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THE FEFFERMAN-STEIN DECOMPOSITION OF SMOOTH FUNCTIONS AND ITS APPLICATION TO $H^p(\mathbf{R}^n)$

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We show the "Fefferman-Stein decomposition" of smooth bump functions. As an application of this we get one result about the singular integral characterization of $H^p(\mathbf{R}^n)$. Our method does not use subharmonicity.

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1. Introduction. In this paper functions considered are complex-valued unless otherwise explicitly stated. Cubes considered have sides parallel to the coordinate axes. For a function $f(x) \in L^1_{\text{loc}}(\mathbf{R}^n)$, let

$$\|f\|_{\text{BMO}} = \sup_I \int_I |f(x) - f_I|\,dx/|I|,$$

where the supremum is taken over all cubes in $\mathbf{R}^n$, $|I|$ denotes the Lebesgue measure of I and

$$f_I = \int_I f(x)\,dx/|I|.$$

A function $f(x)$ is said to belong to $\text{BMO}(\mathbf{R}^n)$ if $\|f\|_{\text{BMO}} < +\infty$.

Let $\theta_1(\xi),\ldots,\theta_m(\xi) \in C^\infty(S_{n-1})$, where

$$S_{n-1} = \{\xi \in \mathbf{R}^n : |\xi| = 1\}$$

and

$$|\xi| = |(\xi_1,\ldots,\xi_n)| = \left(\sum_{j=1}^n \xi_j^2\right)^{1/2}.$$

For $h \in L^2(\mathbf{R}^n)$ let

$$K_j h = \left(\theta_j(\xi/|\xi|)\hat{h}(\xi)\right)^{\vee}, \qquad j = 1,\ldots,m,$$

where $\hat{}$ and $\check{}$ are the Fourier and inverse Fourier transforms. As is well known [see Stein [**29**] p. 75], there exist $\alpha_j \in \mathbf{C}$ and $\Omega_j(x) \in C^\infty(S_{n-1})$ such that

$$\int_{|x|=1} \Omega_j(x) = 0$$

and

$$K_j h(x) = \alpha_j h(x) + \text{P.V.} \int \Omega_j\left(\frac{x-y}{|x-y|}\right)|x-y|^{-n} h(y)\,dy$$

for any $h \in L^2(\mathbf{R}^n)$. For $g \in L^\infty(\mathbf{R}^n)$ let

$$\tilde{K}_j g(x) = \alpha_j g(x) + \text{P.V.} \int \left\{\Omega_j\left(\frac{x-y}{|x-y|}\right)|x-y|^{-n} - \Omega_j\left(\frac{-y}{|y|}\right)|y|^{-n}\chi_{\{|y|>1\}}\right\} g(y)\,dy$$

where χ_E denotes the characteristic function of a set $E \subset \mathbf{R}^n$. In [**32**], the author showed

THEOREM A. *If*

$$\text{rank}\begin{pmatrix} \theta_1(\xi) \cdots \theta_m(\xi) \\ \theta_1(-\xi) \cdots \theta_m(-\xi) \end{pmatrix} \equiv 2 \quad \text{on } S_{n-1}, \tag{1.1}$$

then for any $f \in \mathrm{BMO}(\mathbf{R}^n)$ there exist $g_1,\ldots,g_m \in L^\infty(\mathbf{R}^n)$ such that

$$f = \sum_{j=1}^{m} \tilde{K}_j g_j \quad (\textit{modulo constants})$$

and

$$\sum_{j=1}^{m} \|g_j\|_\infty \le C_{1.1}\|f\|_{\mathrm{BMO}},$$

where $C_{1.1}$ is a constant depending only on $\theta_1,\ldots,\theta_m$.

REMARK 1. The case when $K_1,\ldots,K_{n+1}$ are the Riesz transforms and the identity operator is the case considered by C. Fefferman [**13**] and C. Fefferman-Stein [**14**].

REMARK 2. In [**32**] we assumed that f has compact support. But this restriction can be removed.

Consequently, if (1.1) is satisfied, then the singular integral operators $K_1,\ldots,K_m$ characterize $H^1(\mathbf{R}^n)$. In this paper, we continue this research.

In the following, $\mathbf{f}(x) = (f_1(x),\ldots,f_m(x))$ and $\mathbf{g}(x)$ denote $\mathbf{C}^m$-valued functions. We use the following notations:

$$|\mathbf{f}(x)| = \left(\sum_{j=1}^{m} |f_j(x)|^2\right)^{1/2},$$

$$\mathbf{K}h(x) = (K_1h(x),\ldots,K_mh(x)),$$

$$\mathbf{K}\cdot\mathbf{f}(x) = \sum_{j=1}^{m} K_jf_j(x),$$

$$\mathbf{K}^*\cdot\mathbf{f}(x) = \sum_{j=1}^{m} K_j^*f_j(x),$$

where $K_j^*h(x) = (\bar{\theta}_j(\xi/|\xi|)\hat{h}(\xi))\check{}(x)$. $I(x,t)$ denotes a cube in $\mathbf{R}^n$ with center x and side length t.

DEFINITION 1.1. Let

$$S = \{\mathbf{f} \in L^2(\mathbf{R}^n,\mathbf{C}^m) \colon \mathbf{K}^*\cdot\mathbf{f}(x) \equiv 0\},$$

where $L^2(\mathbf{R}^n,\mathbf{C}^m)$ denotes the set of $\mathbf{C}^m$-valued functions $\mathbf{f}(x)$ with $f_1,\ldots,f_m \in L^2(\mathbf{R}^n)$.

DEFINITION 1.A. [Coifman-Rochberg [**9**].] For a real-valued function $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, let

$$\|f\|_{\mathrm{BLO}} = \sup_I \int_I f(x) - \inf_{y\in I} f(y)\, dx/|I|,$$

where I is taken over all cubes in $\mathbf{R}^n$. A function $f(x)$ is said to belong to $\mathrm{BLO}(\mathbf{R}^n)$ if $\|f\|_{\mathrm{BLO}} < +\infty$. [Note that $\|\cdot\|_{\mathrm{BLO}}$ is not a norm.]

Our main result is the following.

THEOREM 1. *Suppose that* (1.1) *holds. Let* $\mathbf{f} \in C^1(\mathbf{R}^n,\mathbf{C}^m)$,

$$|\mathbf{f}(x)| \le (1+|x|)^{-n-1}, \tag{1.2}$$

and

$$\left|\frac{\partial}{\partial x_j}\mathbf{f}(x)\right| \le (1+|x|)^{-n-2}, \qquad j=1,2,\ldots,n. \tag{1.3}$$

Let $w(x)$ be a nonnegative function defined on $\mathbf{R}^n$ such that

$$\|-\log w\|_{\mathrm{BLO}} \le c_0. \tag{1.4}$$

Then there exists $\mathbf{g} \in L^2(\mathbf{R}^n, \mathbf{C}^m)$ such that

$$\mathbf{f} - \mathbf{g} \in S \tag{1.5}$$

and that

$$|\mathbf{g}(x)| \le C_{1.2} w(x) \left(\int_{I(0,1)} w(y)\,dy \right)^{-1} (1+|x|)^{-n-1/2}, \tag{1.6}$$

where c_0 and $C_{1.2}$ are positive constants depending only on $\theta_1, \ldots, \theta_m$.

REMARK 3. If $\mathbf{f}(x)$ is $\mathbf{R}^m$-valued and if $\theta_j(\xi) \equiv \bar{\theta}_j(-\xi)$ for $j=1,\ldots,m$, then we can take $\mathbf{g}(x)$ to be $\mathbf{R}^m$-valued.

