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TOPOLOGICAL METHODS FOR C^*-ALGEBRAS IV: MOD P HOMOLOGY

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Let h_* be a homology theory on an admissible category of C^*-algebras. We define a homology theory $h_*(-;\mathbf{Z}/n)$ which fits into a Bockstein exact sequence

$$\to h_j(A) \xrightarrow{n} h_j(A) \xrightarrow{\rho_n} h_j(A;\mathbf{Z}/n) \xrightarrow{\beta_n} h_{j-1}(A) \to \cdots.$$

Let p be a prime. If p is odd or if h_* is "good" then $h_*(A;\mathbf{Z}/p)$ is a $\mathbf{Z}/p$-module and (with finiteness assumptions on the torsion of $h_*(A)$) there is a Bockstein spectral sequence with $E^1_* = h_*(A;\mathbf{Z}/p)$ which converges to $(h_*(A)/(\text{torsion})) \otimes \mathbf{Z}/p$. In the special case of K-theory, we show that $K_*(A \otimes N) \cong K_*(A;\mathbf{Z}/n)$, provided that $K_0(N) = \mathbf{Z}/n$, $K_1(N) = 0$, and N is in a certain (large) category $\mathfrak{N}$ of separable nuclear C^*-algebras.

Let h_* be a homology theory on an admissible category of C^*-algebras. This paper has three objects: to define and investigate the properties of the associated mod p homology theory $h_*(-;\mathbf{Z}/p)$, to generalize the notion of Bockstein coboundary homomorphisms and the apparatus of the Bockstein spectral sequence to this setting, and to establish a uniqueness theorem for the introduction of mod p coefficients into K-theory.

DEFINITION [16]. A *homology theory* is a sequence $\{h_n\}$ of covariant functors from an admissible category $\mathcal{C}$ of C^*-algebras to abelian groups which satisfies the following axioms:

Homotopy axiom. Let $h\colon A \to C([0,1], B)$ be a homotopy from $f_0 = p_0 h$ to $f_1 = p_1 h$ $[p_i(\xi) = \xi(i)]$ in $\mathcal{C}$. Then $f_{0*} = f_{1*}\colon h_n(A) \to h_n(B)$ for all n.

Exactness axiom. Let

$$0 \to J \xrightarrow{i} A \xrightarrow{j} B \to 0$$

be a short exact sequence in $\mathcal{C}$. Then there is a map $\partial\colon h_n(B) \to h_{n-1}(J)$ and a long exact sequence

$$\to h_n(J) \xrightarrow{i_*} h_n(A) \xrightarrow{j_*} h_n(B) \xrightarrow{\partial} h_{n-1}(A) \to \cdots.$$

The map ∂ is natural with respect to morphisms of short exact sequences in $\mathcal{C}$.

There is a weaker notion which is also required. Recall **[16]** that a map $g\colon A \to B$ is a *cofibration* if any homotopy $h_t\colon D \to B$ of a composite fg, $f\colon D \to A$, can be extended to a homotopy $H_t\colon D \to A$ with $H_0 = f$ and $gH_t = h_t$.

DEFINITION. A *cofibre homology theory* $\{h_n\}$ is a sequence of covariant functors from an admissible category of C^*-algebras to abelian groups which satisfies the homotopy axiom and the following axioms:

Cofibre axiom. Let $g\colon A \to B$ be a cofibration, and let

$$Cg = \{(\xi, a) \in C([0,1], B) \oplus A \mid \xi(1) = 0, \xi(0) = g(a)\}$$

with $\pi(g)\colon Cg \to A$ by $\pi(g)(\xi, a) = a$ be the mapping cone. Then the sequence

$$h_n(Cg) \xrightarrow{\pi(g)_*} h_n(A) \xrightarrow{g_*} h_n(B)$$

is exact for each n.

Suspension axiom. There is a natural isomorphism

$$\sigma_A\colon h_n(A) \xrightarrow{\cong} h_{n-1}(SA)$$

where $SA = C_0((0,1), A)$ is the suspension of A.

The groups $h_*(A; Z/n)$ are defined in §1 by

$$h_j(A; \mathbf{Z}/n) \equiv h_{j-2}(A \otimes C\Theta_n),$$

where $\Theta_n\colon C_0(\mathbf{R}) \to C_0(\mathbf{R})$ is the canonical map of degree n. Then there is a long exact sequence

$$\to h_j(A) \xrightarrow{n} h_j(A) \xrightarrow{\rho_n} h_j(A; \mathbf{Z}/n) \xrightarrow{\beta_n} h_{j-1}(A) \to \cdots$$

with Bockstein map β_n. It is easy to establish that if h_* is an (additive) (cofibre) homology theory then so is $h_*(-; \mathbf{Z}/n)$. Furthermore, there is a short exact sequence

$$0 \to h_j(A) \otimes \mathbf{Z}/n \xrightarrow{\tilde{\rho}_n} h_j(A; \mathbf{Z}(n)) \to \operatorname{Tor}\bigl(h_{j-1}(A), \mathbf{Z}/n\bigr) \to 0,$$

which is natural in A, known as the Universal Coefficient sequence.

§2 is concerned with the relationship between the theories $h_*(-;\mathbf{Z}/n)$ for various n. There is a serious difficulty. The Universal Coefficient sequence above need not split; $h_*(A;\mathbf{Z}/n)$ need not be a $\mathbf{Z}/n$-module. Fortunately this problem arises and is disposed of already in the algebraic topology setting. Following Deleanu and Hilton, we say that a homology theory h_* is *good* if the Hopf map $S^3 \to S^2$ induces the zero homomorphism

$$h_*\big(A \otimes C_0(\mathbf{R}^2)\big) \to h_*\big(A \otimes C_0(\mathbf{R}^3)\big)$$

for all C^*-algebras A in the category. (For instance, K_* is good). If h_* is good or if p is odd then we show that the Universal Coefficient sequence splits, so that $h_*(A;\mathbf{Z}/p)$ is a graded $\mathbf{Z}/p$-module. As a consequence of our coherence work we introduce other Bockstein maps of the form

$$\beta_{m,n}\colon h_j(A;\mathbf{Z}/n) \to h_{j-1}(A;\mathbf{Z}/m)$$

and

$$b_n\colon h_j(A;\mathbf{Z}/n) \to h_{j-1}(A;\mathbf{Z}/n)$$

with $b_n^2 = 0$.

In §3 we show that corresponding to the relation $b_p^2 = 0$ there is a family of higher order homology operations $\bar{\beta}_{p^n}$ for h_*. Since the Adams operations ψ^k do not extend to K_* for C^*-algebras, the operation b_p on $K_*(-;\mathbf{Z}/p)$ is the first example of a non-trivial homology operation for C^*-algebras. The operations $\bar{\beta}_{p^n}$ are differentials in the Bockstein spectral sequence.

§4 is devoted to the study of torsion in integral homology. The tool is the Bockstein spectral sequence. Fix a prime p and let h_* be a cofibre homology theory such that the p-primary component of the torsion subgroup of $h_j(A)$ is finite for all j and for all requisite A. Assume further that p is odd or that h_* is good. Then there is a spectral sequence with

$$E_j^1 = h_j(A;\mathbf{Z}/p)$$

which converges to

$$E_j^\infty \cong \left(\frac{h_j(A)}{\text{torsion}}\right) \otimes \mathbf{Z}/p.$$

The spectral sequence processes the torsion. For instance, if $y \in h_j(A)$ generates a direct $\mathbf{Z}/p^r$ summand in $h_j(A)$ then $\rho_p(y) \in h_j(A;\mathbf{Z}/p) = E_j^1$ survives to E^r in the spectral sequence and then dies.

