MULTIPLE SERIES ROGERS-RAMANUJAN
TYPE IDENTITIES

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It is shown how each of the classical identities of Rogers-Ramanujan type can be embedded in an infinite family of multiple series identities. The method of construction is applied to four of L. J. Rogers' elegant series related to the quintuple product identity. Other applications are also presented.

1. Introduction. The Rogers-Ramanujan identities [6; Ch. 7] are given analytically as follows:

$$(1.1) \qquad 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

$$(1.2) \qquad 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Numerous authors [18], [19], [16], [12], [13], [22], [23] in the first half of this century found related results connecting q-series resembling those in (1.1) and (1.2) with various modular forms and functions. The culmination of their efforts may be found in the two papers of L. J. Slater [22], [23] wherein over 130 such identities are cataloged.

Within the last decade it has been observed that if one extends the q-series allowed to multiple series then infinite families of Rogers-Ramanujan type identities can be found [4], [5], [14], [17], [25], [26]. For example [4],

$$(1.3) \qquad \sum_{n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}}(q)_{n_{k-2}-n_{k-3}} \cdots (q)_{n_1}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm k \pmod{2k+1}}}^{\infty} (1-q^n)^{-1},$$

where $(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$. Such series have opened several new combinatorial vistas related to such identities [8], [14].

However other identities originally found by Rogers have not been fully generalized to infinite families (work in this direction is given in [17]). The following four results of L. J. Rogers (for a history and combinatorial applications see [11]) are perhaps the best examples of "isolated" Rogers-Ramanujan type identities:

$$
(1.4) \quad 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^{2n})}
$$

$$
= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{2n+1})(1 - q^{20n+4})(1 - q^{20n+8})}
$$

$$
= \prod_{n=0}^{\infty} \frac{(1 - q^{10n+10})(1 - q^{10n+2})(1 - q^{10n+8})(1 - q^{20n+6})(1 - q^{20n+14})}{(1 - q^{n+1})}
$$

$$
(1.5) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(1 - q)(1 - q^2) \cdots (1 - q^{2n+1})}
$$

$$
= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{2n+1})(1 - q^{20n+8})(1 - q^{20n+12})}
$$

$$
= \prod_{n=0}^{\infty} \frac{(1 - q^{10n+10})(1 - q^{10n+4})(1 - q^{10n+6})(1 - q^{20n+2})(1 - q^{20n+18})}{(1 - q^{n+1})}
$$

$$
(1.6) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1 - q)(1 - q^2) \cdots (1 - q^{2n})} = \prod_{\substack{n=1 \\ n \neq 0, \pm 1, \pm 8, \pm 9, 10 \ (\mathrm{mod}\, 20)}}^{\infty} (1 - q^n)^{-1}
$$

$$
= \prod_{n=0}^{\infty} \frac{(1 - q^{20n+10})(1 - q^{10n+1})(1 - q^{10n+9})(1 - q^{20n+8})(1 - q^{20n+12})}{(1 - q^{n+1})}
$$

$$
(1.7) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1 - q)(1 - q^2) \cdots (1 - q^{2n+1})} = \prod_{\substack{n=1 \\ n \neq 0, \pm 3, \pm 4, \pm 7, 10 \ (\mathrm{mod}\, 20)}}^{\infty} (1 - q^n)^{-1}
$$

$$
= \prod_{n=0}^{\infty} \frac{(1 - q^{10n+10})(1 - q^{10n+3})(1 - q^{10n+7})(1 - q^{20n+16})(1 - q^{20n+4})}{(1 - q^{n+1})} .
$$

One reason these four are considered noteworthy is because of the appearance of the Quintuple Product Identity [3; p. 466] in each second line. This is especially intriguing in light of the fact that all previous infinite family extensions [4], [5], [14] have involved only the Triple Product of Jacobi [6; p. 21].

The object of this paper is to show that *all* of the 130 identities given in Slater's compendium [21] can be embedded in infinite families of results. In particular, identities (1.4)–(1.7) can be generalized to infinite families. For example, the generalization of (1.4) is

$$(1.8) \qquad \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}(q)_{2n_1}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \, \pm(k+1) \,(\mathrm{mod}\, 6k+4) \\ n \not\equiv \pm(4k+2) \,(\mathrm{mod}\, 12k+8)}}^{\infty} (1 - q^n)^{-1}.$$

In fact the extensions of (1.4)–(1.7) presented in this paper all rely on the Quintuple Product Identity (see (3.5), (3.9), (3.13)).

In §2, we shall prove Theorem 1 which provides the means for these extensions. In §3 we prove the generalizations of (1.4)–(1.7). In a very natural way certain questions concerning the "reducibility" of Rogers-Ramanujan type identities arise; some initial observations on this topic are provided in §4. In §5 we explore how Theorem 1 can be applied to several identities that seem to fall naturally outside of the domain effectively outlined by L. J. Slater in [22] and [23].