REMARK 4. If we apply Theorem 1 to the case when $K_1 =$ the identity operator and $\mathbf{f}(x) = (f(x), 0, \ldots, 0)$, then (1.5) implies

$$f(x) = g_1(x) + \sum_{j=2}^{m} K_j^* g_j(x).$$

This is the reason why we call Theorem 1 the Fefferman-Stein decomposition of smooth bump functions. The point is the fact that we can dominate $g_1, \ldots, g_m$ pointwise by a "function" on the right-hand side of (1.6).

The idea of this theorem comes from P. W. Jones's recent work "L^∞ estimate for the $\bar{\partial}$ problem in a half-plane" [**25**]. We explain the relation between Theorem 1 and Jones's result in §3.

The proof of Theorem 1 is given in §5. The Main Lemma in §4 is crucial and is itself a partial result related to the Fefferman-Stein decomposition of certain weighted BMO spaces in terms of singular integral operators $K_1, \ldots, K_m$. The Main Lemma is proved in §§6–9. Its proof is a refinement of the argument in [**32**].

As a corollary to Theorem 1, we get one result about the singular integral characterization of $H^p(\mathbf{R}^n)$. Let $\psi \in \mathfrak{D}(\mathbf{R}^n)$ be a fixed real-valued function satisfying $\int \psi(x)\,dx = 1$. For $h \in \mathcal{S}'(\mathbf{R}^n)$, let

$$h^+(x) = \sup_{t>0} |(h * \psi_t)(x)|,$$

where $\psi_t(x) = t^{-n}\psi(x/t)$. For $+\infty > p > 0$, let

$$\|h\|_{H^p} = \|h^+\|_{L^p}.$$

For $\mathbf{h} = (h_1,\dots,h_m) \in \mathcal{S}'(\mathbf{R}^n) \oplus \cdots \oplus \mathcal{S}'(\mathbf{R}^n)$, let

$$\mathbf{h}^+(x) = \sup_{t>0} |(\mathbf{h} * \psi_t)(x)| = \sup_{t>0} |((h_1 * \psi_t)(x),\dots, (h_m * \psi_t)(x))|.$$

It is known that $\|\cdot\|_{H^p}$ is essentially independent of the choice of ψ. [See C. Fefferman-Stein [**14**].]

DEFINITION 1.B. For $q > 0$ and for a measurable function $f(x)$ let

$$M_q f(x) = \sup_{I \ni x} \left(\int_I |f(y)|^q \, dy/|I| \right)^{1/q},$$

where I is taken over all cubes containing x.

THEOREM 2. *If* (1.1) *holds, then there exist* $p_0 \in (0,1)$ *and* $C_{1.3} \in \mathbf{R}$, *depending only on* $\theta_1,\dots,\theta_m$, *such that*

$$(\mathbf{K}h)^+(x) \le C_{1.3} M_{p_0}\big(M_{1/2}(|\mathbf{K}h|)\big)(x)$$

for any $x \in \mathbf{R}^n$ *and any* $h \in L^2(\mathbf{R}^n)$.

REMARK 5. For $h \in L^2(\mathbf{R}^n)$ and $\mathbf{h} \in L^2(\mathbf{R}^n, \mathbf{C}^m)$, let

$$h^{++}(x) = \sup_{\substack{t>0,\\ z\in R^n:\, |x-z|<t}} |(h * P_t)(z)|,$$

$$\mathbf{h}^{++}(x) = \sup_{\substack{t>0,\\ z\in R^n:\, |x-z|<t}} |(\mathbf{h} * P_t)(z)|,$$

where $P_t(x)$ is the Poisson kernel, that is,

$$P_t(x) = c_n t / \left(|x|^2 + t^2\right)^{(n+1)/2}, \quad c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}.$$

Then in the above inequality, we can replace $(\mathbf{K}h)^+(x)$ by $(\mathbf{K}h)^{++}(x)$.

COROLLARY 1. *If* (1.1) *holds and if* $\max(1/2, p_0) < p \le 1$, *then*

$$c_{1.4}\|h\|_{H^p} \le \sum_{j=1}^m \|K_j h\|_{L^p} \le c_{1.5}\|h\|_{H^p} \tag{1.7}$$

for any $h \in L^2(\mathbf{R}^n)$ *and*

$$(1.8) \qquad c_{1.4}\|h\|_{H^p} \leq \sum_{j=1}^{m} \left(\int_{R^n} \left| \lim_{t \to +0} K_j(h * P_t)(x) \right|^p dx \right)^{1/p} \leq c_{1.5}\|h\|_{H^p}$$

for any $h \in H^p(\mathbf{R}^n)$, *where* $c_{1.4}$ *and* $c_{1.5}$ *are positive constants depending only on* $\theta_1, \ldots, \theta_m$ *and* p.

REMARK 6. For $h \in H^p(\mathbf{R}^n)$, $p < 1$, we define $h * P_t$ by $(\hat{h}(\xi)\hat{P}_t(\xi))\check{}$, which is known to belong to $L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \cap C(\mathbf{R}^n)$. It is also known that for any $h \in H^p(\mathbf{R}^n)$, $\lim_{t \to +0} K_j(h * P_t)(x)$ exists almost everywhere. [See Stein [**29**] p. 201.]

REMARK 7. Inequality (1.7) with $p = 1$ holds for any $h \in \mathcal{S}'(\mathbf{R}^n)$, whose Fourier transform is an integrable function on some neighborhood of the origin, if we define $K_j h = (\theta_j \hat{h})\check{}$ in the sense of distributions and if we define

$$\|K_j h\|_{L^1} = +\infty$$

for the distribution $K_j h$ that does not belong to $L^1(\mathbf{R}^n)$. [In Corollary 1 of [**32**], we showed the above. But the statement in [**32**] was somewhat ambiguous.]

As another application of Theorem 2, we get the following extension of the results of Csereteli, Gundy and Varopoulos. [See [**12**], [**18**] and [**34**].]

COROLLARY 2. *Let*

$$(1.9) \qquad \sum_{j=1}^{m} \left|\theta_j(\xi) - \theta_j(-\xi)\right| \neq 0 \quad \text{for any } \xi \in S_{n-1}.$$

Let h *be a finite complex measure on* $\mathbf{R}^n$ *and let* $dh = f\,dx + ds$, *where* $f \in L^1(\mathbf{R}^n)$ *and* s *is singular. Then*

$$\liminf_{\lambda \to +\infty} \lambda \left| \left\{ x \in \mathbf{R}^n \colon \sum_{j=1}^{m} \left| \lim_{t \to +0} K_j(h * P_t)(x) \right| > \lambda \right\} \right| \geq C_{1.6}\|s\|_M,$$

where $C_{1.6}$ *is a positive constant depending only on* $\theta_1, \ldots, \theta_m$ *and where* $\|s\|_M$ *is the total variation of* s *on* $\mathbf{R}^n$.

REMARK 8. It is known that for any finite measure h

$$\lim_{t\to+0} K_j(h*P_t)(x)$$

exists almost everywhere.

Proofs of Theorem 2 and corollaries are given in §2.

