of (d), Theorem 2.9 can be applied to obtain a group B such that typeset$(B) = S_1$. Furthermore, B can be assumed to satisfy OT$(B) \leq \sigma_0$ by (b) and Lemma 2.2. By Theorem 2.6, there is a rank-2 group A such that typeset$(A) =$ typeset(B) and OT$(A) = \sigma_0$. By (c) and Proposition 1.1(e), cotypeset$(A) = \{\sigma_1, \sigma_2, \ldots\} = S_2$.

EXAMPLE 2.12. Let τ_i for $i \geq 1$ be defined as in Example 1.5. Let $S = \{\tau_i \mid i \geq 1\} \cup \{\text{type}(Z)\}$. Then there is a rank-2 group A with typeset$(A) = S$, by Corollary 2.10.a. On the other hand, by Corollary 2.10.b there is no completely anisotropic rank-2 group A with typeset$(A) = S$ (compare Example 1.5).

EXAMPLE 2.13. (Ito [9].) Let $\tau_1, \tau_2, \ldots$ be given by $\tau_i = [h_i]$, where $h_1 = (0, 1, 0, 1, 0, 1, \ldots)$; $h_2 = (\infty, 0, 0, \ldots)$; $h_3 = (0, \infty, 0, \ldots)$; $h_4 = (0, 0, \infty, \ldots)$; $\ldots$. Let $S = \{\tau_i \mid i \geq 1\}$. Then $\{\tau_{2i} \mid i \geq 1\}$ is an infinite subset of S with no snarls in S. By Corollary 2.10(b), there is a completely anisotropic rank-2 group A with typeset$(A) = S$. Similarly, there is a completely anisotropic rank-2 group A with typeset$(A) = S \cup \{\text{type}(Z)\}$.

COROLLARY 2.14. *Let $S = \{\tau_i \mid i \geq 1\}$ be a set of types with $\tau_0 = \inf\{\tau_i, \tau_j\}$ whenever $i \neq j$.*

(a) (Beaumont-Pierce [3]) *If S is finite and $\tau_0 \in S$ then there is a rank-2 group A with* typeset$(A) = S$.

(b) (Ito [9]) *If there is $h_i \in \tau_i$ for $i \geq 0$ with $h_0 = \inf\{h_i, h_j\}$ for each $i \neq j$ then there is a rank-2 group A with* typeset$(A) = S$ *and* $\mathrm{OT}(A) = [\sup\{h_i \mid i \geq 1\}]$.

Proof. By Lemma 1.3, it suffices to assume that $\tau_0 = \text{type}(Z)$. In either case S has no snarls in S. Now apply Corollary 2.10 to get a rank-2 group A with typeset$(A) = S$. This group is constructed via Theorem 2.9 so that $\mathrm{OT}(A) \leq [\sup\{h_i \mid i \geq 1\}]$. By Theorem 2.6, A may be chosen with $\mathrm{OT}(A) = \sup[\{h_i \mid i \geq 1\}]$.

The next example shows that the hypothesis of Corollary 2.10 that $\{\tau \in S \mid \tau \text{ is very large}\}$ has no snarls in S is not necessary.

EXAMPLE 2.15. There is a rank-2 group A such that $\mathrm{IT}(A) = \text{type}(Z)$ and $\{\tau \mid \tau \in \text{typeset}(A) \text{ and } \tau \text{ very large}\}$ is infinite with infinitely many snarls in typeset(A).

Proof. Let $S = \{\tau_i \mid i \geq 1\}$ where $\tau_i = [h_i]$ and h_i is defined by:

$$h_1 = (1,1,1,\ldots,\infty,\infty,\infty,\ldots,0,0,0,\ldots,0,0,0,\ldots,0,0,0,\ldots)$$
$$h_2 = (\infty,0,0,\ldots,0,0,0,\ldots,1,1,1,\ldots,\infty,\infty,\infty,\ldots,0,0,0,\ldots)$$
$$h_3 = (0,\infty,0,\ldots,0,0,0,\ldots,\infty,0,0,\ldots,0,0,0,\ldots,1,1,1,\ldots), \text{ etc.}$$

Apply Theorem 2.9(a) to obtain a rank-2 group A with typeset$(A) = \{\tau_i' \mid i \geq 1\}$, $\tau_i' \geq \tau_i$ for each i, and $\{\tau \mid \tau \in \text{typeset}(A) \text{ and } \tau \text{ very large}\}$ is infinite with infinitely many snarls in typeset(A).