The results we prove are most closely related to S. Milne's striking and elegant multiple series expansion for the general Rogers-Selberg function [17]. Indeed Milne sketches applications using the quintuple product identity [17; pp. 641–642], and presumably the specializations he gives in [17; eq. (3.12) abcd] make his function $_{(a)}\phi_{1,1}(x; q)$ summable to yield our (1.3), (3.5), (3.9) and (3.13). The primary advantage of our approach is that it allows us to avoid having to sum special cases of $_{(a)}\phi_{1,1}(x; q)$ and instead lets us quote known summations from Slater's extended tables [22]. Indeed one might say that Milne's work represents the full generalization of Rogers' second proof of (1.1) and (1.2) [20] while the results in this paper fully generalize Rogers' first proof [18].

2. The main theorem.   The germ of our work lies in W. N. Bailey's fundamental papers [12], [13] on Rogers-Ramanujan type identities. In particular, the very brief §4 of Bailey's paper [13] provides the mechanism for producing an infinite family of identities out of one identity. To give Bailey's results we require the following standard notation [24; p. 89]:

$$(2.1) \qquad (a; q)_\infty = (a)_\infty = \prod_{n=0}^{\infty} (1 - aq^n),$$

$$(2.2) \quad (a; q)_n = (a)_n = (a; q)_\infty / (aq^n; q)_\infty$$

$$\left(= (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \text{ for } n \text{ a nonnegative integer}\right).$$

The assertion by Bailey in [**13**; §4] is:

BAILEY'S LEMMA. *If for $n \geq 0$*

$$(2.3) \qquad \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},$$

then

$$(2.4) \qquad \sum_{n \geq 0} \left(\frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \alpha_n}{(aq/\rho_1)_n (aq/\rho_2)_n} \right) \frac{1}{(q)_{N-n}(aq)_{N+n}}$$

$$= \sum_{n \geq 0} \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \beta_n (aq/\rho_1\rho_2)_{N-n}}{(q)_{N-n}(aq/\rho_1)_N(aq/\rho_2)_N}.$$

To obtain the above result one merely takes the results from [**13**; §4] and fills them into the identity in [**13**; §2]. Finally algebraic simplification provides the above form of the result.

The important observation is that (2.4) is again an instance of (2.3) with now

$$(2.5) \qquad \alpha'_r = \frac{(\rho_1)_r (\rho_2)_r (aq/\rho_1\rho_2)^r \alpha_r}{(aq/\rho_1)_r (aq/\rho_2)_r},$$

and

$$(2.6) \qquad \beta'_N = \sum_{n \geq 0} \frac{(aq/\rho_1\rho_2)_{N-n}(\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \beta_n}{(q)_{N-n}(aq/\rho_1)_N(aq/\rho_2)_N}.$$

Thus Bailey's Lemma can be iterated ad infinitum, and this iteration is what is made explicit in the following result.

THEOREM 1. *Let the sequences α_n and β_n be related by* (2.3), *then*

$$(2.7) \quad \sum_{n \geq 0} \frac{(b_1)_n (c_1)_n (b_2)_n (c_2)_n \cdots (b_k)_n (c_k)_n (q^{-N})_n}{(aq/b_1)_n (aq/c_1)_n (aq/b_2)_n (aq/c_2)_n \cdots (aq/b_k)_n (aq/c_k)_n (aq^{N+1})_n}$$

$$\times \left(\frac{a^k q^{k+N}}{b_1 c_1 \cdots b_k c_k} \right)^n q^{-\binom{n}{2}} (-1)^n \alpha_n$$

$$= \frac{(aq)_N (aq/b_k c_k)_N}{(aq/b_k)_N (aq/c_k)_N} \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_k)_{n_k} (c_k)_{n_k} \cdots (b_1)_{n_1} (c_1)_{n_1}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}}$$

$$\times \frac{(q^{-N})_{n_k}(aq/b_{k-1}c_{k-1})_{n_k - n_{k-1}} \cdots (aq/b_1 c_1)_{n_2 - n_1}}{(b_k c_k q^{-N}/a)_{n_k}(aq/b_{k-1})_{n_k}(aq/c_{k-1})_{n_k} \cdots (aq/b_1)_{n_2}(aq/c_1)_{n_2}}$$

$$\times q^{n_1 + n_2 + \cdots + n_k} a^{n_1 + \cdots + n_{k-1}} (b_{k-1} c_{k-1})^{-n_{k-1}} \cdots (b_1 c_1)^{-n_1} \beta_{n_1}.$$

Proof. We proceed by mathematical induction. When $k = 1$, (2.7) reduces to

$$(2.8) \quad \sum_{n \geq 0} \frac{(b_1)_n (c_1)_n (q^{-N})_n}{(aq/b_1)_n (aq/c_1)_n (aq^{N+1})_n} \left(\frac{aq^{1+N}}{b_1 c_1} \right)^n (-1)^n q^{-\binom{n}{2}} \alpha_n$$

$$= \frac{(aq)_N (aq/b_1 c_1)_N}{(aq/b_1)_N (aq/c_1)_N} \sum_{n_1 \geq 0} \frac{(b_1)_{n_1} (c_1)_{n_1} (q^{-N})_{n_1} q^{n_1} \beta_{n_1}}{(b_1 c_1 q^{-N}/a)_{n_1}},$$

which is (2.4) (with $\rho_1 = b_1$, $\rho_2 = c_1$) multiplied by $(q)_N (aq)_N$ and simplified. Hence the case $k = 1$ is merely Bailey's Lemma.