NOTATION. A dyadic cube is a cube of the form

$$\prod_{j=1}^{n}\left[k_j 2^{-k},(k_j+1)2^{-k}\right],$$

where $k_1,\ldots,k_n$ and k are integers. For a cube I, x_I, $l(I)$ and $Q(I)$ denote the center of I, the side length of I and

$$\{(x,t)\in\mathbf{R}^{n+1}: x\in I, t\in(0,l(I))\},$$

respectively. For $\alpha>0$, αI denotes a cube concentric with I and with $l(\alpha I)=\alpha l(I)$. Σ_{2m-1} denotes $\{\boldsymbol{\nu}=(\nu_1,\ldots,\nu_m)\in\mathbf{C}^m: \sum_{j=1}^m|\nu_j|^2=1\}$. $|\boldsymbol{\nu}|$ denotes $(\sum_{j=1}^m|\nu_j|^2)^{1/2}$. For $\boldsymbol{\nu}\in\mathbf{C}^m\backslash\{\mathbf{0}\}$, $U(\boldsymbol{\nu})$ denotes $\boldsymbol{\nu}/|\boldsymbol{\nu}|$. [For the sake of convenience, let $U(\mathbf{0})=(1,0,\ldots,0)$.] For $\boldsymbol{\nu}$ and $\boldsymbol{\mu}\in\mathbf{C}^m$, $\langle\boldsymbol{\nu},\boldsymbol{\mu}\rangle$ denotes $\sum_{j=1}^m(\operatorname{Re}\nu_j\operatorname{Re}\mu_j+\operatorname{Im}\nu_j\operatorname{Im}\mu_j)$, i.e., the inner product in $\mathbf{R}^{2m}$. For $\theta\in C^\infty(S_{n-1})$ and $\xi\in\mathbf{R}^n\backslash\{0\}$, $\theta(\xi)$ denotes $\theta(\xi/|\xi|)$. The letter C denotes various positive constants depending only on $\theta_1,\ldots,\theta_m$.

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2. Proofs of Theorem 2 and Corollaries.

LEMMA 2.A. [*See Coifman-Rochberg* [9].] *If* $h(x)\not\equiv 0$ *and if* $M_1h(x)\not\equiv+\infty$, *then*

$$\|\log M_1h\|_{\mathrm{BLO}}\leq C_{2.1}.$$

Proof of Theorem 2. By dilation and translation the proof of Theorem 2 can be reduced to the inequality

$$\left|\int\mathbf{K}h(x)\psi(x)\,dx\right|\leq C_{1.3}M_{p_0}\big(M_{1/2}(|\mathbf{K}h|)\big)(0). \tag{2.1}$$

Put $\varepsilon = c_0/2C_{2.1}$. Take any $\boldsymbol{\nu} \in \Sigma_{2m-1}$. Applying Theorem 1 to $\mathbf{f}(x) = \psi(x)\boldsymbol{\nu}$ and $w(x) = M_{1/2}(|\mathbf{K}h|)(x)^{-\varepsilon}$, we get $\mathbf{g}(x)$ such that

$$\mathbf{K}^* \cdot (\psi\boldsymbol{\nu} - \mathbf{g}) \equiv 0$$

and such that

$$\begin{aligned} |\mathbf{g}(x)| &\leq CM_{1/2}(|\mathbf{K}h|)(x)^{-\varepsilon}\left(\int_{I(0,1)} M_{1/2}(|\mathbf{K}h|)(y)^{-\varepsilon}\,dy\right)^{-1} \\ &\qquad \times (1+|x|)^{-n-1/2} \\ &\leq CM_{1/2}(|\mathbf{K}h|)(x)^{-\varepsilon}M_{1/2}(|\mathbf{K}h|)(0)^{\varepsilon}(1+|x|)^{-n-1/2}. \end{aligned}$$

Thus

$$\begin{aligned} \left|\int \mathbf{K}h(x)\psi(x)\,dx \cdot \boldsymbol{\nu}\right| &= \left|\int \mathbf{K}h(x)\cdot \mathbf{g}(x)\,dx\right| \leq \int |\mathbf{K}h(x)|\,|\mathbf{g}(x)|dx \\ &\leq CM_{1/2}(|\mathbf{K}h|)(0)^{\varepsilon}\int M_{1/2}(|\mathbf{K}h|)(x)^{1-\varepsilon}(1+|x|)^{-n-1/2}\,dx \\ &\leq CM_{1-\varepsilon}\big(M_{1/2}(|\mathbf{K}h|)\big)(0). \end{aligned}$$

[In the first and the second formulae of the last string of inequalities, $\cdot$ denotes the inner product in $\mathbf{C}^m$.] This concludes the proof of (2.1). Remark 5 follows from the same argument. □

Proof of Corollary 1. Let $h \in L^2$ and $\max(1/2, p_0) < p \leq 1$. From Theorem 2 and the Hardy-Littlewood maximal theorem, it follows that

$$c_p\|\mathbf{K}h\|_{H^p} \leq \|\mathbf{K}h\|_{L^p} \leq \|\mathbf{K}h\|_{H^p}, \tag{2.2}$$

where

$$\|\mathbf{K}h\|_{H^p} = \|(\mathbf{K}h)^+\|_{L^p}.$$

From the boundedness of singular integral operators on H^p, it follows that

$$\|\mathbf{K}h\|_{H^p} \leq c_p\|h\|_{H^p}. \tag{2.3}$$

On the other hand, by (1.1) there exist multipliers homogeneous of degree zero

$$\Theta_1(\xi), \ldots, \Theta_m(\xi) \in C^\infty(S_{n-1})$$

such that

$$\sum_{j=1}^m \theta_j(\xi)\Theta_j(\xi) \equiv 1 \quad \text{on } S_{n-1}.$$

So

$$(2.4)\quad \|h\|_{H^p} = \left\| \sum \left(\Theta_j(\xi)(K_jh)\hat{}(\xi)\right)\check{}\right\|_{H^p} \leq c_p \sum \left\|K_jh\right\|_{H^p} \leq c_p \|\mathbf{K}h\|_{H^p}.$$

Thus, from (2.2)–(2.4), we get (1.7).

Let $h \in H^p$. Applying (1.7) to $h * P_t$, we get

$$(2.5)\qquad c_{1.4}\|h * P_t\|_{H^p} \leq \sum_{j=1}^{m} \left\|K_j(h * P_t)\right\|_{L^p} \leq c_{1.5}\|h * P_t\|_{H^p}.$$

It is known that

$$h * P_t \to h \quad \text{in } H^p \quad \text{as } t \to +0$$

and that

$$\sup_{t>0} \left|K_j(h * P_t)(x)\right| \in L^p.$$

Thus by the Lebesgue dominated convergence theorem, we get

$$\left\|K_j(h * P_t)\right\|_{L^p} \to \left(\int \left| \lim_{t\to +0} K_j(h * P_t)(x)\right|^p dx\right)^{1/p} \quad \text{as } t \to +0.$$

Therefore, letting $t \to +0$ in (2.5), we get (1.8). □

LEMMA 2.1. *Let $u(x, t)$ be a nonnegative function defined on $\mathbf{R}^n \times [0, +\infty)$ and continuous on $\mathbf{R}^n \times (0, +\infty)$. Let $q > 1$. If*

$$(2.6)\qquad u(x,0) = \lim_{t\to +0} u(x, t) \quad a.e.\ x$$

and if

$$(2.7)\qquad \left|\left\{x \in \mathbf{R}^n\colon \sup_{t\geq 0} u(x, t) > \lambda\right\}\right| \leq \lambda^{-q}$$

for any $\lambda > 0$, then

$$(2.8)\qquad \lim_{t\to +0} M_1(u(\cdot, t))(x) = M_1(u(\cdot,0))(x) \quad a.e.\ x.$$