These results carry over without difficulty to cohomology theories h^*; in §5 we indicate the result.

§6 deals with K-theory. We report on joint work with J. Cuntz. If N is a C^*-algebra in a certain large category $\mathfrak{N}$ with $K_0(N) = \mathbf{Z}/n$, $K_1(N) = 0$, then there is a natural equivalence of homology theories

$$K_*(A;\mathbf{Z}/n) \cong K_*(A \otimes N)$$

so that, in this sense at least, $K_*(\ ;\mathbf{Z}/n)$ is unique.

The problem of introducing coefficients into a cohomology theory on spaces has attracted much attention. Our work builds upon that of Araki-Toda [**1**], Browder [**5**], Deleanu-Hilton [**9**], and Maunder [**12**]. We have learned much from conversations and correspondence with Larry Brown, Joachim Cuntz, and Jonathan Rosenberg, to whom we are most grateful.

This paper is closely related to [**16**] and inadvertently undefined terms in this paper are defined there.

1. Construction of $h_*(A;\mathbf{Z}/n)$. This section is devoted to the construction of the (cofibre) homology theory $h_*(-;\mathbf{Z}/n)$ associated to a given (cofibre) homology theory h_*.

Let Θ_n: $C_0(\mathbf{R}) \to C_0(\mathbf{R})$ denote the canonical map of degree n. Write $Cn = C\Theta_n$ for the mapping cone, with cone sequence

$$0 \to C_0(\mathbf{R}^2) \xrightarrow{i(\Theta_n)} Cn \xrightarrow{\pi(\Theta_n)} C_0(\mathbf{R}) \to 0. \tag{1.1}$$

DEFINITION 1.2. The groups $h_j(A;\mathbf{Z}/n)$ are defined by

$$h_j(A;\mathbf{Z}/n) = h_{j-2}(A \otimes Cn).$$

As Cn is commutative the tensor product is unambiguous. It is clear that for each j the correspondence $A \rightsquigarrow h_j(A;\mathbf{Z}/n)$ is a homotopy-invariant covariant functor defined on the same admissible category $\mathcal{C}$ as h_*. (We assume here and henceforth that $\mathcal{C}$ is closed under the operation $A \rightsquigarrow A \otimes Cn$.)

The long exact sequence associated to the mapping cone of the map

$$SA \equiv A \otimes C_0(\mathbf{R}) \xrightarrow{1\otimes\Theta_n} A \otimes C_0(\mathbf{R}) \equiv SA$$

has the form

(1.3)

$$\begin{tikzcd}
\arrow[r] & h_{j-2}(S^2A) \arrow[r, "(1\otimes i(\Theta_n))_*"] & h_{j-2}(A \otimes Cn) \arrow[r, "(1\otimes\pi(\Theta_n))_*"] & h_{j-2}(SA) \arrow[r, "\partial"] & h_{j-3}(S^2A) \\
& h_j(A) \arrow[u, "\sigma_2", "\cong"'] & h_j(A;\mathbf{Z}/n) \arrow[u, "\equiv"'] & h_{j-1}(A) \arrow[u, "\sigma", "\cong"'] & h_{j-1}(A) \arrow[u, "\cong", "\sigma_2"']
\end{tikzcd}$$

Define the *reduction* map

$$\rho_n: h_j(A) \to h_j(A;\mathbf{Z}/n)$$

by

$$\rho_n = (1 \otimes i(\Theta_n))_* \sigma_2.$$

Define the *Bockstein* map

$$\beta_n: h_j(A;\mathbf{Z}/n) \to h_{j-1}(A)$$

by

$$\beta_n = \sigma^{-1}(1 \otimes \pi(\Theta_n))_*.$$

LEMMA 1.4. *The composite $\sigma_2^{-1}\partial\sigma$: $h_j(A) \to h_j(A)$ is given by*

$$(\sigma_2^{-1}\partial\sigma)(x) = nx.$$

(We write this as n: $h_j(A) \to h_j(A)$ henceforth.)

Proof. Expand to the diagram

$$\begin{tikzcd} h_{j-1}(A) \arrow[r] \arrow[d, "\sigma"] & h_{j-1}(A) \arrow[d, "\sigma"] \\ h_{j-2}(SA) \arrow[r] \arrow[dr, "\partial"'] & h_{j-2}(SA) \arrow[d, "\sigma"] \\ & h_{j-3}(S^2A) \end{tikzcd}$$

It thus suffices to show that $\sigma^{-1}\partial = n$. However

$$\begin{aligned} \sigma^{-1}\partial &= (1 \otimes \Theta_n)_* && \text{by [16, 3.5b]} \\ &= n && \text{by [16, Remark 8.7].} \end{aligned}$$

□

Rewriting the long exact sequence (1.3) yields the long exact sequence

$$(1.5) \quad \longrightarrow h_j(A) \xrightarrow{n} h_j(A) \xrightarrow{\rho_n} h_j(A;\mathbf{Z}/n) \xrightarrow{\beta_n} h_{j-1}(A) \longrightarrow \cdots,$$

which is called the Bockstein exact sequence. To summarize:

PROPOSITION 1.6. *Given a (cofibre) homology theory h_*, there are canonical natural transformations ρ_n, β_n such that the Bockstein sequence (1.5) is exact.* □

PROPOSITION 1.7. *Let h_* be an (additive) (cofibre) homology theory. Then so is $h_*(-;\mathbf{Z}/n)$.*

Proof. The homotopy axiom is clear. Suppose h_* is a cofibre homology theory. Let $f\colon A \to B$ be a cofibration. Then so is $f \otimes 1\colon A \otimes Cn \to B \otimes Cn$ and there is a natural isomorphism $C(f \otimes 1) \cong Cf \otimes Cn$. The diagram

$$\begin{tikzcd}
h_j(Cf;\mathbf{Z}/n) \arrow[r, "\pi(f)_*"] \arrow[d, "\equiv"] & h_j(A;\mathbf{Z}/n) \arrow[r, "f_*"] & h_j(B;\mathbf{Z}/n) \\
h_{j-2}(Cf \otimes Cn) \arrow[d, "\cong"] & & \\
h_{j-2}(C(f \otimes 1)) \arrow[r, "\pi(f\otimes 1)_*"] & h_{j-2}(A \otimes Cn) \arrow[uu, "\equiv"] \arrow[r, "(f\otimes 1)_*"] & h_j(B \otimes Cn) \arrow[uu, "\equiv"]
\end{tikzcd}$$

commutes, the vertical maps are isomorphisms, and the lower row is exact by the cofibre axiom for h_*. Hence the upper row is exact. Thus $h_*(-;\mathbf{Z}/n)$ satisfies the cofibre axiom. The suspension axiom is obvious:

$$h_j(A;\mathbf{Z}/n) \equiv h_{j-2}(A \otimes Cn) \cong h_{j-3}(SA \otimes Cn) \cong h_{j-1}(A;\mathbf{Z}/n)$$

and, hence, $h_*(-;\mathbf{Z}/n)$ is a cofibre homology theory.

If h_* is a homology theory and

$$0 \to J \to A \to B \to 0$$

is exact, then the sequence

$$0 \to J \otimes Cn \to A \otimes Cn \to B \otimes Cn \to 0$$

is exact since Cn is nuclear. Apply h_* and one obtains the long exact sequence

$$\cdots \to h_j(J;\mathbf{Z}/n) \to h_j(A;\mathbf{Z}/n) \to h_j(B;\mathbf{Z}/n) \xrightarrow{\partial} h_{j-1}(J;\mathbf{Z}/n) \to \cdots$$

as required. Thus $h_*(-;\mathbf{Z}/n)$ is a homology theory.