Now assume the result true up to but not including a particular k. Then the case $k - 1$ is merely a new instance of (2.3) with

$$(2.9) \quad \alpha_n^* = \frac{(b_1)_n (c_1)_n \cdots (b_{k-1})_n (c_{k-1})_n}{(aq/b_1)_n (aq/c_1)_n \cdots (aq/b_{k-1})_n (aq/c_{k-1})_n}$$

$$\times \left(\frac{a^{k-1} q^{k-1}}{b_1 c_1 \cdots b_{k-1} c_{k-1}} \right)^n \alpha_n,$$

$$(2.10) \quad \beta_n^* = \frac{(aq/b_{k-1} c_{k-1})_n}{(q)_n (aq/b_{k-1})_n (aq/c_{k-1})_n}$$

$$\times \sum_{n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_{k-1})_{n_{k-1}} (c_{k-1})_{n_{k-1}} \cdots (b_1)_{n_1} (c_1)_{n_1}}{(q)_{n_{k-1}-n_{k-2}} \cdots (q)_{n_2-n_1}}$$

$$\times \frac{(q^{-n})_{n_{k-1}} (ab/b_{k-2} c_{k-2})_{n_{k-1}-n_{k-2}} \cdots (aq/b_1 c_1)_{n_2-n_1}}{(b_{k-1} c_{k-1} q^{-n}/a)_{n_{k-1}} (aq/b_{k-2})_{n_{k-1}} (aq/c_{k-2})_{n_{k-1}} \cdots (aq/b_1)_{n_2} (aq/c_1)_{n_2}}$$

$$\times q^{n_1+n_2+\cdots+n_{k-1}} a^{n_1+\cdots+n_{k-2}} (b_{k-2} c_{k-2})^{-n_{k-2}} \cdots (b_1 c_1)^{-n_1} \beta_{n_1}.$$

Hence we may substitute α_n^* and β_n^* into (2.4) with $\rho_1 = b_k$, $\rho_2 = c_k$ and then multiply through by $(q)_N (aq)_N$. The resulting lefthand side of (2.4) is obviously the lefthand side of (2.7). The resulting righthand side is

$$(2.11) \quad (q)_N (aq)_N \sum_{n_k \geq 0} \frac{(b_k)_{n_k} (c_k)_{n_k} (aq/b_k c_k)^{n_k} \beta_{n_k}^* (aq/b_k c_k)_{N-n_k}}{(q)_{N-n_k} (aq/b_k)_N (aq/c_k)_N}$$

$$= \frac{(aq)_N (aq/b_k c_k)_N}{(aq/b_k)_N (aq/c_k)_N} \sum_{n_k \geq 0} \frac{(b_k)_{n_k} (c_k)_{n_k} (q^{-N})_{n_k} q^{n_k} \beta_{n_k}^*}{(b_k c_k q^{-N}/a)_{n_k}}$$

$$= \frac{(aq)_N(aq/b_kc_k)_N}{(aq/b_k)_N(aq/c_k)_N} \sum_{n_k \geq 0} \frac{(aq/b_{k-1}c_{k-1})_{n_k}q^{n_k}}{(q)_{n_k}(aq/b_{k-1})_{n_k}(aq/c_{k-1})_{n_k}}$$

$$\times \sum_{n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_k)_{n_k}(c_k)_{n_k}\cdots(b_1)_{n_1}(c_1)_{n_1}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}}$$

$$\times \frac{(q^{-N})_{n_k}(q)_{n_k}(aq/b_{k-1}c_{k-1})_{n_k-n_{k-1}}}{(b_kc_kq^{-N}/a)_{n_k}(q)_{n_k-n_{k-1}}(aq/b_{k-1}c_{k-1})_{n_k}}\left(\frac{a}{b_{k-1}c_{k-1}}\right)^{n_{k-1}}$$

$$\times \frac{(aq/b_{k-2}c_{k-2})_{n_{k-1}-n_{k-2}}\cdots(aq/b_1c_1)_{n_2-n_1}}{(aq/b_{k-2})_{n_{k-1}}(aq/c_{k-2})_{n_{k-1}}\cdots(aq/b_1)_{n_2}(aq/c_1)_{n_2}}$$

$$\times q^{n_1+\cdots+n_{k-1}}a^{n_1+\cdots+n_{k-2}}(b_{k-2}c_{k-2})^{-n_{k-2}}\cdots(b_1c_1)^{-n_1}\beta_{n_1}$$

$$= \frac{(aq)_N(aq/b_kc_k)_N}{(aq/b_k)_N(aq/c_k)_N} \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_k)_{n_k}(c_k)_{n_k}\cdots(b_1)_{n_1}(c_1)_{n_1}}{(q)_{n_k-n_{k-1}}\cdots(q)_{n_2-n_1}}$$