Proof. Take any $\varepsilon > 0$. By (2.6) and (2.7) there exists $t_0 > 0$ such that $|G| < \varepsilon$, where

$$G = \left\{x \in \mathbf{R}^n\colon \sup_{t\in[0,\, t_0]} |u(x, t) - u(x,0)| > \varepsilon\right\}.$$

Since

$$\int_G \sup_{t\geq 0} u(x, t)\, dx < C\varepsilon^{1-1/q}$$

by (2.7), there exists a measurable set E such that

$$|E| < C\varepsilon^{(1-1/q)/2},$$
$$|M_1(u(\cdot,t))(x) - M_1(u(\cdot,0))(x)| < C\varepsilon^{(1-1/q)/2} + \varepsilon$$

for any $x \in E^c$ and any $t \in [0, t_0]$. Since $\varepsilon > 0$ is arbitrary, we get (2.8). □

Proof of Corollary 2. Put $\theta_0 \equiv 1$ and $K_0 =$ the identity operator. By the usual argument about maximal singular integral operators and the Hardy-Littlewood maximal theorem, we get

$$\lambda\left|\left\{x \in \mathbf{R}^n \colon \sum_{j=0}^{m} \sup_{t>0} |K_j(h * P_t)(x)| > \lambda\right\}\right| \le C\|h\|_M$$

for any $\lambda > 0$ and

$$\limsup_{\lambda\to+\infty} \lambda\left|\left\{x \in \mathbf{R}^n \colon \sum_{j=0}^{m} \sup_{t>0} |K_j(h * P_t)(x)| > \lambda\right\}\right| \le C\|s\|_M. \tag{2.9}$$

It is also known that

$$\kappa_j(x) = \lim_{t\to+0} K_j(h * P_t)(x)$$

exists almost everywhere and that $\kappa_0(x) = f(x)$ a.e. By (2.9)

$$\limsup_{\lambda\to+\infty} \lambda\left|\left\{x \in \mathbf{R}^n \colon M_{1/2}\left(\sum_{j=0}^{m} |\kappa_j|\right)(x) > \lambda\right\}\right| \le C\|s\|_M. \tag{2.10}$$

Applying Lemma 2.1 to

$$u(x,t) = \begin{cases} \left(\sum_{j=0}^{m} |K_j(h * P_t)(x)|\right)^{1/2} & \text{if } t > 0, \\ \left(\sum_{j=0}^{m} |\kappa_j(x)|\right)^{1/2} & \text{if } t = 0, \end{cases}$$

and $q = 2$, we get

$$M_{1/2}\left(\sum_{j=0}^{m} |K_j(h * P_t)|\right)(x) \to M_{1/2}\left(\sum_{j=0}^{m} |\kappa_j|\right)(x) \quad \text{a.e. } x \quad \text{as } t \to +0.$$

Similarly

$$M_{p_0}\left(M_{1/2}\left(\sum_{j=0}^{m} |K_j(h * P_t)|\right)\right)(x) \to M_{p_0}\left(M_{1/2}\left(\sum_{j=0}^{m} |\kappa_j|\right)\right)(x) \quad \text{a.e. } x \quad \text{as } t \to +0. \tag{2.11}$$

Since $\theta_1, \ldots, \theta_m$ satisfy (1.9), $\theta_0, \ldots, \theta_m$ satisfy (1.1). By Remark 5,

$$\begin{aligned}(h * P_t)^{++}(x) &= (K_0(h * P_t))^{++}(x) \\ &\leq C M_{p_0}\left(M_{1/2}\left(\sum_{j=0}^{m} |K_j(h * P_t)| \right)\right)(x)\end{aligned}$$

for any $t > 0$. Letting $t \to +0$, we get

$$h^{++}(x) \leq C M_{p_0}\left(M_{1/2}\left(\sum |\kappa_j| \right)\right)(x) \quad \text{a.e. } x \tag{2.12}$$

form (2.11).

On the other hand, **[18]** and **[34]** showed

$$\liminf_{\lambda \to +\infty} \lambda |\{x \in \mathbf{R}^n : h^{++}(x) > \lambda\}| \geq c\|s\|_M, \tag{2.13}$$

where $c > 0$ depends only on the dimension.Thus, for a sufficient large λ, we have

$$\begin{aligned}\|s\|_M &\leq C\lambda \left| \left\{ x \in \mathbf{R}^n : M_{p_0}\left(M_{1/2}\left(\sum |\kappa_j| \right)\right)(x) > \lambda \right\} \right| \\ &\leq C\lambda^{1-p_0} \int_{\{M_{1/2}(\sum|\kappa_j|)(x) > \lambda/2\}} M_{1/2}\left(\sum |\kappa_j| \right)(x)^{p_0}\, dx \\ &\leq C\lambda^{1-p_0} \left| \left\{ M_{1/2}\left(\sum |\kappa_j| \right)(x) > \lambda/2 \right\} \right|^{1-p_0} \|s\|_M^{p_0}\end{aligned}$$

by (2.10). Therefore

$$\lambda \left| \left\{ M_{1/2}\left(\sum |\kappa_j| \right)(x) > \lambda \right\} \right| \geq C\|s\|_M \quad \text{as } \lambda \to +\infty.$$

Repeating the same argument, we get

$$\lambda \left| \left\{ \sum_{j=0}^{m} |\kappa_j(x)| > \lambda \right\} \right| \geq C\|s\|_M \quad \text{as } \lambda \to +\infty.$$

Since $\lambda\{|\kappa_0(x)| > \lambda\}| \to 0$ as $\lambda \to +\infty$, we get Corollary 2. □

3. Jones's formula. In this section, we explain the relation between Theorem 1 and Jones's recent work **[25]**.

DEFINITION 3.A. A complex measure on the upper half-plane $\mathbf{R}^2_+ = \{(x, t) : x \in \mathbf{R}, t > 0\}$ is called a Carleson measure if

$$\sup_I |\mu|(Q(I))/|I| = \|\mu\|_c < +\infty,$$

where $|\mu|$ is the total variation of μ, I is taken over all intervals.

Suppose that $\|\mu\|_c \leq 1$. It has been shown by Carleson [3] [see also Hörmander [19]] that there exists $F \in L^\infty(\mathbf{R})$ such that

$$\|F\|_{L^\infty} \leq C \tag{3.1}$$

and such that

$$\int_R f(x)F(x)\,dx = \iint_{R_+^2} f(x,t)\,d\mu(x,t) \tag{3.2}$$

for any $f \in L^1(\mathbf{R})$ with $\operatorname{supp} \hat{f} \subset [0, +\infty)$, where

$$f(x,t) = \int_R P_t(y) f(x-y)\,dy,$$
$$P_t(y) = t/\left(\pi(y^2+t^2)\right).$$

Recently, Jones [25] gave an explicit formula for the construction of F.

DEFINITION 3.B. [Jones [25].] For a measure μ on $\mathbf{R}_+^2$ let

$$J(\mu, x, \zeta) = \frac{1}{\pi}\frac{\operatorname{Im}\zeta}{(x-\zeta)(x-\bar{\zeta})} \tag{3.3}$$
$$\times \exp\left(\iint_{0<\operatorname{Im}\eta\leq\operatorname{Im}\zeta} \frac{-i}{x-\bar{\eta}} + \frac{i}{\zeta-\bar{\eta}}\,d|\mu|(\eta)\right)$$

where $i = (-1)^{1/2}$, and ζ and η are complex numbers. [We identify η with $(\operatorname{Re}\eta, \operatorname{Im}\eta) \in \mathbf{R}^2$.]