If h_* is additive then there are natural isomorphisms

$$\begin{aligned}
h_j\Big(\bigoplus_i A_i;\mathbf{Z}/n\Big) &\equiv h_j\Big(\Big(\bigoplus_i A_i\Big) \otimes Cn\Big) \cong h_j\Big(\bigoplus_i (A_i \otimes Cn)\Big) \\
&\cong \bigoplus_i h_j(A_i \otimes Cn) \quad \text{since } h_* \text{ is additive} \\
&\equiv \bigoplus_i h_j(A_i;\mathbf{Z}/n)
\end{aligned}$$

as required. □

Note that Proposition 1.7 implies $h_*(-;\mathbf{Z}/n)$ satisfies the homology of a triple axiom, has an appropriate Mayer-Vietoris sequence, and (if additive) commutes with limits, by the main results of [**16**].

PROPOSITION 1.8. *There is a natural short exact sequence*

$$0 \to h_j(A) \otimes \mathbf{Z}/n \xrightarrow{\tilde{\rho}_n} h_j(A;\mathbf{Z}/n) \xrightarrow{\tilde{\beta}_n} \operatorname{Tor}\big(h_{j-1}(A),\mathbf{Z}/n\big) \to 0.$$

This is referred to as the Universal Coefficient Theorem (UCT) for $h_*(A;\mathbf{Z}/n)$.

Proof. Consider the Bockstein sequence; it unsplices as shown in the diagram below:

$$\begin{tikzcd}
h_j(A) \arrow[r, "n"] & h_j(A) \arrow[rr, "\rho_n"] \arrow[dr] & & h_j(A;\mathbf{Z}/n) \arrow[rr, "\beta_n"] \arrow[dr, "\tilde{\beta}_n"'] & & h_{j-1}(A) \\
& & \operatorname{Cok}(n) \arrow[ur, "\tilde{\pi}_n"] \arrow[dr] & & \operatorname{Ker}(n) \arrow[ur] \arrow[dr] & \\
& 0 \arrow[ur] & & 0 \quad 0 \arrow[ur] & & 0
\end{tikzcd}$$

It is routine to complete the argument, for

$$G \otimes \mathbf{Z}/n \cong \operatorname{Cok}(n\colon G \to G)$$

and

$$\operatorname{Tor}(G,\mathbf{Z}/n) \cong \operatorname{Ker}(n\colon G \to G). \qquad \square$$

2. Relations between theories; good theories. Next we consider the relations between the theories $h_*(-;\mathbf{Z}/n)$ for various n. Let $\Theta_{n,m}\colon Cm \to Cn$ be a map making the diagram

$$\begin{tikzcd}
C_0(\mathbf{R}^2) \arrow[r, "i(\Theta_n)"] \arrow[d, "S\Theta_{n/(n,m)}"] & Cm \arrow[r, "\pi(\Theta_n)"] \arrow[d, "\Theta_{n,m}"] & C_0(\mathbf{R}) \arrow[r, "\Theta_m"] \arrow[d, "\Theta_{m/(n,m)}"] & C_0(\mathbf{R}) \arrow[d, "\Theta_{n/(n,m)}"] \\
C_0(\mathbf{R}^2) \arrow[r, "i(\Theta_n)"] & Cn \arrow[r, "\pi(\Theta_n)"] & C_0(\mathbf{R}) \arrow[r, "\Theta_n"] & C_0(\mathbf{R})
\end{tikzcd}$$

commute, where (n, m) is the greatest common divisor of n and m. Define

$$\kappa_{n,m} = (1 \otimes \Theta_{n,m})_*\colon h_j(A;\mathbf{Z}/m) \to h_j(A;\mathbf{Z}/n).$$

This is evidently a natural transformation of theories.

PROPOSITION 2.1.
(1) $\beta_n\kappa_{n,m} = (m/(n,m))\beta_m$.
(2) $\kappa_{n,m}\rho_m = \rho_n(n/(n,m))$.
(3) $\kappa_{k,n}\kappa_{n,m} = (n(k,m)/((k,n)(n,m)))\kappa_{k,m}$
and, *in particular*, $\kappa_{n,n} = 1$.

Proof. These follow directly from the definitions at the level of homotopy before applying h_*. See Deutz **[10]** for details. □

PROPOSITION 2.2. *The diagram*

$$\begin{tikzcd}
0 \arrow[r] & h_j(A) \otimes \mathbf{Z}/m \arrow[r, "\tilde{\rho}_m"] \arrow[d, "\kappa'_{n,m}"] & h_j(A;\mathbf{Z}/m) \arrow[r, "\tilde{\beta}_m"] \arrow[d, "\kappa_{n,m}"] & \mathrm{Tor}(h_{j-1}(A),\mathbf{Z}/m) \arrow[r] \arrow[d, "\kappa''_{n,m}"] & 0 \\
0 \arrow[r] & h_j(A) \otimes \mathbf{Z}/n \arrow[r, "\tilde{\rho}_n"] & h_j(A;\mathbf{Z}/n) \arrow[r, "\tilde{\beta}_n"] & \mathrm{Tor}(h_{j-1}(A),\mathbf{Z}/n) \arrow[r] & 0
\end{tikzcd}$$

commutes, where κ' *and* κ'' *are induced by multiplications by* $n/(n, m)$ *and* $m/(n, m)$, *respectively.*

We omit the proof.

Note that $h_j(A) \otimes \mathbf{Z}/n$ and $\mathrm{Tor}(h_{j-1}(A), \mathbf{Z}/n)$ are Z/n-modules. One might hope, then, that $h_j(A; \mathbf{Z}/n)$ would be a $\mathbf{Z}/n$-module. This turns out to be true if n is odd. If n is even (the case $n = 2$ illustrates the difficulty) then the UCT need not split.

Recall that the Hopf map η: $S^3 \to S^2$ is a generator of $\pi_3(S^2) = \mathbf{Z}$. Writing

$$S^3 = \left\{(z_1, z_2) \in \mathbf{C}^2 \,\middle|\, |z_1|^2 + |z_2|^2 = 1\right\},$$

$$S^2 = \mathbf{C}P^1 = \mathbf{C}^2 - \{0\}/\sim \quad \text{where } (z_1, z_2) \sim (\lambda z_1, \lambda z_2), \lambda \in \mathbf{C} - \{0\},$$

then $\eta(z_1, z_2) = [z_1, z_2]$.

DEFINITION 2.3. (Deleanu and Hilton **[9]**.) A homology h_* is *good* if the Hopf map η: $S^3 \to S^2$ induces the zero homomorphism

$$h_*(A \otimes C_0(\mathbf{R}^2)) \to h_*(A \otimes C_0(\mathbf{R}^3))$$

for all C^*-algebras A in the category.

Note that η induces a map $h_j(A) \to h_{j+1}(A)$. If h_* has the property that $h_{2j+1}(A) = 0$ for all A, then h_* is good. K-theory itself is good, for the Künneth Theorem applied to the situation yields a commuting diagram

$$\begin{tikzcd}
\bigoplus_{i+j\equiv 0} K_i(A) \otimes K_j(C_0(\mathbf{R}^2)) \arrow[r, "\cong"] \arrow[d, "1 \otimes \eta^*"] & K_j(S^2A) \arrow[d, "\eta^*"] \\
\bigoplus_{i+j\equiv 0} K_i(A) \otimes K_j(C_0(\mathbf{R}^3)) \arrow[r, "\cong"] & K_j(S^3A)
\end{tikzcd}$$

and $K_*(C_0(\mathbf{R}^2)) \to K_*(C_0(\mathbf{R}^3))$ is the zero map on trivial algebraic grounds.