$$\times \frac{(q^{-N})_{n_k}(aq/b_{k-1}c_{k-1})_{n_k-n_{k-1}}\cdots(aq/b_1c_1)_{n_2-n_1}}{(b_kc_kq^{-N}/a)_{n_k}(aq/b_{k-1})_{n_k}(aq/c_{k-1})_{n_k}\cdots(aq/b_1)_{n_2}(aq/c_1)_{n_2}}$$

$$\times q^{n_1+\cdots+n_k}a^{n_1+\cdots+n_{k-1}}(b_{k-1}c_{k-1})^{-n_{k-1}}\cdots(b_1c_1)^{-n_1}\beta_{n_1},$$

which is the desired form of the right hand side of (2.7). Thus Theorem 1 is proved.   □

While Theorem 1 looks cumbersome it has in fact a number of famous and important specializations. The most important occurs for

$$(2.12) \qquad\qquad \alpha_n = \frac{(1-aq^{2n})(a)_n(-1)^nq^{\binom{n}{2}}}{(1-a)(q)_n},$$

$$(2.13) \qquad\qquad \beta_n = \delta_{n,0} = \begin{cases} 1 & \text{if} \quad n=0, \\ 0 & \quad n > 0. \end{cases}$$

The fact that α_n and β_n satisfy (2.3) is immediate from the following result (with $M = n$) which is easily proved by mathematical induction on M.

$$(2.14) \qquad \sum_{j=0}^{M} \frac{(1-aq^{2j})(q^{-n})_j(a)_jq^{nj}}{(1-a)(aq^{n+1})_j(q)_j} = \frac{(aq)_Mq^{nM}(q^{1-n})_M}{(q)_M(aq^{n+1})_M},$$

a result derived originally by R. P. Agarwal [1; p. 444, final equation].

The case $k = 1$ of Theorem 1 with α_n given by (2.12) and β_n given by (2.13) is the terminating case of the limiting form of Jackson's theorem [24; p. 96]. The case $k = 2$ is Watson's celebrated q-analog of Whipple's Theorem [28] which implies (among many other results) the Rogers-Ramanujan identities (equations (1.1) and (1.2)).

3. q-Series extensions of (1.3)–(1.7). We have chosen four identities from among Slater's list of 130. However it should be emphasized that *Slater derives all* 130 *identities from* (α_n, β_n) *pairs* applied to (2.4) with $N \to \infty$. We may therefore derive for each (α_n, β_n) pair given by Slater in [22] an infinite family of identities corresponding to $k = 1, 2, 3, \ldots$ in Theorem 1. We have chosen to generalize (1.4)–(1.7) as striking examples of how this works. To simplify our proceedings we consider a special limiting case of Theorem 1.

THEOREM 2. *Let α_n and β_n be related by* (2.3). *Then*

$$(3.1) \quad \frac{1}{(aq)_\infty} \sum_{n \geq 0} q^{kn^2} a^{kn} \alpha_n$$

$$= \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{a^{n_1 + \cdots + n_k} q^{n_1^2 + n_2^2 + \cdots + n_k^2} \beta_{n_1}}{(q)_{n_k - n_{k-1}} (q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}} .$$

Proof. Let $N, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k$ all tend to infinity in Theorem 1. $\square$

THEOREM 3. *Identity* (1.8) *is valid.*

REMARK. All we need do for this result and the remaining results in this section is to choose the appropriate (α_n, β_n) pair given by Slater [22] and fill it in to (3.1).

Proof. Since (1.4) appears as identity (98) in Slater [23] we see that the requisite (α_n, β_n) pair in [22] is given by her $A(1)$, i.e.

$$(3.2) \quad \alpha_m = \begin{cases} -q^{6n^2 - 5n + 1}, & m = 3n - 1 > 0, \\ q^{6n^2 - n} + q^{6n^2 + n}, & m = 3n > 0, \\ -q^{6n^2 + 5n + 1}, & m = 3n + 1 > 0, \\ 1, & m = 0, \end{cases}$$

$$(3.3) \qquad\qquad \beta_n = 1/(q)_{2n},$$

and furthermore $a = 1$ (Slater's x is our aq^{-1}). Hence by Theorem 2 (with $a = 1$),

$$(3.4) \quad \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}(q)_{2n_1}}$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=1}^\infty q^{k(3n-1)^2}\left(-q^{6n^2 - 5n + 1}\right) + 1 + \sum_{n=1}^\infty q^{k(3n)^2}\left(q^{6n^2 - n} + q^{6n^2 + n}\right) \right.$$

$$\left. + \sum_{n=0}^\infty q^{k(3n+1)^2}\left(-q^{6n^2 + 5n + 1}\right) \right)$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^\infty q^{(9k+6)n^2 + n} - q^{k+1} \sum_{n=-\infty}^\infty q^{(9k+6)n^2 + (6k+5)n}$$

$$= \frac{1}{(q)_\infty} \prod_{n=0}^\infty \left(1 - q^{(6k+4)(n+1)}\right)\left(1 - q^{k+1+(6k+4)n}\right)\left(1 - q^{5k+3+(6k+4)n}\right)$$

$$\times \left(1 - q^{4k+2+(12k+8)n}\right)\left(1 - q^{8k+6+(12k+8)n}\right)$$

$$\text{(by [6; p. 21, Th. 2.8])}$$

$$= \prod_{\substack{n=1 \\ n \neq 0, \pm(k+1) \,(\mathrm{mod}\, 6k+4) \\ n \not\equiv \pm(4k+2)\,(\mathrm{mod}\,12k+k)}}^\infty (1 - q^n)^{-1},$$

as desired. $\square$

THEOREM 4.