THEOREM 3.A. [*Jones* [25].] *Let* $\|\mu\|_c \leq 1$. *Set*

$$F(x) = \iint_{R_+^2} J(\mu, x, \zeta)\,d\mu(\zeta).$$

Then,

$$\|F\|_{L^\infty} \leq C$$

and (3.2) *holds.*

Our Theorem 1 can be regarded as a generalization of the formula (3.3). In Jones's argument, we can replace the formula (3.3) by Theorem 1. In the following, we sketch it.

Let H be the Hilbert transform, that is

$$Hf = \left(-i(\operatorname{sign}\xi)\hat{f}(\xi)\right)^{\vee}.$$

For $t > 0$ set

$$u_t(x) = \iint_{\substack{0<s\leq t,\\ y\in R}} s^{-1}(1+|x-y|/s)^{-3/2}\, d|\mu|(y,s).$$

LEMMA 3.A. [*See* [9] *and* [22].] *Let* $\|\mu\|_c \leq 1$. *Then,*

$$\|u_t\|_{\mathrm{BLO}} \leq C_{3.1}$$

and

$$\int_I u_t\, dx \leq C_{3.1}|I|$$

for any cube I with $l(I) \geq t$.

Set $\varepsilon = c_0/C_{3.1}$. Then $e^{-\varepsilon u_t(x)}$ satisfies (1.4) and

$$\int_{y-t}^{y+t} e^{-\varepsilon u_t(x)}\, dx/t \geq C$$

for any $y \in \mathbf{R}$. So, by applying Theorem 1 and Remark 3 to $K_1 =$ the identity operator and $K_2 = -H$ and by using dilation and translation, for each $(y,t) \in \mathbf{R}^2_+$ we get real-valued functions $g_{1,(y,t)}(x)$ and $g_{2,(y,t)}(x)$ such that

$$P_t(y-x) - g_{1,(y,t)}(x) - Hg_{2,(y,t)}(x) \equiv 0,$$
$$|g_{j,(y,t)}(x)| \leq Ce^{-\varepsilon u_t(x)}t^{-1}(1+|y-x|/t)^{-3/2} \qquad (j=1,2).$$

Set

$$F(x) = \iint_{R^2_+} g_{1,(y,t)}(x) + ig_{2,(y,t)}(x)\, d\mu(y,t).$$

Then

$$\begin{aligned}
|F(x)| &\leq C\iint_{R^2_+} e^{-\varepsilon u_t(x)}t^{-1}(1+|y-x|/t)^{-3/2}\, d|\mu|(y,t)\\
&= C\iint \exp\left(-\varepsilon \iint_{\substack{0<s\leq t,\\ v\in R}} s^{-1}(1+|x-v|/s)^{-3/2}\, d|\mu|(v,s)\right)\\
&\qquad\qquad \cdot t^{-1}(1+|y-x|/t)^{-3/2}\, d|\mu|(y,t)\\
&\leq C\varepsilon^{-1}\left[\exp\left(-\varepsilon\iint_{\substack{0<s\leq t,\\ v\in R}} s^{-1}(1+|x-v|/s)^{-3/2}\, d|\mu|(v,s)\right)\right]_{t=+\infty}^{t=0}\\
&\leq C\varepsilon^{-1}
\end{aligned}$$

and

$$\int f(x)F(x)\,dx = \iint_{R^2_+} d\mu(y,t)\int \left(g_{1,(y,t)}(x) + ig_{2,(y,t)}(x)\right)f(x)\,dx$$

$$= \iint_{R^2_+} d\mu(y,t)\int P_t(y-x)f(x)\,dx = \iint_{R^2_+} f(y,t)\,d\mu(y,t)$$

for any $f \in L^1(\mathbf{R})$ with $\operatorname{supp}\hat{f} \subset [0, +\infty)$.

4. Weighted BMO. In the following, we assume (1.4).

DEFINITION 4.1. For a measurable set E let

$$m_w(E) = \int_E w(x)\,dx$$

and

$$w(E) = \sup_{x\in E} w(x).$$

DEFINITION 4.2. For $\mathbf{f}(x) \in L^1_{\mathrm{loc}}(\mathbf{R}^n, \mathbf{C}^m)$, let

$$\|\mathbf{f}\|_{\mathrm{BMO}\,w} = \sup_I \int_I |\mathbf{f}(x) - \mathbf{f}_I|\,dx/m_w(I),$$

where the supremum is taken over all cubes in $\mathbf{R}^n$ and $\mathbf{f}_I = \int_I \mathbf{f}\,dx/|I|$.

For the scalar-valued case, this definition is due to Muckenhoupt-Wheeden **[26]**–**[27]**.

We prepare some easy lemmas.

LEMMA 4.A. *If* $\|\mathbf{f}\|_{\mathrm{BMO}\,w} \le 1$, *then for any cube* I *and any* $\lambda > 0$,

$$\left|\left\{x \in I\colon |\mathbf{f}(x) - \mathbf{f}_I| > \lambda\right\}\right|/|I| \le C_{4.1}e^{-C_{4.2}\lambda/w(I)}.$$

LEMMA 4.B. *For any cube* I *and any* $\lambda > 0$

$$|\{x \in I\colon -\log w(x) > -\log w(I) + \lambda\}|/|I| \le C_{4.1}e^{-C_{4.2}\lambda/c_0}.$$

These follow from **[21]**, (1.4) and **[9]**.

LEMMA 4.1. *For any cubes* I *and* J *and for any* $t > 0$,

$$|\{x \in I\colon w(x) \le tw(I)\}|/|I| \le C_{4.1}t^{C_{4.2}/c_0}, \tag{4.1}$$

(4.2) $\int_I w(I) - w(x)\, dx/m_w(I) \leq Cc_0 \quad i.e.\ (1 + Cc_0)m_w(I) \geq w(I)|I|,$

(4.3) $\quad \text{if } J \supset I, \text{ then} \quad w(J)/w(I) \leq C(|J|/|I|)^{c_0/C_{4.2}},$

(4.4) $\quad \text{if } |I| = |J|, \text{ then} \quad w(J)/w(I) \leq C(1 + |x_I - x_J|/l(I))^{c_0 n/C_{4.2}}.$

LEMMA 4.2.

$$\|w\|_{\mathrm{BMO}\,w} \leq Cc_0.$$

The above two lemmas are easy consequences of Lemma 4.B.

DEFINITION 4.3. For $0 < \varepsilon \leq 1$, let

$$\|\mathbf{f}\|_{\mathrm{Lip}\,\varepsilon} = \sup_{x,y:\, x \neq y} |\mathbf{f}(x) - \mathbf{f}(y)|/|x-y|^{\varepsilon},$$

$$\|\mathbf{f}\|_{\mathrm{Lip}\,2} = \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j}\mathbf{f} \right\|_{\mathrm{Lip}\,1}.$$

LEMMA 4.3. *If* $1 \geq \varepsilon \geq c_0 n/C_{4.2}$ *and if* $\operatorname{supp}\mathbf{f} \subset I(0, t)$, *then*

$$\|\mathbf{f}\|_{\mathrm{BMO}\,w} \leq Ct^{\varepsilon}\|\mathbf{f}\|_{\mathrm{Lip}\,\varepsilon}/w(I(0,t)).$$

Proof. We may assume $t = 1$. Take any cube I in $\mathbf{R}^n$. If $l(I) > 1$ and $I \cap I(0,1) \neq \varnothing$, then

$$\int_I |\mathbf{f}(x) - \mathbf{f}_I|\, dx/m_w(I) \leq C\|\mathbf{f}\|_{L^1}/m_w(I) \leq C\|\mathbf{f}\|_{\mathrm{Lip}\,\varepsilon}/w(I(0,1)).$$