The difficulty in proceeding without "goodness" arises as follows. Let $h_j(A; \mathbf{Z}) = h_j(A)$ and let

$$h_j(A; G \oplus H) = h_j(A; G) \oplus h_j(A; H)$$

so $h_*(A; G)$ is defined for all finitely generated abelian groups G. Is the assignment

$$G \rightsquigarrow h_*(A; G)$$

natural in G? No, not in general, even on cyclic groups of order p^r, and there lies the difficulty. Let $\{A, B\} = \lim_{k\to\infty} [S^k A, S^k B]$ denote stable homotopy. There is a natural map

$$d\colon \{Cm, Cn\} \to \hom(\mathbf{Z}/m, \mathbf{Z}/n)$$

given by $d(f) = f_*\colon K_0(Cm) \to K_0(Cn)$. It is not an isomorphism. Barratt **[2]** shows that there is an exact sequence

$$0 \to \mathbf{Z}/m \otimes \mathbf{Z}/n \otimes \mathbf{Z}/2 \to \{Cm, Cn\} \xrightarrow{d} \mathrm{Hom}(\mathbf{Z}/m; \mathbf{Z}/n) \to 0.$$

If both m and n are even then d is not injective. The kernel is related to the map $\eta\colon S^3 \to S^2$ and thus goodness enters.

We summarize and refer to Deleanu-Hilton **[9]** for details.

PROPOSITION 2.4. *Let h_* be a homology theory and suppose h_* is good or that n is odd. Then the Universal Coefficient sequence*

$$0 \to h_j(A) \otimes \mathbf{Z}/n \to h_j(A; \mathbf{Z}/n) \to \mathrm{Tor}\big(h_{j-1}(A), \mathbf{Z}/n\big) \to 0$$

is natural in Z/n and the sequence splits (unnaturally).

Proof (Hilton-Deleanu). For good homology theories or in the world of odd primes the Universal Coefficient sequence is natural. Regard

$$h_j(A) \otimes (-) \longrightarrow h_j(A; -) \longrightarrow \mathrm{Tor}\big(h_{j-1}(A), -\big)$$

as an additive functor to a category of short exact sequences. An algebraic argument which uses the fact that $h_j(A) \otimes (-)$ is right exact implies that the Universal Coefficient sequence is pure exact. The group $\mathrm{Tor}(h_{j-1}(A), \mathbf{Z}/n)$ is the direct sum of cyclic groups, which implies that the Universal Coefficient sequence splits. □

The splitting is obtained on algebraic grounds and is certainly unnatural. However, an argument of Bodigheimer **[3]**, **[4]** carries over to produce a better splitting result. Fix a prime p. Then there is an inverse system of Universal Coefficient sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & h_j(A)\otimes \mathbf{Z}/p & \longrightarrow & h_j(A;\mathbf{Z}/p) & \longrightarrow & \mathrm{Tor}(h_{j-1}(A),\mathbf{Z}/p) & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & h_j(A)\otimes \mathbf{Z}/p^n & \xrightarrow{\tilde\rho_{p^n}} & h_j(A;\mathbf{Z}/p^n) & \xrightarrow{\tilde\beta_{p^n}} & \mathrm{Tor}(h_{j-1}(A),\mathbf{Z}/p^n) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle \kappa'_{p^n,p^{n+1}}} & & \uparrow{\scriptstyle \kappa_{p^n,p^{n+1}}} & & \uparrow{\scriptstyle \kappa''_{p^n,p^{n+1}}} & & \\
0 & \longrightarrow & h_j(A)\otimes \mathbf{Z}/p^{n+1} & \xrightarrow{\tilde\rho_{p^{n+1}}} & h_j(A;\mathbf{Z}/p^{n+1}) & \xrightarrow{\tilde\beta_{p^{n+1}}} & \mathrm{Tor}(h_{j-1}(A),\mathbf{Z}/p^{n+1}) & \longrightarrow & 0
\end{array}$$

that is, a short exact sequence of inverse systems. The following proposition asserts that the short exact sequence of inverse systems splits.

PROPOSITION 2.5 (Bodigheimer [3, 2.8]). *Let h_* be a homology theory. Let p be a prime and suppose p is odd or that h_* is good. For every $n \geq 1$ there is a homomorphism*

$$s_{p^n}\colon \mathrm{Tor}(h_{i-1}(A),\mathbf{Z}/p^n) \to h_i(A;\mathbf{Z}/p^n)$$

such that

$$\tilde\beta_{p^n}s_{p^n} = 1$$

so s_{p^n} is a splitting of the UCT for $h_*(A;\mathbf{Z}/p^n)$, and the s_{p^n} are coherent in that

$$\kappa_{p^n,p^{n+1}}s_{p^{n+1}} = s_{p^n}\kappa''_{p^n,p^{n+1}}.$$

The proof may be taken verbatim from [3]. □

Suppose we are given whole numbers n and m. Then there are maps

$$h_j(A;\mathbf{Z}/m) \xrightarrow{\kappa_{mn,m}} h_j(A;\mathbf{Z}/mn) \xrightarrow{\kappa_{n,mn}} h_j(A;\mathbf{Z}/n)$$

and one immediately is led to expect a long exact coefficient sequence analogous to that in ordinary homology of spaces. Note that $\kappa_{n,mn}\kappa_{mn,m} = (n, m)\kappa_{n,m}$ by (2.1), and $(n, m)\kappa_{n,m} = 0$ by direct computation.

PROPOSITION 2.6. *Let m and n be whole numbers. Then there is a natural transformation of theories*

$$\beta_{m,n}\colon h_j(A;\mathbf{Z}/n) \to h_{j-1}(A;\mathbf{Z}/m)$$

with the following properties:

(1) *There is a natural long exact sequence*

$$\to h_j(A;\mathbf{Z}/m) \xrightarrow{\kappa_{mn,m}} h_j(A;\mathbf{Z}/mn) \xrightarrow{\kappa_{m,mn}} h_j(A;\mathbf{Z}/n) \xrightarrow{\beta_{m,n}} h_{j-1}(A;\mathbf{Z}/m) \to \cdots .$$

(2) *The diagram*

$$\begin{tikzcd} h_j(A;\mathbf{Z}/n) \arrow[r, "\beta_n"] \arrow[dr, "\beta_{m,n}"'] & h_{j-1}(A) \arrow[d, "\rho_m"] \\ & h_{j-1}(A;\mathbf{Z}/m) \end{tikzcd}$$

commutes (*providing an alternate definition of* $\beta_{m,n}$).

(3) $\beta^2_{n,n}$: $h_j(A;\mathbf{Z}/n) \to h_{j-2}(A;\mathbf{Z}/n)$ *is the zero homomorphism.*

Note that (2) implies (3) since

$$\beta^2_{n,n} = \rho_n\beta_n\rho_n\beta_n = 0 \qquad \text{since } \beta_n\rho_n = 0.$$

Proof. We apply Verdier's axiom [**16**, 2.10] to the commutative triangle

$$\begin{tikzcd} A \otimes C_0(\mathbf{R}) \arrow[rr, "1\otimes\Theta_{mn}"] \arrow[dr, "1\otimes\Theta_m"'] & & A \otimes C_0(\mathbf{R}) \\ & A \otimes C_0(\mathbf{R}) \arrow[ur, "1\otimes\Theta_n"'] & \end{tikzcd}$$

and obtain a cofibre sequence of the form

$$\longrightarrow A \otimes SCn \xrightarrow{\gamma} A \otimes Cm \xrightarrow{1\otimes\Theta_{mn,m}} A \otimes Cmn \xrightarrow{1\otimes\Theta_{n,mn}} A \otimes Cn$$

such that the map γ is the composite

$$A \otimes SCn \xrightarrow{1\otimes S\pi(\Theta_n)} A \otimes C_0(\mathbf{R}^2) \xrightarrow{1\otimes i(\Theta(m))} Cm.$$