3. Realization of cotypesets.

THEOREM 3.1 (Vinsonhaler-Wickless [12]). *Let $S = \{\sigma_1, \sigma_2, \ldots\}$ be a set of types with $\sigma_0 = \sup\{\sigma_i, \sigma_j\}$ for each $i \neq j$ and $\sigma_0 \in S$ if S is finite.*

(a) *There is $s_i \in \sigma_i$ for $i \geq 0$ such that $s_0 = \max\{s_i, s_j\}$ if $i \neq j$.*

(b) *There is a rank*-2 *group* A *with* cotypeset(A) = S, OT(A) = σ_0, IT(A) = $[\inf\{s_i \mid i \geq 1\}]$, *and* typeset($A$) = $\{\sigma_0 - \sigma_i + \mathrm{IT}(A) \mid i \geq 1\}$.

Proof. The following proof is a simplification of the arguments given in Vinsonhaler-Wickless [**12**].

(a) Given $s_0, s_1, \ldots, s_{n-1}$ with $s_i \in \sigma_i$ for $0 \leq i \leq n-1$ and $s_0 = \max\{s_i, s_j\}$ for $1 \leq i \neq j \leq n-1$ choose $s_n \in \sigma_n$ with $s_0 = \sup\{s_i, s_n\}$ for $1 \leq i \leq n-1$.

(b) Define $t_0' = \min\{s_i \mid i \geq 1\}$, $\tau_0' = [t_0']$, and $\gamma_i = \sigma_i - \tau_0'$ for $i \geq 0$. Note that $\gamma_i = [s_i - t_0']$ for each $i \geq 0$. Now $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ with $\gamma_0 = \sup\{\gamma_i, \gamma_j\}$ if $i \neq j$ and $\gamma_0 \in \Gamma$ if Γ is finite. Define $\tau_i = \gamma_0 - \gamma_i$ for $i \geq 0$.

The next step is to show that there is a rank-2 group B with typeset(B) = $\{\tau_i \mid i \geq 1\}$ and cotypeset(B) = Γ. For each i, let $t_i = (s_0 - t_0') - (s_i - t_0') \in \tau_i = \gamma_0 - \gamma_i$. Note that:

(i) if $t_i(p) = \infty$ then $s_0(p) = \infty$, $t_0'(p) < \infty$, and $s_i(p) < \infty$.

(ii) if $0 < t_i(p) < \infty$ then $s_0(p) < \infty$.

(iii) $t_i(p) = s_0(p) - s_i(p)$.

(iv) $\inf\{t_i, t_j\} = 0$ whenever $i \neq j$.

By (iv) and Corollary 2.14.b there is a rank-2 group B with typeset(B) = $\{\tau_i \mid i \geq 1\}$, OT(B) = $[\sup\{t_i \mid i \geq 1\}]$, and IT($B$) = τ_0 = type(Z). By (iii), $\sup\{t_i \mid i \geq 1\} = s_0 - t_0'$ so that OT(B) = γ_0. Therefore, cotypeset(B) = $\{\gamma_0 - \tau_i \mid i \geq 1\}$ by Proposition 1.1.e.

Furthermore, $\gamma_0 - \tau_i = \gamma_i$ for each $i \geq 0$. To see this, note that $\gamma_0 - \tau_i = [(s_0 - t_0') - (s_0 - s_i)]$, by (iii), and $\gamma_i = [s_i - t_0']$. The only non-trivial case is $t_0'(p) \leq s_i(p) < s_0(p) = \infty$, in which case $s_j(p) = s_0(p)$ for $j \neq i$ (by (a)), $t_0'(p) = s_i(p)$, and $(s_0(p) - t_0'(p)) - (s_0(p) - s_i(p)) = 0 = s_i(p) - t_0'(p)$. Consequently, cotypeset(B) = $\{\gamma_i \mid i \geq 1\}$.

By Lemma 1.3, there is a rank-2 group A with typeset(A) = $\{\tau_i + \tau_0' \mid i \geq 1\}$, IT($A$) = τ_0', cotypeset(A) = $\{\gamma_i + \tau_0' \mid i \geq 1\} = S$, and OT($A$) = $\gamma_0 + \tau_0' = \sigma_0$. Finally, $\tau_i + \tau_0' = \sigma_0 - \sigma_i + \tau_0'$ by (iii).

COROLLARY 3.2. *Let* $S = \{\sigma_1, \sigma_2, \ldots\}$ *be a set of types. There is a rank*-2 *group* A *with* cotypeset(A) = S *iff there is* $\sigma_0 = \sup\{\sigma_i, \sigma_j\}$ *whenever* $i \neq j$ *and* $\sigma_0 \in S$ *if* S *is finite*.

REMARK. Vinsonhaler-Wickless [**12**] have given necessary and sufficient conditions for a set of types to be the cotypeset of a finite rank torsion free group, with Corollary 3.2 as a special case.