If $l(I) \leq 1$ and $I \cap I(0,1) \neq \varnothing$, then

$$\int_I |\mathbf{f}(x) - \mathbf{f}_I|\, dx/m_w(I) \leq Cl(I)^{\varepsilon}\|\mathbf{f}\|_{\mathrm{Lip}\,\varepsilon}/w(I) \leq C\|\mathbf{f}\|_{\mathrm{Lip}\,\varepsilon}/w(I(0,1))$$

by (4.3). □

MAIN LEMMA. *Let* $c_0 > 0$ *be small enough depending only on* $\theta_1, \ldots, \theta_m$. *Let* $t > 0$. *Suppose that* (1.4),

(4.5) $\quad \|\mathbf{f}\|_{\mathrm{BMO}\,w} < c_0$

and

(4.6) $\quad \operatorname{supp}\mathbf{f} \subset I(0, t)$

hold. Then there exists $\mathbf{g}(x)$ *such that*

$$|\mathbf{g}(x)| \leq w(x)(1+|x|/t)^{-n-1/2} \tag{4.7}$$

and

$$\mathbf{f} - \mathbf{g} \in S. \tag{4.8}$$

We prove this Main Lemma in §9.

5. Proof of Theorem 1. Let $h(t) \in C^\infty([0, +\infty))$ be such that

$$h(t) \geq 0, \qquad \operatorname{supp} h \subset [1/4, 1], \tag{5.1}$$

and

$$\sum_{k=1}^{\infty} h_k(t) = 1 \quad \text{on } [1, +\infty),$$

where

$$h_k(t) = h(2^{-k}t) \quad \text{for } k = 1, 2, 3, \ldots . \tag{5.2}$$

Set

$$h_0(t) = 1 - \sum_{k=1}^{\infty} h_k(t). \tag{5.3}$$

Then

$$\mathbf{f}(x) = \sum_{k=0}^{\infty} h_k(|x|)\mathbf{f}(x)$$

and

$$\|h_k(|x|)\mathbf{f}(x)\|_{\mathrm{BMO}\,w} \leq C2^k \|h_k \mathbf{f}\|_{\mathrm{Lip}\,1}/w(I(0, 2^k)) \leq C2^{-k(n+1)}/w(I(0,1))$$

by (1.2), (1.3) and Lemma 4.3.

Applying the Main Lemma in §4 to each $h_k\mathbf{f}$, we get $\mathbf{g}_k$ such that

$$h_k\mathbf{f} - \mathbf{g}_k \in S,$$

$$|\mathbf{g}_k(x)| \leq c_0^{-1}C2^{-k(n+1)}w(x)(1+2^{-k}|x|)^{-n-1/2}/w(I(0,1)).$$

Set

$$\mathbf{g}(x) = \sum_{k=0}^{\infty} \mathbf{g}_k(x).$$

Then (1.5) is clear and (1.6) follows from

$$\sum_{k=0}^{\infty} |\mathbf{g}_k(x)| \leq c_0^{-1}Cw(x)\sum 2^{-k(n+1)}(1+2^{-k}|x|)^{-n-1/2}/w(I(0,1))$$

$$\leq c_0^{-1}Cw(x)(1+|x|)^{-n-1/2}/w(I(0,1)). \qquad \square$$

6. The property of the space S. The hard part in our argument is the problem, "What property does the space S have?" Since $\theta_1,\dots,\theta_m$ satisfy (1.1), $\bar\theta_1,\dots,\bar\theta_m$ satisfy (1.1). Then by Lemma 2.2 of **[32]** there exist functions

$$\Theta_j(\xi,\boldsymbol{\nu}) \in C^\infty\big(S_{n-1}\times \textstyle\sum_{2m-1}\big), \qquad 1\le j\le m,$$

such that

$$\sum_{j=1}^m \bar\theta_j(\xi)\Theta_j(\xi,\boldsymbol{\nu}) \equiv 1,$$

$$\operatorname{Re}\sum_{j=1}^m \bar\nu_j\big(\Theta_j(\xi,\boldsymbol{\nu}) + \Theta_j(-\xi,\boldsymbol{\nu})\big) \equiv 0,$$

$$\operatorname{Im}\sum_{j=1}^m \bar\nu_j\big(\Theta_j(\xi,\boldsymbol{\nu}) - \Theta_j(-\xi,\boldsymbol{\nu})\big) \equiv 0.$$

This fact tells us that for any $\boldsymbol{\nu}\in\Sigma_{2m-1}$ the set of real-valued functions

$$\{\langle \mathbf{p}(x),\boldsymbol{\nu}\rangle : \mathbf{p}\in S\}$$

is a sufficiently large class of functions. More precisely, we obtain

LEMMA 6.1. *Let $\boldsymbol{\nu}\in\Sigma_{2m-1}$. Let I be a cube. Let $b(x)$ be a real-valued function such that*

$$\operatorname{supp} b \subset 3I, \tag{6.1}$$

$$\int b(x)\,dx = 0, \tag{6.2}$$

$$\|b\|_{\mathrm{Lip}\,2} \le l(I)^{-2}. \tag{6.3}$$

Then there exists a $\mathbf{C}^m$-valued function $\mathbf{p}(x)$ such that

$$\mathbf{p}\in S, \tag{6.4}$$

$$\int \mathbf{p}(x)\,dx = 0, \tag{6.5}$$

$$\langle \mathbf{p}(x),\boldsymbol{\nu}\rangle \equiv b(x), \tag{6.6}$$

$$|\mathbf{p}(x)| \le C\big(1+|x-x_I|/l(I)\big)^{-n-1}, \tag{6.7}$$

$$\left|\frac{\partial}{\partial x_j}\mathbf{p}(x)\right| \le Cl(I)^{-1}\big(1+|x-x_I|/l(I)\big)^{-n-2}, \qquad j=1,\dots,n. \tag{6.8}$$

Proof. Set

$$\tilde p_j(x) = -\Big(\Theta_j(\xi,\boldsymbol{\nu})\big(\operatorname{Re}(\mathbf{K}^*\cdot(b\boldsymbol{\nu}))\big)^\wedge(\xi)\Big)^\vee(x)$$
$$-i\Big(\Theta_j(\xi,i\boldsymbol{\nu})\big(\operatorname{Im}(\mathbf{K}^*\cdot(b\boldsymbol{\nu}))\big)^\wedge(\xi)\Big)^\vee(x)$$

and

$$\tilde{\mathbf{p}}(x) = (\tilde{p}_1(x), \dots, \tilde{p}_m(x)).$$

By the properties of $\{\Theta_j\}$ and by the same argument as in Lemma 2.3 of **[32]**,

$$\mathbf{K}^* \cdot \tilde{\mathbf{p}} = -\mathbf{K}^* \cdot (b\boldsymbol{\nu}), \quad \int \tilde{\mathbf{p}}(x)\,dx = 0, \quad \langle \tilde{\mathbf{p}}(x), \boldsymbol{\nu} \rangle \equiv 0.$$

Set

$$\mathbf{p}(x) = \tilde{\mathbf{p}}(x) + b(x)\boldsymbol{\nu}.$$

Then (6.4)–(6.6) hold. Since $\tilde{p}_j(x)$ can be written in the form of a linear combination of b and its images by Calderon-Zygmund singular integral operators with smooth kernels [see Stein **[29]** p. 75], (6.7)–(6.8) follow from (6.1)–(6.3). See Lemma 2.3 of **[32]** for details. □