Apply the functor h_* to obtain the long exact sequence

$$\longrightarrow h_j(A;\mathbf{Z}/m) \xrightarrow{(1\otimes\Theta_{mn,m})_*} h_j(A;\mathbf{Z}/mn) \xrightarrow{(1\otimes\Theta_{n,mn})_*} h_j(A;\mathbf{Z}/n) \xrightarrow{\gamma_*} h_{j-1}(A;\mathbf{Z}/m) \longrightarrow$$

Since $\kappa_{r,s} = (1 \otimes \Theta_{r,s})_*$, if we *define* $\beta_{m,n} = \gamma_*$ then the exact sequence (1) has been established. Furthermore,

$$\beta_{m,n} = \gamma_* = (1 \otimes i\Theta(m))_*(1 \otimes S\pi\Theta(n))_* = \rho_m\beta_n,$$

so (2) is also satisfied and the proof is complete. □

3. Higher order homology operations. Let us write

$$b_p = \beta_{p,p}\colon h_j(A;\mathbf{Z}/p) \to h_{j-1}(A;\mathbf{Z}/p)$$

for a fixed prime p. Suppose we are given an element $x \in h_j(A;\mathbf{Z}/p)$ such that $b_p(x) = 0$. Then we would expect some sort of secondary homology operation to be defined on x, associated with the relation $b_p^2 = 0$, and having some connection with $\beta_{p^2}\colon h_j(A;\mathbf{Z}/p^2) \to h_{j-1}(A)$, since $x = \kappa_{p,p^2}(y)$ for some $y \in h_j(A;\mathbf{Z}/p^2)$. If this secondary operation vanishes on x, a tertiary operation ought to be defined, and so on.

This is indeed the case, just as in the topological setting. Following Maunder **[12]**, we define a series of operations $\bar{\beta}_{p^s}$ $(s \geq 1)$ as follows. Given $x \in h_j(A;\mathbf{Z}/p)$, suppose $\beta_p(x) \in h_{j-1}(A)$ is divisible by p^{s-1}. Define $\bar{\beta}_{p^s}(x)$ to be the set of $\rho_p(y)$, for all $y \in h_{j-1}(A)$ such that $p^{s-1}y = x$. It is best to think of $\bar{\beta}_{p^s}$ as a relation

$$\begin{tikzcd} h_j(A;\mathbf{Z}/p) \arrow[r, "\beta_p"] & h_{j-1}(A) & \\ & h_{j-1}(A) \arrow[u, "p^{s-1}"'] \arrow[r, "\rho_p"] & h_{j-1}(A;\mathbf{Z}/p) \end{tikzcd}$$

with $\bar{\beta}_{p^s}(x)$ defined for some subset of $h_j(A;\mathbf{Z}/p)$ and for each x having value a certain subset $\bar{\beta}_{p^s}(x) \subset h_{j-1}(A;\mathbf{Z}/p)$.

To show the connection between $\bar{\beta}_{p^s}$ and b_{p^s}, we note first that since $\rho_{p^{s-1}}\beta_p = \beta_{p^{s-1},p}$, $\bar{\beta}_{p^s}(x)$ is defined if and only if $\beta_{p^{s-1},p}(x) = 0$, i.e., $x = \kappa_{p,p^s}(z)$ for some $z \in h_j(A;\mathbf{Z}/p^s)$. Thus the domain of $\bar{\beta}_{p^s}$ is exactly the kernel of $\beta_{p^{s-1},p}$

$$\beta_{p^{s-1},p}\colon h_j(A;\mathbf{Z}/p) \to h_j(A;\mathbf{Z}/p^{s-1}),$$

which is the image of the map

$$\kappa_{p,p^s}\colon h_j(A;\mathbf{Z}/p^s) \to h_j(A;\mathbf{Z}/p).$$

Now

$$p^{s-1}\beta_{p^s} = \beta_p\kappa_{p,p^s} \qquad \text{by (2.1)}$$

so that

$$p^{s-1}\beta_{p^s}(z) = \beta_p(x),$$

and thus

$$\rho_p\beta_{p^s}(z) \in \bar{\beta}_{p^s}(x).$$

In fact we could have defined $\bar{\beta}_{p^s}(x)$ to be the set

$$\bar{\beta}_{p^s}(x) = \{\kappa_{p,p^s} b_{p^s}(z) \mid \kappa_{p,p^s}(z) = x\}$$

using the relation

$$\begin{array}{ccc} h_j(A;\mathbf{Z}/p) & \xleftarrow{\kappa_{p,p^s}} & h_j(A;\mathbf{Z}/p^s) & & \\ & & \big\downarrow b_{p^s} & & \\ & & h_{j-1}(A;\mathbf{Z}/p^s) & \xrightarrow{\kappa_{p,p^s}} & h_j(A;\mathbf{Z}/p) \end{array}$$

since with either definition, the indeterminacy of $\bar{\beta}_{p^s}(x)$ is

$$\rho_p \beta_{p^{s-1}} h_j(A;\mathbf{Z}/p^{s-1}) \tag{3.2}$$

(using the relation $\beta_p \kappa_{p,p^{s-1}} = \beta_{p^{s-1}}$).

The operations $\bar{\beta}_{p^s}$ ($s \geq 1$) have all the properties one would expect of higher operations.

THEOREM 3.3. *The operations $\bar{\beta}_{p^s}$ have the following properties:*

(1) $\bar{\beta}_p = b_p$.

(2) $\bar{\beta}_{p^s}(x)$ *is defined for those elements* $x \in h_j(A;\mathbf{Z}/p)$ *such that* $\bar{\beta}_{p^r}(x)$ *contains zero, for* $s > r > 0$.

(3) *If x is such an element, then $\bar{\beta}_{p^s}(x)$ is an equivalence class of elements of $h_{j-1}(A;\mathbf{Z}/p)$, two elements being equivalent if they differ by an element of* $\operatorname{Im} \bar{\beta}_{p^{s-1}}$ *(the set of all elements in $\bar{\beta}_{p^{s-1}}(y)$, as y runs over all elements of $h_j(A;\mathbf{Z}/p)$ such that $\bar{\beta}_{p^{s-1}}(y)$ is defined.)*

(4) *Given a map $f\colon A \to B$ and x as in* (2), *then*

$$f_*[\bar{\beta}_{p^s}(x)] \subset \bar{\beta}_{p^s}(f_*(x)).$$

(5) *If $\sigma\colon h_t(A;\mathbf{Z}/p) \to h_{t-1}(SA;\mathbf{Z}/p)$ is the suspension isomorphism then*

$$\sigma[\bar{\beta}_{p^s}(x)] = [\bar{\beta}_{p^s}(\sigma x)].$$

The proofs are elementary except for (3) which follows from our discussion of indeterminacy at (3.2). □

In the following section these operations will appear as differentials in a Bockstein spectral sequence.