4. Locally completely decomposable groups. A finite rank torsion free group A is *locally completely decomposable* if $A_p = Z_p \otimes_Z A$ is the direct sum of a free Z_p-module and a divisible Z_p-module for each prime p, where Z_p is the localization of Z at p.

Let A be a finite rank torsion free group. Recall that *typeset*$(A) = \{\text{type}(X) \mid X \text{ is a pure rank-1 subgroup of } A\}$ and *cotypeset*$(A) = \{\text{type}(Y) \mid Y \text{ is a rank-1 torsion free quotient of } A\}$.

THEOREM 4.1. *Assume that A is a finite rank torsion free group.*

(a) *There is a finite rank torsion free locally completely decomposable group B with $B \subseteq A$, A/B torsion, and* typeset(B) = typeset(A).

(b) (Vinsonhaler-Wickless [**12**]). *There is a finite rank torsion free locally completely decomposable group B with $A \subseteq B$, B/A torsion, and* cotypeset(A) = cotypeset(B).

(c) *Further assume that* rank$(A) = 2$, typeset$(A) = \{\tau_i \mid i \geq 1\}$, $h_i \in \tau_i$ *for each i and $s_0 \in \mathrm{OT}(A)$. Then A is locally completely decomposable iff whenever $s_0(p) = \infty$ then $h_i(p) = \infty$ for some i.*

Proof. (a) Let $A_1, A_2, \ldots$ be a listing of the pure rank-1 subgroups of A, let p_i be the ith prime and choose a free subgroup F of A with A/F torsion.

Define $B_{p_i} = F_{p_i} + d(A_{p_i}) + (A_1)_{p_i} + \cdots + (A_i)_{p_i}$ where $d(A_p)$ is the maximal divisible Z_p-submodule of A_p. Define $B = \bigcap_p B_p$. Then $F = \bigcap_p F_p \subseteq B \subseteq A = \bigcap_p A_p$ and A/B is torsion. Let $X = A_i$. Then $X_p \subseteq B_p$ for almost all p. If $X_p \simeq Q$ then $X_p \subseteq d(A_p) \subseteq B_p$. Otherwise, $X_p/(X_p \cap B_p)$ is finite. Hence $X/X \cap B$ is finite since $(X/X \cap B)_p \cong X_p/X_p \cap B_p = 0$ for almost all p and $X_p/(X_p \cap B_p)$ is finite otherwise. Consequently, $X \cong X \cap B$ and so typeset(A) = typeset(B). Finally, B is locally completely decomposable since for each p, $B_p/d(B_p)$ is a finitely generated free Z_p-module.

(b) The following is a rank-2 version of the proof in Vinsonhaler-Wickless [**13**]. Define

$$B = \bigcap \{f^{-1}f(A) \mid f \in \mathrm{Hom}(QA, Q)\} \subseteq QA = Q \otimes_Z A$$

where A is regarded as a subgroup of QA. Then $A \subseteq B \subseteq QA$ and B/A is torsion. Suppose that $g(A) = Y \subseteq Q$ for some $g\colon A \to Q$. Then $g\colon QA \to Q$ and $g(A) \subseteq g(B) \subseteq g(g^{-1}g(A)) = g(A)$. Conversely, if $g(B) = Y \subseteq Q$ for some $g\colon B \to Q$ then $g\colon QA \to Q$ since $QA = QB$ and $g(A) \subseteq g(B) \subseteq g(g^{-1}g(A)) = g(A)$. Consequently, cotypeset(A) = cotypeset(B), noting that each rank-1 torsion free group is isomorphic to a subgroup of Q.

To show that B is locally completely decomposable suppose that $B_p = X \oplus Y$ where X has no rank-1 summands and Y is the direct sum of a free and a divisible Z_p-module. Let $0 \neq f \in \mathrm{Hom}(QA, Q)$. If $f(X) \neq 0$, then $f(X) = Q = f(B_p)$, since otherwise $f(X) \simeq Z_p$ and X has no rank-1 summands. But $\operatorname{Ker} f$ is divisible so $B_p \subseteq f^{-1}f(B_p) = \operatorname{Ker} f \oplus H$ where $f(H) = f(B_p) = Q$. Thus H is divisible so that $f^{-1}f(B_p) = f^{-1}f(A_p) = QA \supseteq QX$ in this case. Now assume that $f(X) = 0$. Then $QX \subseteq \operatorname{Ker} f \subseteq f^{-1}f(A_p)$. Thus, $QX \subseteq f^{-1}f(A_p)$ for all $f \in \operatorname{Hom}(QA, Q)$. But

$$B_p = \left(\bigcap \{f^{-1}f(A) | f \in \operatorname{Hom}(QA, Q)\}\right)_p$$

$$= \bigcap \left\{f^{-1}f(A_p) | f \in \operatorname{Hom}(QA, Q)\right\} \supseteq QX$$

so that $QX = X$ is divisible. Since X has no rank-1 summands, $X = 0$ as desired.