LEMMA 6.2. *The function* $\mathbf{p}(x)$ *of Lemma* 6.1 *can be decomposed as follows*:

$$\mathbf{p}(x) = \sum_{j=4}^{\infty} 2^{-j(n+1)} \boldsymbol{\beta}_j(x), \quad \operatorname{supp} \boldsymbol{\beta}_j \subset 2^j I,$$

$$\|\boldsymbol{\beta}_j\|_{\mathrm{Lip}\,1} \leq C/\left(2^j l(I)\right),$$

$$\int \boldsymbol{\beta}_j(x)\,dx = 0,$$

$$\left\langle \boldsymbol{\beta}_j(x), \boldsymbol{\nu} \right\rangle \equiv 0 \quad \text{if } j > 4, \quad \left\langle \boldsymbol{\beta}_4(x), \boldsymbol{\nu} \right\rangle \equiv b(x).$$

Proof. Let $h_k(x)$ be as in (5.2)–(5.3). Then

$$\begin{aligned} \mathbf{p}(x) = &\left\{ h_0(2^{-4}|x|)\mathbf{p}(x) + h_4(|x|) \int \sum_{k=5}^{\infty} h_k(|y|)\mathbf{p}(y)\,dy \Big/ \int h_4(|y|)\,dy \right\} \\ &+ \sum_{j=5}^{\infty} \left\{ h_j(|x|)\mathbf{p}(x) - h_{j-1}(|x|) \int \sum_{k=j}^{\infty} h_k(|y|)\mathbf{p}(y)\,dy \Big/ \int h_{j-1}(|y|)\,dy \right. \\ &\left. + h_j(|x|) \int \sum_{k=j+1}^{\infty} h_k(|y|)\mathbf{p}(y)\,dy \Big/ \int h_j(|y|)\,dy \right\} \end{aligned}$$

gives the desired decomposition. See Lemma 3.5 of **[32]** for details. □

7. Weighted Carleson measures. We continue to assume (1.4).

DEFINITION 7.1. For a measure μ defined on $\mathbf{R}^{n+1}_+$, let

$$\|\mu\|_{c,w} = \sup_I |\mu|(Q(I))/m_w(I),$$

where I is taken over all cubes in $\mathbf{R}^n$.

We prepare some easy lemmas.

LEMMA 7.1. *If* $\|\mu\|_{c,w} \leq 1$, *then for any cube* I

$$\iint_{Q(I)} w(I(x,t))^{-1}\, d|\mu|(x,t) \leq C|I|.$$

Proof. [For the definition of $w(I(x,t))$ recall Definition 4.1.] We may assume that I is a closed dyadic cube. Let $\{I_{k,j}\}_{j=1}^{\infty}$ be the maximal closed dyadic subcubes of I such that

$$w(I_{k,j}) \leq 2^{-k} w(I).$$

By (4.1)

$$\sum_j |I_{k,j}| \leq C_{4.1} 2^{-C_{4.2}k/c_0}|I|.$$

So

$$\begin{aligned}\iint_{Q(I)} w(I(x,t))^{-1}\, d|\mu| &\leq C \sum_{k=0}^{\infty} \sum_j \iint_{Q(I_{k,j})} 2^{k+1} w(I)^{-1}\, d|\mu| \\ &\leq C w(I)^{-1} \sum_k \sum_j 2^{k+1} m_w(I_{k,j}) \\ &\leq C w(I)^{-1} \sum_k 2^{k+1} \sum_j 2^{-k} w(I)|I_{k,j}| \\ &\leq C \sum_k C_{4.1} 2^{-C_{4.2}k/c_0}|I| \leq C|I|. \qquad \square\end{aligned}$$

DEFINITION 7.2. For nonnegative real numbers $\{\lambda_I\}_I$, where I is taken over all dyadic cubes, set

$$\eta_k(x) = \sum_{I:\, l(I)=2^{-k}} \lambda_I \left(1 + 2^k|x - x_I|\right)^{-n-1},$$

$$\varepsilon_k(x) = \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j \eta_{k-j}(x).$$

LEMMA 7.2.

$$\lambda_I \le \left(1 + 2^k|x - x_I|\right)^{n+1}\eta_k(x) \quad \text{if } l(I) = 2^{-k}, \tag{7.1}$$

$$\eta_k(x) \le \left(1 + 2^k|x - y|\right)^{n+1}\eta_k(y), \tag{7.2}$$

$$\varepsilon_k(x) \le \left(1 + 2^k|x - y|\right)^{n+1}\varepsilon_k(y). \tag{7.3}$$

Since this is easy, we omit the proof.

LEMMA 7.3. *Let $c_0 > 0$ be small enough in* (1.4). *Let*

$$\left\| \sum \lambda_I^2 |I| \delta_{(x_I, l(I))} \right\|_{c, w^2} \le 1. \tag{7.4}$$

Then

$$\eta_k(x) \le \varepsilon_k(x) \le Cw\left(I(x, 2^{-k})\right), \tag{7.5}$$

$$\left\| \sum_{k=-\infty}^{+\infty} \varepsilon_k(x)^2 \delta_{t=2^{-k}} \right\|_{c, w^2} \le C, \tag{7.6}$$

where $\delta_{(x,t)}$ is the Dirac measure concentrated at the point $(x, t) \in \mathbf{R}_+^{n+1}$ and $\delta_{t=a}$ denotes the measure induced from n-dimensional Lebesgue measure on the hyperplane $t = a$ in $\mathbf{R}_+^{n+1}$.

Proof. Since $\lambda_J \le Cw(J)$,

$$\begin{aligned} \eta_k(x) &\le C \sum_{J:\, l(J) = 2^{-k}} \left(1 + 2^k \operatorname{dist}(x, J)\right)^{-n-1} w(J) \\ &\le C \sum (\cdots)^{-n-1+Cc_0} w\left(I(x, 2^{-k})\right) \quad \text{by (4.4)} \\ &\le Cw\left(I(x, 2^{-k})\right). \end{aligned}$$

So,

$$\begin{aligned} \varepsilon_k(x) &\le C \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j w\left(I(x, 2^{-k+j})\right) \\ &\le C \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j 2^{jc_0C} w\left(I(x, 2^{-k})\right) \quad \text{by (4.3)} \\ &\le Cw\left(I(x, 2^{-k})\right). \end{aligned}$$

Condition (7.6) follows from almost the same argument as Lemma 3.2 of **[32]** with slight additional estimates about the order of growth of w as in the proof of (7.5). We omit the proof. □

8. The decomposition of weighted BMO functions. We continue to assume (1.4).

Following Chang-R. Fefferman [7], we decompose a weighted BMO function $\mathbf{f}(x)$ and the weight function $w(x)$.