REMARK 3.4. Our work shows that $h_*(-\mathbf{Z}/p)$ does have at least one possibly non-trivial homology operation, namely

$$b_p\colon h_j(A;\mathbf{Z}/p) \to h_{j-1}(A;\mathbf{Z}/p).$$

For example, take $h = K$, $A = Cp$. Then b_p corresponds to

$$b_p^{(0)}\colon K_0(Cp; \mathbf{Z}/p) \to K_1(Cp; \mathbf{Z}/p),$$

and

$$b_p^{(1)}\colon K_1(Cp; \mathbf{Z}/p) \to K_0(Cp; \mathbf{Z}/p).$$

The UCT implies $K_j(Cp; \mathbf{Z}/p) \cong \mathbf{Z}/p, j = 0, 1$. The operation $b_p^{(j)}$ factors through $K_{j-1}(Cp)$, and $K_1(Cp) = 0$. Thus $b_p^0 = 0$. What about $b_p^{(1)}$? Since $b_p^{(1)} = \rho_p \beta_p$ and ρ_p is an isomorphism in this case, we must study β_p: $K_1(Cp; \mathbf{Z}/p) \to K_0(Cp)$. But $\operatorname{Ker} \beta_p = \operatorname{Im}(\rho_p\colon K_1(Cp) \to K_1(Cp; \mathbf{Z}/p)) = 0$ since $K_1(Cp) = 0$. Thus β_p is an isomorphism. So the operation $b_p^{(1)}$: $K_1(Cp; \mathbf{Z}/p) \to K_0(Cp; \mathbf{Z}/p)$ is an isomorphism. Similar reasoning proves that

$$b_p\colon K_1\big(\mathcal{O}_{p+1}; \mathbf{Z}/p\big) \to K_0\big(\mathcal{O}_{p+1}; \mathbf{Z}/p\big)$$

is an isomorphism, where $\mathcal{O}_{p+1}$ is the Cuntz algebra. Note that if $x \in K_0(Cp; \mathbf{Z}/p)$ or if $x \in K_0(\mathcal{O}_{p+1}; \mathbf{Z}/p)$, then the higher operations $\bar{\beta}_{p^s}(x)$ are inductively defined and contain zero, since x is the reduction of an integral class.

4. The Bockstein spectral sequence. In the simplest cases, $h_*(A)$ is a free abelian group. However, one generally expects torsion in $h_*(A)$, and there is no reason in general not to expect higher (p^r) torsion for various primes.

The Bockstein spectral sequence (one for each prime p) focuses attention upon and sorts out p^r-torsion. In the special case where every element of $h_*(A)$ has order p^r for some r and r is bounded, the spectral sequence has $E^1 = h_*(A; \mathbf{Z}/p)$ (which has elements only of order p if h_* is good or p is odd) and E^∞ is zero. Roughly, summands of order p die going to E^2, summands of order p^2 die going to E^3, etc. If $h_*(A)$ has no p^i-torsion for $i \geq r$ then $E^r = E^\infty$.

Here is a statement of the basic result. Fix a prime p. We assume h_* is a cofibre homology theory such that the p-primary component of the torsion subgroup $Th_j(A)$ of $h_j(A)$ is finite for all A in the category, and that either p is odd or h_* is good. (This is to avoid convergence difficulties.)

THEOREM 4.1. *There exists a spectral sequence* $\{E^r, d^r\colon r \geq 1\}$ *with differentials* $d^r\colon E_j^r \to E_{j-1}^r$ *such that*:

(1) $E_j^1 = h_j(A; \mathbf{Z}/p)$.

(2) *For each j there is a short exact sequence*

$$0 \to Th_j(A) + ph_j(A) \to h_j(A) \to E_j^\infty \to 0,$$

so, in particular,

$$E_j^\infty \cong \big(h_j(A)/Th_j(A)\big) \otimes \mathbf{Z}/p.$$

(3) *The spectral sequence is natural; that is, a map $f\colon A \to B$ induces homomorphisms $f^r\colon E^r(A) \to E^r(B)$ $(r \geq 1)$ which commute with differentials, $f^1 = f_*\colon h_*(A;\mathbf{Z}/p) \to h_*(B;\mathbf{Z}/p)$, and f^∞ is induced by $f_*\colon h_*(A) \to h_*(B)$.*

(4) *The spectral sequence is stable; that is, the suspension isomorphism σ induces an isomorphism of spectral sequences*

$$E_s^r(A) \cong E_{s-1}^r(SA).$$

Proof. We have an exact triangle

$$\begin{tikzcd} h_*(A) \arrow[rr, "p"] & & h_*(A) \arrow[dl, "\rho_p"] \\ & h_*(A; Z/p) \arrow[ul, "\beta_p"] & \end{tikzcd}$$

The associated exact couple is defined as follows:

$$E_j^1 = h_j(A;\mathbf{Z}/p),$$
$$D_j^1 = h_j(A);$$
$$i^1\colon D_j^1 \to D_j^1$$

is given by $i^1(x) = px$,

$$j^1\colon D_j^1 \to E_j^1$$

is given by ρ_p, and

$$\partial^1\colon E_j^1 \to D_{j-1}^1$$

is given by β_p.

The associated spectral sequence is the Bockstein spectral sequence. Properties (1), (3) and (4) are immediate from the construction.

In order to verify (2) and for subsequent propositions, we examine the terms more carefully. The first differential d^1 is easy to identify:

$$\begin{aligned} d^1 &= j^1\partial^1 \qquad \text{for any exact couple} \\ &= \rho_p\beta_p = b_p. \end{aligned}$$

Recall that the differential $d^r\colon E^r_j \to E^r_{j-1}$ is induced by the map $\rho_p^{(r)}\beta_p^{(r)}$. In particular, if $x \in h_j(A; \mathbf{Z}/p) = E^1_j$ is the reduction of an integral class then $d^r x = 0$ for all r; x is a permanent cycle. Thus ρ^1_p: $h_j(A) \to E^1_j$ induces maps $\rho_{(r)}\colon h_j(A) \to E^r_j$. The E^r term is given by

$$E^r_j = \beta_p^{-1}\big(p^{r-1}h_{j-1}(A)\big)/\rho_p(\operatorname{Ker} p^{r-1}) \tag{4.2}$$

and

$$D^r_j = p^{r-1}h_j(A). \tag{4.3}$$

For $r \geq \bar{r}$ the map $p\colon D^{\bar{r}} \to D^r$ is an injection, and this implies $\beta_p^{(\bar{r})} \equiv 0$, so $E^{\bar{r}} = E^\infty$. In that case $\operatorname{Ker} p^{\bar{r}-1} = T_p h_j(A)$ (where T_p denotes p-primary torsion) and

$$\rho_p(\operatorname{Ker} p^{\bar{r}-1}) = \rho_p\big(Th_j(A)\big)$$

since $h_*(A;\, Z/p)$ has no torsion prime to p. Thus the sequence

$$0 \to \rho_p\big(Th_j(A)\big) \to \beta_p^{-1}\big(p^{r-1}h_{j-1}(A)\big) \to E^\infty_j \to 0 \tag{4.4}$$

is exact. An easy exact sequence argument yields (2). □

PROPOSITION 4.5. *Let $0 \neq x \in E^r_j$. Then there is an element $w \in h_j(A; \mathbf{Z}/p^r)$ such that $x = [\kappa_{p,p^r} w]$.*

Proof. Represent x as $x \in \beta_p^{-1}(p^{r-1}h_{j-1}(A))$. That is, $x \in h_j(A; \mathbf{Z}/p)$ with $\beta_p(x) = p^{r-1}y \in p^{r-1}h_{j-1}(A)$. Then $\beta_{p^{r-1},p}\colon h_j(A; \mathbf{Z}/p) \to h_{j-1}(A; \mathbf{Z}/p^{r-1})$ kills x:

$$\beta_{p^{r-1},p}(x) = \rho_{p^{r-1}}\beta_p(x) = \rho_{p^{r-1}}p^{r-1}y = 0.$$

Thus $x \in \operatorname{Ker}(\beta_{p^{r-1},p})$ which coincides with the image of the map κ_{p,p^r}: $h_j(A; \mathbf{Z}/p^r) \to h_j(A; \mathbf{Z}/p)$. Hence $x = \kappa_{p,p^r} w$ as required. □