$$(3.5) \quad \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2 + 2n_1 + n_2 + \cdots + n_k}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}(q)_{2n_1 + 1}}$$

$$= \prod_{\substack{n=1 \\ n \neq 0, \pm(2k+2)\,(\mathrm{mod}\,6k+4) \\ n \not\equiv \pm 2k\,(\mathrm{mod}\,12k+8)}}^\infty (1 - q^n)^{-1}.$$

Proof. This result reduces to (1.5) when $k = 1$, and since (1.5) appears as identity (96) in Slater [23] we see that the requisite (α_n, β_n) pair in [22] is given by her $A(4)$, i.e.

$$(3.6) \quad \alpha_m = \begin{cases} q^{6n^2 - 4n}, & m = 3n - 1 > 0, \\ q^{6n^2 + 4n}, & m = 3n > 0, \\ -q^{6n^2 + 8n + 2} - q^{6n^2 + 4n}, & m = 3n + 1 > 0, \\ 1, & m = 0, \end{cases}$$

(3.7) $$\beta_n = q^n/(q^2; q)_{2n},$$

and furthermore $a = q$.

Hence by Theorem 2, with $a = q$ and both sides multiplied by $(1 - q)^{-1}$,

$$(3.8) \quad \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2 + 2n_1 + n_2 + \cdots + n_k}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}(q)_{2n_1 + 1}}$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=1}^\infty q^{k(3n-1)^2 + k(3n-1)} \left(q^{6n^2 - 4n} \right) + 1 + \sum_{n=1}^\infty q^{k(3n)^2 + 3kn} \left(q^{6n^2 + 4n} \right) \right.$$

$$\left. + \sum_{n=0}^\infty q^{k(3n+1)^2 + k(3n+1)} \left(-q^{6n^2 + 8n + 2} - q^{6n^2 + 4n} \right) \right)$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^\infty q^{(9k+6)n^2 + (3k+4)n} - q^{2k} \sum_{n=-\infty}^\infty q^{(9k+6)n^2 + (9k+4)n} \right)$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^\infty q^{(9k+6)n^2 + (3k+4)n} - q^{2k+2} \sum_{n=-\infty}^\infty q^{(9k+6)n^2 + (9k+8)n} \right)$$

(replacing n by $-n - 1$ in the second sum)

$$= \frac{1}{(q)_\infty} \prod_{n=0}^\infty \left(1 - q^{(6k+4)(n+1)} \right) \left(1 - q^{2k+2 + (6k+4)n} \right) \left(1 - q^{4k+2 + (6k+4)n} \right)$$

$$\times \left(1 - q^{10k+8 + (12k+8)n} \right) \left(1 - q^{2k + (12k+8)n} \right)$$

(by [6; p. 21, Th. 2.8])

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k+2) \,(\mathrm{mod}\,6k+4) \\ n \not\equiv \pm 2k \,(\mathrm{mod}\,12k+8)}}^\infty (1 - q^n)^{-1},$$

as desired. □

THEOREM 5.

$$(3.9) \quad \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2 + n_1}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}(q)_{2n_1}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm k \,(\mathrm{mod}\,6k+4) \\ n \not\equiv \pm(4k+4) \,(\mathrm{mod}\,12k+8)}}^\infty (1 - q^n)^{-1}$$

Proof. This result reduces to (1.6) when $k = 1$, and since (1.6) appears as identity (99) in Slater [23] we see that the requisite (α_n, β_n) pair in [22]

is given by her $A(3)$, i.e.

$$(3.10) \qquad \alpha_m = \begin{cases} -q^{6n^2-2n}, & m = 3n - 1 > 0, \\ q^{6n^2-2n} + q^{6n^2+2n}, & m = 3n > 0, \\ -q^{6n^2+2n}, & m = 3n + 1 > 0, \\ 1, & m = 0, \end{cases}$$

$$(3.11) \qquad\qquad\qquad \beta_n = q^n/(q)_{2n},$$

and furthermore $a = 1$.