(c) ($\Rightarrow$) Let $A_p = X_1 \oplus X_2$. Then there are pure rank-1 subgroups A_i of A with $(A_i)_p = X_i$. If $s_0(p) = \infty$ then $(A/A_i)_p \simeq Q$ for some i, say $i = 1$. But $(A/A_1)_p \simeq A_p/(A_1)_p \simeq X_2$ so that $X_2 \simeq Q$ and A_2 is p-divisible. Therefore, for some k, $\tau_k = \operatorname{typeset}(A_2)$ and if $h_k \in \tau_k$ then $h_k(p) = \infty$.

($\Leftarrow$) Assume that $s_0(p) < \infty$ and that A_i is a pure rank-1 subgroup of A. Then $0 \to (A_i)_p \to A_p \to (A/A_i)_p \to 0$ is exact with $(A/A_i)_p \simeq Z_p$ since $\operatorname{type}(A/A_1) \leq \operatorname{OT}(A)$. Thus $A_p \simeq Z_p \oplus (A_i)_p$ is completely decomposable. Now assume that $s_0(p) = \infty = h_i(p)$. Let A_i be a pure rank-1 subgroup of A with $\tau_i = \operatorname{type}(A_i)$. Then $(A_i)_p \simeq Q$ so that $A_p \simeq Q \oplus (A/A_i)_p$ is completely decomposable as desired.

REMARK. In view of Theorem 4.1(c), each rank-2 group constructed in Theorem 2.5 is locally completely decomposable, noting that the $h_i \in \tau_i$ in this construction are chosen to satisfy (a) of Lemma 2.1.

COROLLARY 4.2. *Let* $S_1 = \{\tau_1, \tau_2, \ldots\}$ *be a set of types with* $\inf\{\tau_i, \tau_j\} = \operatorname{type}(Z)$ *whenever* $i \neq j$ *and assume that* $\{\tau_j | \tau_j$ *very large*$\}$ *has no snarls in* S_1. *Let* $S_2 = \{\sigma_1, \sigma_2, \ldots\}$ *be another set of types. Then there is a rank*-2 *locally complete decomposable group* A *with* $\operatorname{typeset}(A) = S_1$ *and* $\operatorname{cotypeset}(A) = S_2$ *iff*

(a) *There is a type* σ_0 *such that* $\sigma_0 = \sup\{\sigma_i, \sigma_j\}$ *for* $i \neq j$;

(b) $\tau_i \leq \sigma_0$ *for each* i;

(c) $\sigma_i = \sigma_0 - \tau_i$ *for each* i;

(d) *Either* $\operatorname{type}(Z) \in S_1$ *or else* S_1 *has an infinite subset with no snarls in* S_1;

(e) *If* $s_0 \in \sigma_0$, $h_i \in \tau_i$ *for each* i, *and* $s_0(p) = \infty$ *then* $h_i(p) = \infty$ *for some* i.

Proof. A consequence of Corollary 2.11, Theorem 4.1(c), and the preceding remark.

5. Open questions.

(5.1) Is it true that Corollaries 2.10 and 2.11 are true without the hypothesis that $\{\tau \in S \mid \tau \text{ is very large}\}$ has no snarls in S?

As noted earlier, Example 2.15 shows that this hypothesis is not necessary. In fact, it is unknown whether or not the set S of types in Example 2.15 may be realized as the typeset of a rank-2 group.

The construction of Theorem 2.9 uses Lemma 2.8. Consequently, some strengthened, possibly infinite, version of Lemma 2.8 would be needed to make the construction of Theorem 2.9 work without the hypothesis that $\{\tau_j \mid \tau_j \text{ very large}\}$ has no snarls in T.

(5.2) Are the results of the paper true for modules over an arbitrary principal ideal domain?

The results of this paper use a version of the prime number theorem (Lemma 0.1(b)) which is not applicable for arbitrary principal ideal domains. De Munter-Kuyl [**4**], claims that Ito's Theorem (Corollary 2.14(b)) is true for arbitrary principal ideal domains. However, the construction is incorrect, even in the case of groups. For example, the construction fails for a set of types $= \{\tau_i \mid i \geq 1\}$ where $\tau_i = [h_i]$ and $h_1 = (\infty, 0, \ldots)$, $h_2 = (0, \infty, 0, \ldots)$, $h_3 = (0, 0, \infty, \ldots), \ldots,$ even though Ito's theorem is true for groups.

The answer to (5.2) may depend on:

(5.3) Can the results of this paper be proved without appealing to some version of the prime number theorem?

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