PROPOSITION 4.6. *If y generates a direct $\mathbf{Z}/p^r$ summand in $h_j(A)$ then $\rho_{(r)}(y) \neq 0$ in E^r_{j+1}.*

Proof. The element $p^{r-1}y \in p^{r-1}h_j(A)$ is non-zero, and $p(p^{r-1}y) = 0$, so there is some $z \in h_{j+1}(A; \mathbf{Z}/p)$ with $\beta_p(z) = p^{r-1}y$. Thus $z \in \beta_p^{-1}(p^{r-1}h_*(A))$ represents y in E^r_{j+1}. Is $\rho_{(r)}(y) \equiv [z] = 0$ in E^r_{j+1}? Suppose that $z \in \rho_p(\operatorname{Ker} p^{r-1})$, so that $[z] = 0$. Then there is an element $w \in h_{j+1}(A)$ with $p^{r-1}w = 0$ and $\rho_p(w) = z$. But then

$$p^{r-1}y = \beta_p(z) = \beta_p\rho_p(w) = 0 \qquad (\beta_p\rho_p = 0),$$

which is a contradiction. Thus $\rho_{(r)}(y) \neq 0$. □

COROLLARY 4.7. *$h_*(A)$ has no p^i-torsion for $i \geq r$ if and only if $E^r = E^\infty$.* □

THEOREM 4.8. *The differentials in the spectral sequence are given by the Bockstein operations $\bar{\beta}_{p^r}$.*

Proof. Let $x \in p^{r-1}h_*(A)$ be of the form $x = p^{r-1}y$. Then $\rho_{(r)}(x)$ is the coset of $\rho_p(y)$, independent of the choice of y. If $\alpha \in E^r$ is represented by $z \in h_*(A; \mathbf{Z}/p)$, then $\beta_p^{(r)}(\alpha) = \beta_p(z) \in p^{r-1}h_*(A)$. Thus

$$d^r(\alpha) = \rho_p^{(r)}\beta_p^{(r)}(\alpha) = \rho_p^{(r)}\beta_p(z),$$

which is exactly the coset of $\bar{\beta}_{p^r}(x)$, as required. □

REMARK 4.9. To illustrate the spectral sequence, take $h = K$ and $A = \mathcal{O}_{p+1}$ or $A = Cp$. Then $K_*(A)$ consists entirely of torsion elements, and $E^\infty \equiv 0$. By Remark 3.4 we see that $E_j^1 = \mathbf{Z}/p$ for all j, so $E^1 \neq E^\infty$. Using Theorem 4.8 and Remark 3.4 we see that the differential

$$d^1: E_{2j}^1 \to E_{2j-1}^1$$

is identically zero, whereas the differential

$$d^1: E_{2j+1}^1 \to E_{2j}^1$$

is an isomorphism. Thus $E^2 = H(E^1, d^1) \equiv 0$, and $E^1 \neq E^2 = E^\infty$. The fact that there is no p^2-torsion in $K_*(A)$ corresponds by Corollary 4.7 to the fact that $E^2 = E^\infty$.

5. Cohomology. The situation in cohomology is so similar to that in homology that very little need be said. Suppose h^* is a cofibre cohomology theory. Define

$$h^n(A; \mathbf{Z}/n) = h^{n-1}(A \otimes Cn).$$

The cone exact sequence

$$0 \to S^2A \xrightarrow{1 \otimes i(\Theta_n)} A \otimes Cn \xrightarrow{\pi(\Theta_n)} SA \to 0$$

yields a long exact sequence

$$\begin{tikzcd}
\arrow[r] & h^{j-1}(SA) \arrow[r] \arrow[d, "\cong"] & h^{j-1}(A \otimes Cn) \arrow[r] \arrow[d, "\equiv"'] & h^{j-1}(S^2A) \arrow[r] \arrow[d, "\cong"] & h^j(SA) \arrow[r] \arrow[d, "\cong"] & {} \\
\arrow[r] & h^j(A) \arrow[r, "\rho^n"] & h^j(A; \mathbf{Z}/n) \arrow[r, "\beta^n"] & h^{j+1}(A) \arrow[r, "n"] & h^{j+1}(A) \arrow[r] & {}
\end{tikzcd}$$

with Bockstein and reduction maps as required. If h^* is additive then

$$\begin{aligned} h^*\big(\oplus A_j; \mathbf{Z}/n\big) &= h^*\big(\big(\oplus A_j\big) \otimes Cn\big) = h^*\big(\oplus \left(A_j \otimes Cn\right)\big) \\ &= \prod_j h^*\left(A_j \otimes Cn\right) = \prod_j h^*\left(A_j; \mathbf{Z}/n\right). \end{aligned}$$

Thus $h^*(-;\, Z/n)$ is an additive cohomology theory.

Similarly the entire discussion of higher Bockstein operations and the Bockstein spectral sequence goes through. We omit details, except to note that $d_r\colon E_r^j \to E_r^{j+1}$ corresponds to $\rho^p_{(j)}\beta^p_{(j)}$ as in homology. The spectral sequence converges, given finiteness assumptions, to

$$E_\infty^j = \big(h^j(A)/Th^j(A)\big) \otimes \mathbf{Z}/p.$$

6. Uniqueness of $K_*(-;\mathbf{Z}/n)$. The C^*-algebra Cn is clearly not the only C^*-algebra which may be used to define $h_*(A;\mathbf{Z}/n)$. For instance, Cn could be replaced by $Cn \oplus D$, where D is a contractible C^*-algebra, or (by adjusting the grading) Cn could be replaced by SCn. Consider the problem from the opposite perspective. Let N be a nuclear C^*-algebra, and suppose there is a natural long exact Bockstein sequence of the form

$$\to h_j(A) \xrightarrow{n} h_j(A) \to h_{j-2}(A \otimes N) \to h_{j-1}(A) \to \cdots. \tag{6.1}$$

Then there is a natural short exact sequence

$$0 \to h_j(A) \otimes \mathbf{Z}/n \to h_{j-2}(A \otimes N) \to \operatorname{Tor}\big(h_{j-1}(A), \mathbf{Z}/n\big) \to 0. \tag{6.2}$$

Suppose (6.2) splits unnaturally. Then there is an *unnatural* isomorphism

$$h_j(A;\mathbf{Z}/n) \cong h_{j-2}(A \otimes N). \tag{6.3}$$

Can this result be improved to yield a *natural* isomorphism? Not without more structure or more information.

In the special case of K-theory it turns out that much more can be said. Let $\mathfrak{N}$ be the smallest category of C^*-algebras which contains the separable Type I C^*-algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and (reduced) crossed products by $\mathbf{Z}^k$ and by $\mathbf{R}$. The following theorem is joint work with J. Cuntz.

THEOREM 6.4. *Let $N \in \mathfrak{N}$ with $K_0(N) = \mathbf{Z}/n$ and $K_1(N) = 0$. Then there is a natural equivalence of homology theories*

$$K_*(A;\mathbf{Z}/n) \cong K_*(A \otimes N).$$

Note that since $N \in \mathfrak{N}$ the Künneth theorem [**15**] applies to $K_*(A \otimes N)$ and yields the natural short exact sequence

(6.5) $$0 \to K_j(A) \otimes \mathbf{Z}/n \to K_j(A \otimes N) \to \operatorname{Tor}\big(K_{j-1}(A), \mathbf{Z}/n\big) \to 0$$

for $j \in \mathbf{Z}/2$ which splits unnaturally, by Deutz [**10**]. Thus if one omits the word "natural" then the theorem is true with no further proof required.