Hence by Theorem 2, with $a = 1$

$$(3.12) \quad \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2 + n_1}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}(q)_{2n_1}}$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=1}^\infty q^{k(3n-1)^2}\left(-q^{6n^2-2n}\right) + 1 \right.$$

$$\left. + \sum_{n=1}^\infty q^{k(3n)^2}\left(q^{6n^2-2n} + q^{6n^2+2n}\right) + \sum_{n=0}^\infty q^{k(3n+1)^2}\left(-q^{6n^2+2n}\right) \right)$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^\infty q^{(9k+6)n^2-2n} - q^k \sum_{n=-\infty}^\infty q^{(9k+6)n^2+(6k+2)n} \right)$$

$$= \frac{1}{(q)_\infty} \prod_{n=0}^\infty \left(1 - q^{(6k+4)(n+1)}\right)\left(1 - q^{k+(6k+4)n}\right)\left(1 - q^{(5k+4)+(6k+4)n}\right)$$

$$\times \left(1 - q^{4k+4+(12k+8)n}\right)\left(1 - q^{8k+4+(12k+8)n}\right)$$

(by [6; p. 21, Th. 2.8])

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm k \,(\mathrm{mod}\, 6k+4) \\ n \not\equiv \pm(4k+4)\,(\mathrm{mod}\, 12k+8)}}^\infty (1 - q^n)^{-1},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Finally we have

THEOREM 5.

$$(3.13) \quad \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2 + n_1 + n_2 + \cdots + n_k}}{(q)_{n_k - n_{k-1}}(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1}(q)_{2n_1+1}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k+1)\,(\mathrm{mod}\, 6k+4) \\ n \not\equiv \pm(2k+2)\,(\mathrm{mod}\, 12k+8)}}^\infty (1 - q^n)^{-1}.$$

Proof. This result reduces to (1.7) when $k = 1$, and since (1.7) appears as identity (94) in Slater [23] we see that the requisite (α_n, β_n) pair in [22] is given by her $A(2)$, i.e.

$$(3.13) \qquad \alpha_m = \begin{cases} q^{6n^2 - n}, & m = 3n - 1 > 0, \\ q^{6n^2 + n}, & m = 3n \geq 0, \\ -q^{6n^2 + 5n + 1} - q^{6n^2 + 7n + 2}, & m = 3n + 1 > 0, \end{cases}$$

$$(3.14) \qquad\qquad\qquad \beta_n = 1/(q^2; q)_{2n},$$

and here $a = q$.

Hence by Theorem 2 with $a = q$ and both sides multiplied by $(1 - q)^{-1}$

$$(3.15) \qquad \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_k^2 + n_1 + n_2 + \cdots + n_k}}{(q)_{n_k - n_{k-1}} (q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1} (q)_{2n_1 + 1}}$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=1}^\infty q^{k(3n-1)^2 + k(3n-1)} \left(q^{6n^2 - n} \right) + \sum_{n=0}^\infty q^{k(3n)^2 + k3n} \left(q^{6n^2 + n} \right) \right.$$

$$\left. + \sum_{n=0}^\infty q^{k(3n+1)^2 + k(3n+1)} \left(-q^{6n^2 + 5n + 1} - q^{6n^2 + 7n + 2} \right) \right)$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^\infty q^{(9k+6)n^2 + 3kn + n} - q^{2k+1} \sum_{n=-\infty}^\infty q^{(9k+6)n^2 + (9k+5)n} \right)$$

$$= \frac{1}{(q)_\infty} \prod_{n=0}^\infty \left(1 - q^{(6k+4)(n+1)}\right)\left(1 - q^{2k+1+(6k+4)n}\right)\left(1 - q^{4k+3+(6k+4)n}\right)$$

$$\times \left(1 - q^{2k+2+(12k+8)n}\right)\left(1 - q^{10k+6+(12k+8)n}\right)$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k+1) \,(\mathrm{mod}\, 6k+4) \\ n \not\equiv \pm(2k+2) \,(\mathrm{mod}\, 12k+8)}}^\infty (1 - q^n)^{-1},$$

as desired.   □

As is obvious from the above, each pair (α_n, β_n) given by Slater in [22] can be inserted in Theorem 2 or Theorem 1. The four examples we have chosen constitute applications to probably the most elegant results considered by Rogers et al. beyond the original Rogers-Ramanujan identities.

4. The pairs of sequences (α_n, β_n). Often in this subject one may be given only one of the sequences α_n or β_n. If the α_n are given then the β_n are determined by (2.3). Less well known but still of importance is the "inversion" of (2.3). Namely [7; p. 8, Lemma 3]

$$(4.1) \qquad \alpha_n = \left(1 - aq^{2n}\right) \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j} q^{\binom{n-j}{2}}}{(q)_{n-j}} \beta_j.$$