Let $\mathcal{O}_n$ and $\mathcal{O}_\infty$ denote the Cuntz algebras [**6**], [**7**]. Recall that these lie in $\mathfrak{N}$, that $K_0(\mathcal{O}_{n+1}) = \mathbf{Z}/n$, $K_1(\mathcal{O}_{n+1}) = 0$, and that $A \to A \otimes \mathcal{O}_\infty$ defined by $a \rightsquigarrow a \otimes 1$ induces an isomorphism $K_j(A) \xrightarrow{\cong} K_j(A \otimes \mathcal{O}_\infty)$.

PROPOSITION 6.6 (*J. Cuntz*). *Let N be a C^*-algebra and let $z \in K_0(N)$ be some element with $nz = 0$.*

(a) *If N is unital then there is a map $f\colon \mathcal{O}_{n+1} \to N \otimes \mathcal{O}_\infty$ such that $f_*[1] = z$.*

(b) *If N is not unital then there is a map $f\colon \mathcal{O}_{n+1} \to N^+ \otimes \mathcal{O}_\infty$ such that $f_*[1] = z$.*

Proof. With no loss of generality we may assume $N \cong N \otimes \mathcal{O}_\infty$. Suppose N is unital. Given projections p, q, write $p \prec q$ if there is some projection q' with $p \sim q' \leq q$. Let $\mathcal{P}$ be the collection of projections $p \in N \cong N \otimes \mathcal{O}_\infty$ with $1 \otimes 1 \prec p$. Then [**7**, Theorem 1.4] every element of $K_0(N)$ is represented by some projection in $\mathcal{P}$. Two projections in $\mathcal{P}$ define the same element in $K_0(N)$ if and only if they are equivalent. Moreover, for any sequence $\{z_i\}$ in $K_0(B)$, there is a sequence $\{e_i\}$ of pairwise orthogonal projections in $\mathcal{P}$ with $[e_i] = z_i$.

Suppose $z \in K_0(N)$ with $nz = 0$. Find pairwise orthogonal projections $\{e_1, \ldots, e_{n+1}\}$ in $\mathcal{P}$ with $[e_j] = z$ and with

$$e_j \sim \sum_{i=1}^{n+1} e_i \equiv e.$$

Thus there are elements $x_1, \ldots, x_{n+1} \in N$ with $x_j^* x_j = e$ and $x_j x_j^* = e_j$. The map $S_j \mapsto x_j$ $(j = 1, \ldots, n)$ extends to a homomorphism $f\colon \mathcal{O}_{n+1} \to N$ with $f_*[1] = z$.

If N is non-unital then the first part of the proof yields a homomorphism $f\colon \mathcal{O}_{n+1} \to N^+ \otimes \mathcal{O}_\infty$ such that $\varphi_*[1] = z$. The map f_* factors as required as

$$\begin{tikzcd}
K_0(\mathcal{O}_{n+1}) \arrow[r, "f_*"] \arrow[dr, dashed] & K_0(B^+ \otimes \mathcal{O}_\infty) \\
& K_0(B \otimes \mathcal{O}_\infty) \arrow[u]
\end{tikzcd}$$

since $z \in K_0(B \otimes \mathcal{O}_\infty)$, and this completes the proof. $\square$

LEMMA 6.7. *Let $f\colon N_1 \to N_2$ be a map of C^*-algebras in $\mathfrak{N}$ such that $f_*\colon K_*(N_1) \to K_*(N_2)$ is an isomorphism. Then*

$$(1 \otimes f)_*\colon K_*(A \otimes N_1) \to K_*(A \otimes N_2)$$

is a natural equivālence of theories for all C^*-algebras A.

Proof. This is immediate from the Künneth theorem [**15**]. □

All the tools have been assembled to prove Theorem 6.4. Let $N \in \mathfrak{N}$ be unital with $K_0(N) = \mathbf{Z}/n$, $K_1(N) = 0$. There is a sequence of maps

$$Cn \xrightarrow{f_1} Cn \otimes \mathcal{O}_\infty \xleftarrow{f_2} \mathcal{O}_{n+1} \xrightarrow{f_3} N \otimes \mathcal{O}_\infty \xleftarrow{f_4} N$$

and each f_i induces an isomorphism on K-theory. The lemma implies that there is a sequence of *natural* isomorphisms

$$\begin{tikzcd}
K_*(A;\mathbf{Z}/n) \arrow[r, "(1\otimes f_1)_*"] \arrow[drr, dashed] & K_*(A \otimes Cn \otimes \mathcal{O}_\infty) & K_*(A \otimes \mathcal{O}_{n+1}) \arrow[l, "(1\otimes f_2)_*"'] \arrow[d, "(1\otimes f_3)_*"] \\
 & & K_*(A \otimes N \otimes \mathcal{O}_\infty) \\
 & & K_*(A \otimes N) \arrow[u, "(1\otimes f_4)_*"']
\end{tikzcd}$$

and the dotted composite is the required equivalence. The argument in the non-unital case is essentially the same with the minor modification required by the use of (6.6b) rather than (6.6a). □

REMARK 6.8. For some purposes it might be more natural to start with $\mathcal{O}_{n+1}$ as the basic C^*-algebra rather than with Cn, as this proof illustrates. At this time our knowledge of Cn is more extensive than our knowledge of $\mathcal{O}_{n+1}$, but it may be that in a few years $\mathcal{O}_{n+1}$ will prove to be much more convenient. The first real test is in the analysis of admissible multiplications [**1**].

Here is a generalization of Theorem 6.4 to the homology theories generated by the Kasparov functors.

THEOREM 6.9. *Fix C^*-algebras $D \in \mathfrak{N}$ and $N \in \mathfrak{N}$ with $K_0(N) = \mathbf{Z}/n$, $K_1(N) = 0$. Then for all separable nuclear C^*-algebras A there is an equivalence of theories*

$$KK_*(D, A \otimes Cn) \xrightarrow{\cong} KK_*(D, A \otimes N).$$

Proof. The sequence of K_*-isomorphisms

$$Cn \to Cn \otimes \mathcal{O}_\infty \leftarrow \mathcal{O}_{n+1} \to N \otimes \mathcal{O}_\infty \leftarrow N$$

induces a sequence of K_*-isomorphisms

$$(6.10) \quad A \otimes Cn \to A \otimes Cn \otimes \mathcal{O}_\infty \leftarrow A \otimes \mathcal{O}_{n+1} \to A \otimes N \otimes \mathcal{O}_\infty \leftarrow A \otimes N$$

for all C^*-algebras A, since $\mathcal{O}_{n+1}$, $\mathcal{O}_\infty$, N, and Cn are in $\mathfrak{N}$. The Universal Coefficient Theorem for the KK-groups **[13]** asserts that there is a natural short exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbf{Z}}\big(K_*(D), K_*(B)\big) \to KK_*(D, B) \to \operatorname{Hom}\big(K_*(D), K_*(B)\big) \to 0$$

for appropriate B. Feeding in the sequence (6.10) and applying the five lemma yields the result. □

REMARK. An (additive) homology operation ν: $h \to k$ is a natural transformation of functors which commutes with suspension/boundary homomorphisms. There are some very interesting cohomology operations ψ^k on $K^*(X)$, called the Adams operations. These do not extend to operations on K_*. In fact there are *no* non-trivial operations $K_* \to K_0$. Here is a proof, due to J. F. Adams. Let ψ: $K_* \to K_*$ be an operation and let $x \in K_0(A)$. Then $x = f_*y$ for some $y \in K_0(\mathbf{C})$ and f: $\mathbf{C} \to A \otimes \mathcal{K}$ by geometric realization **[15]**. Then $\psi(x) = \psi(f_*(y)) = f_*\psi(y) = f_*(\lambda y) = \lambda x$ for some integer λ.

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