Also of interest is the creation of (α_n', β_n') given by Bailey's Lemma. Furthermore if (α_n', β_n') are given then one can immediately determine α_n from (2.5), and one can solve for $\beta_0, \beta_1, \ldots, \beta_N$ from the diagonal system of equations (2.6). Thus the operation of forming the sequence of pairs:

$$(4.2) \qquad\qquad (\alpha_n, \beta_n) \to (\alpha_n', \beta_n') \to (\alpha_n'', \beta_n'') \to$$

which was codified explicitly in Theorem 1 can, in fact, be extended to the left as well. In actual fact, if we consider $\alpha_n = \alpha_n^0$, $\beta_n = \beta_n^0$, then we can form

$$(4.3) \qquad \cdots \to \left(\alpha_n^{(-2)}, \beta_n^{(-2)}\right) \to \left(\alpha_n^{(-1)}, \beta_n^{(-1)}\right) \to \left(\alpha_n^0, \beta_n^0\right)$$

$$\to \left(\alpha_n^1, \beta_n^1\right) \to \cdots$$

where each pair is related to the next through instances of (2.5) and (2.6). In the following, we shall refer to a pair of sequences (α_n, β_n) related by (2.3) to be a *Bailey pair*. We shall call (4.2) an *ordinary Bailey chain*, and (4.3) a *bilateral Bailey chain*. For any given Bailey chain one would like to designate a "simplest" Bailey pair as the "reduced" element of the chain. At the current state of knowledge it does not appear to be fruitful (or perhaps possible) to provide a rigorous definition of "reduced". Suffice it to say that the Bailey pair given by (2.12) and (2.13) are the obvious "reduced" pair in their family.

Another important way to produce new Bailey pairs (α_n, β_n) from previous ones is through the replacements $a \to a^{-1}$, $q \to q^{-1}$. Thus if $\alpha_n = \alpha_n(a, q)$, $\beta_n = \beta_n(a, q)$,

$$A_k(a, q) = A_k = a^k q^{k^2} \alpha_k\left(a^{-1}, q^{-1}\right),$$

$$B_k(a, q) = B_k = a^{-k} q^{-k^2 - k} \beta_k\left(a^{-1}, q^{-1}\right),$$

then the replacement of a by a^{-1} and q by q^{-1} in (2.3) yields

$$(4.4) \qquad\qquad B_n = \sum_{r=0}^{n} \frac{A_r}{(q)_{n-r}(aq)_{n+r}}.$$

Thus the substitution $a \to a^{-1}$, $q \to q^{-1}$ yields a new pair satisfying (2.3). We call (A_n, B_n) the *dual Bailey* pair of (α_n, β_n).

For example, the Bailey pair (α_n, β_n) given by (2.12) and (2.13) is self-dual. On the other hand Slater's A-Table [22] of eight pairs (α_n, β_n) consists of the four pairs we utilized in §3 and the 4 duals of these pairs.

5. New applications. The q-series identities that have previously appeared primarily have their genesis in the work of L. J. Rogers [18], [19]. Indeed the majority of Bailey pairs given by Slater in [22] arise, as she notes, from Rogers' pathbreaking papers. In fact, the Bailey pairs used in the classical development of this subject have always taken α_n as zero or plus or minus a power of q. The β_n have always been simple finite q-products.

Let us start by delving into the work of I. Schur on the Rogers-Ramanujan type identities. Schur in his famous initial paper [21] on this topic provided three polynomial identities that fit the Bailey pair criterion. Namely he proved (where $\begin{bmatrix} A \\ B \end{bmatrix} = (q)_A/((q)_B(q)_{A-B})$ and $[x]$ is the greatest integer in x)

$$(5.1) \qquad \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{\lambda(3\lambda+1)/2} \left[\begin{bmatrix} n \\ \left[\dfrac{n-3\lambda}{2}\right] \end{bmatrix} \right] = 1,$$

$$(5.2) \qquad \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{\lambda(5\lambda+1)/2-2\alpha\lambda} \left[\begin{bmatrix} n \\ \left[\dfrac{n-5\lambda}{2}\right] + \alpha \end{bmatrix} \right]$$

$$= \sum_{j\geq 0} q^{j^2+\alpha j} \begin{bmatrix} n - \alpha - j \\ j \end{bmatrix},$$

where in (5.2) $\alpha = 0$ or 1. Actually (5.2) first appears in [2]; however, as noted there, the essential observations for this identity go back to Schur. From (5.1) and (5.2) one can obtain 12 Bailey pairs: the first 6 come from n even or odd, $\alpha = 0$ or 1. The last 6 are the dual Bailey pairs of the first 6. In fact, Rogers had proved (5.1) in [18] since when n is even (5.1) is Bailey pair $A(1)$ of [22] and when n is odd (5.1) is Bailey pair $A(2)$ of [22]. As mentioned previously, Rogers also had the dual Bailey pairs of $A(1)$ and $A(2)$. Of the remaining eight possible Bailey pairs implied by (5.2), we choose the case n even, $\alpha = 0$. In this instance the Bailey pair is

$$(5.3) \qquad \alpha_n = \begin{cases} q^{m(10m-1)} + q^{m(10m+1)}, & n = 5m > 0, \\ 1, & n = 0, \\ -q^{10m^2+11m+3}, & n = 5m + 3 > 0, \\ -q^{10m^2-11m+3}, & n = 5m - 3 > 0. \end{cases}$$

$$(5.4) \qquad \beta_n = \frac{1}{(q)_{2n}} \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} 2n - j \\ j \end{bmatrix},$$

where $a = 1$.

When this pair is used in conjunction with Theorem 2, the resulting identity is

$$(5.5) \quad \frac{1}{(q)_\infty} \left(\prod_{m=0}^{\infty} \left(1 - q^{(m+1)(50k+20)}\right)\left(1 + q^{(25k+9)+(50k+20)m}\right)\left(1 + q^{(25k+11)+(50k+20)m}\right) \right.$$

$$\left. - q^{4k+2} \prod_{m=0}^{\infty} \left(1 - q^{(m+1)(50k+20)}\right)\left(1 + q^{5k+1+(50k+20)m}\right)\left(1 + q^{45k+19+(50k+20)m}\right) \right)$$

$$= \sum_{n_k \geq \cdots \geq n_1 \geq n_0 \geq 0} \frac{q^{n_k^2 + \cdots + n_1^2 + n_0^2}(q)_{2n_1 - n_0}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2 - n_1}(q)_{2n_1 - 2n_0}(q)_{2n_1}(q)_{n_0}}.$$

D. Bressoud [15], in a study building on Rogers' work, has found several other new Bailey pairs. As an example, we list

$$(5.6) \qquad \alpha_n = \begin{cases} (-1)^m \left(q^{m(5m+1)/2} + q^{m(5m-1)/2}\right), & n = 2m, \\ 1, & n = 0, \\ 0, & n \text{ odd}, \end{cases}$$

$$(5.7) \qquad \beta_n = \frac{1}{(q)_{2n}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} q^{j^2},$$

where $a = 1$.

Hence by Theorem 2,

$$(5.8) \qquad \sum_{n_k \geq \cdots \geq n_1 \geq n_0 \geq 0} \frac{q^{n_k^2 + \cdots + n_1^2 + n_0^2}(q)_{n_1}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2 - n_1}(q)_{n_1 - n_0}(q)_{n_0}(q)_{2n_1}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(4k+2)\,(\text{mod }8k+5)}}^{\infty} (1 - q^n)^{-1},$$

which, of course, reduces to (1.1) for $k = 0$.

Finally, and most off the beaten track, we shall see that the Wall polynomials [27] from group theory fit into the Bailey pair scheme.

The Wall polynomials were first defined in [27] in order to determine the generating functions for the numbers of conjugacy classes in the symplectic and orthogonal groups over finite fields of characteristic 2. They are defined by the following recurrences:

$$(5.9) \qquad X_{2n+1} = X_{2n} + q^{2n+1}X_{2n-1},$$

$$(5.10) \quad X_{2n+2} = X_{2n+1} + q^{n+1}\big(1 + q^{n+1}\big)\Big(X_{2n+1} + \big(1 - q^{2n+1}\big)X_{2n-1}\Big),$$

where $X_{-1} = X_{-1}(a, b, q) = a$ and $X_0 = X_0(a, b, q) = b$.

Recently it has been shown that [10] among other results

$$(5.11) \qquad X_{2n-1}(1,1,q) = \sum_{j=-\infty}^{\infty} q^{j^2}\left[\begin{array}{c} 2n \\ n + 2j \end{array} \right],$$

$$(5.12) \qquad X_{2n-1}(0,1,q) = \sum_{j \geq 0} q^{j^2+j}\left[\begin{array}{c} 2n \\ n - 2j - 1 \end{array} \right].$$

Hence $\beta_n = X_{2n-1}(1,1,q)/(q)_{2n}$ forms a Bailey pair with

$$(5.13) \qquad\qquad \alpha_n = \begin{cases} 2q^{j^2}, & n = 2j, \\ 1, & n = 0, \\ 0, & n \text{ odd}, \end{cases}$$

and $a = 1$.

Thus by Theorem 2

$$(5.14) \qquad \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{n_k^2 + \cdots + n_1^2} X_{2n_1-1}(1,1,q)}{(q)_{n_k-n_{k-1}} \cdots (q)_{n_2-n_1}(q)_{2n_1}}$$

$$= \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} q^{(4k+1)j^2}.$$

Finally $\beta_n = X_{2n-1}(0,1,q)/(q)_{2n}$ forms a Bailey pair with

$$(5.15) \qquad\qquad \alpha_n = \begin{cases} q^{j^2+j}, & n = 2j + 1, \\ 0, & n \text{ even}. \end{cases}$$

with $a = 1$.

Hence by Theorem 2

$$(5.16) \qquad \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{n_k^2 + \cdots + n_1^2} X_{2n_1-1}(0,1,q)}{(q)_{n_k-n_{k-1}} \cdots (q)_{n_2-n_1}(q)_{2n_1}}$$

$$= \frac{\sum_{j=0}^{\infty} q^{k(2j+1)^2+j^2+j}}{(q)_\infty}.$$

6. Conclusion. Our object has not been to add heaps of further Rogers-Ramanujan identities to the already extended lists. Rather it has been to exhibit clearly mechanisms for their construction. The problems

of additive number theory [8], combinatorics and special function [5], and physics [9] will presumably dictate which Bailey pairs and Bailey chains will be of most interest. Presumably we have here provided the framework for their construction.

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