LEMMA 8.1. *Suppose that* $\operatorname{supp}\mathbf{f} \subset I(0,1)$ *and* $\|\mathbf{f}\|_{\mathrm{BMO}\,w} \leq 1$. *Then there exist* $\mathbf{C}^m$*-valued functions* $\{\mathbf{b}_I(x)\}_I$ *and nonnegative real numbers* $\{\lambda_I\}_I$, *where I is taken over all dyadic cubes in* $\mathbf{R}^n$, *such that*

$$\mathbf{f} = \sum_I \lambda_I \mathbf{b}_I, \tag{8.1}$$

$$\lambda_I = 0 \quad \text{if } 3I \cap I(0,1) = \varnothing, \tag{8.2}$$

$$\operatorname{supp}\mathbf{b}_I \subset 3I, \tag{8.3}$$

$$\int \mathbf{b}_I \, dx = 0, \tag{8.4}$$

$$\|\mathbf{b}_I\|_{\mathrm{Lip}\,2} \leq Cl(I)^{-2}, \tag{8.5}$$

$$\left\| \sum_I \lambda_I^2 |I| \delta_{(x_I, l(I))} \right\|_{c, w^2} \leq C. \tag{8.6}$$

Proof. We use the idea of Chang-R. Fefferman [7]. Take a real-valued function $\varphi(x) \in \mathfrak{D}(\mathbf{R}^n)$ such that

$$\operatorname{supp}\varphi \subset \{x \in \mathbf{R}^n : |x| < 1\},$$

$$\int_0^{+\infty} \hat{\varphi}(\xi t)^2 t^{-1} \, dt \equiv 1 \quad \text{for any } \xi \in \mathbf{R}^n \setminus \{0\}.$$

Set

$$\lambda_I = |I|^{-1/2} \left(\iint_{T(I)} |\varphi_t * \mathbf{f}(y)|^2 t^{-1} \, dt \, dy \right)^{1/2}$$

and

$$\mathbf{b}_I(x) = \iint_{T(I)} \varphi_t(x-y)(\varphi_t * \mathbf{f})(y) t^{-1} \, dt \, dy / \lambda_I,$$

where we define $0/0 = 0$ and

$$T(I) = \{(x,t) : x \in I, t \in (l(I)/2, l(I))\}.$$

Then (8.2) is clear. Conditions (8.3)–(8.5) follow from the same argument as in Lemma 3.1 and Remark 3.1 of [32]. See [32] for details.

Since

$$\sum_{I\subset J} \lambda_I^2|I| = \iint_{Q(J)} |\varphi_t * \mathbf{f}(y)|^2 t^{-1}\,dt\,dy$$
$$\leq C\iint_{3J} |\mathbf{f}(x) - \mathbf{f}_J|^2\,dx \leq C|J|w(J)^2$$

for any dyadic cube J by Lemma 4.A, (8.6) holds. □

LEMMA 8.2. *Let* $k > 0$. *In Lemma* 8.1 *set*

$$\mathbf{f}_k(x) = \sum_{I:\, l(I)\geq 2^{-k}} \lambda_I \mathbf{b}_I(x).$$

Then

$$\operatorname{supp}\mathbf{f}_k \subset I(0,3), \tag{8.7}$$

$$|\mathbf{f}_k(x) - \mathbf{f}_k(y)| \leq Cw\big(I(x,2^{-k})\big)2^k|x-y| \quad \text{provided } |x-y| < 2^{-k}. \tag{8.8}$$

Proof. Set

$$\Phi = \left(\int_1^\infty \hat{\varphi}(t\xi)^2 t^{-1}\,dt\right)^{\check{}}$$
$$= \lim_{\varepsilon\to+0 \text{ in } \mathcal{S}'} \left(\delta_0 - \int_\varepsilon^1 \varphi_t * \varphi_t t^{-1}\,dt\right),$$

where δ_0 is the dirac measure concentrated at the origin. Since

$$\mathbf{f}_k = \mathbf{f} * 2^{kn}\Phi(2^k\,\cdot),$$

(8.7) is clear. (8.8) follows from $\|\mathbf{f}\|_{\mathrm{BMO}\,w} \leq 1$ and from

$$\mathbf{f}_k(x) - \mathbf{f}_k(y) = \int \mathbf{f}(z)2^{kn}\big(\Phi(2^k(x-z)) - \Phi(2^k(y-z))\big)\,dz. \quad \square$$

From Lemmas 8.1–8.2 we get

LEMMA 8.3. *Let* $\|\mathbf{f}\|_{\mathrm{BMO}\,w} \leq c_0$. *Let* $\operatorname{supp}\mathbf{f} \subset I(0,1)$. *Let M be a positive integer. Then there exist* $\mathbf{f}_M(x)$, $\{\mathbf{b}_I(x)\}_{I:\ \text{dyadic}}$ *and nonnegative real numbers* $\{\lambda_{f,I}\}_{I:\ \text{dyadic}}$ *such that*

$$\mathbf{f} = \sum_I \lambda_{f,I}\mathbf{b}_I + \mathbf{f}_M, \tag{8.9}$$

(8.10) $\lambda_{f,I} = 0 \quad \textit{if } 3I \cap I(0,1) = \varnothing \textit{ or if } l(I) \geq 2^{-M},$

(8.3)–(8.5),

(8.6)′ $$\Big\| \sum_I \lambda_{f,I}^2 |I| \delta_{(x_I, l(I))} \Big\|_{c,w^2} \leq C c_0^2,$$

(8.7)′ $$\operatorname{supp} \mathbf{f}_M \subset I(0,3),$$

(8.8)′ $$|\mathbf{f}_M(x) - \mathbf{f}_M(y)| \leq C c_0 w(I(x, 2^{-M})) 2^M |x-y|$$

provided $|x-y| < 2^{-M}$.

LEMMA 8.4. *Let $c_0 > 0$ be small enough in* (1.4). *Let M be a positive integer. Then there exist real-valued functions $w_M(x)$, $\{b_I(x)\}_{I:\text{ dyadic}}$ and nonnegative real numbers $\{\lambda_{w,I}\}_{I:\text{ dyadic}}$ such that*

(8.11) $$w = \sum_I \lambda_{w,I} b_I + w_M,$$

(8.12) $$\lambda_{w,I} = 0 \quad \textit{if} \quad l(I) \geq 2^{-M},$$

(8.13) $$\operatorname{supp} b_I \subset 3I,$$

(8.14) $$\int b_I \, dx = 0,$$

(8.15) $$\|b_I\|_{\mathrm{Lip}\,2} \leq C l(I)^{-2},$$

(8.16) $$\Big\| \sum_I \lambda_{w,I}^2 |I| \delta_{(x_I, l(I))} \Big\|_{c,w^2} \leq C c_0^2,$$

(8.17) $$w_k(x) \geq 3w(I(x, 2^{-k}))/4,$$

where $k \geq M$ and

$$w_k(x) = \sum_{I:\, 2^{-M} > l(I) \geq 2^{-k}} \lambda_{w,I} b_I(x) + w_M(x).$$

Proof. Take the same $\varphi(x)$ as in the proof of Lemma 8.1. If $l(I) < 2^{-M}$, then set

$$\lambda_{w,I} = |I|^{-1/2} \bigg(\iint_{T(I)} |\varphi_t * w(y)|^2 t^{-1} \, dt \, dy \bigg)^{1/2}$$

and

$$b_I(x) = \iint_{T(I)} \varphi_t(x-y)(\varphi_t * w)(y) t^{-1} \, dt \, dy / \lambda_{w,I}.$$

If $l(I) \geq 2^{-M}$, then set $\lambda_{w,I} = 0$ and $b_I(x) = 0$. Set

$$w_M(x) = w(x) - \sum_{I:\, l(I)<2^{-M}} \lambda_{w,I} b_I(x).$$

Then (8.11)–(8.12) are clear. Conditions (8.13)–(8.16) follow from Lemma 4.2 and the same argument as in the proof of Lemma 8.1.

Let $k \geq M$. Take the same Φ as in the proof of Lemma 8.2. Then

$$w_k = w * 2^{kn}\Phi(2^k).$$

Put $J = I(x, 2^{-k})$. Since

$$\begin{aligned} |w(J) - w_k(x)| &= \left| \int (w(J) - w(y)) 2^{kn}\Phi(2^k(x-y))\, dy \right| \\ &\leq C \int_{2J} |w(J) - w(y)|\, dy/|J| \leq Cc_0 w(J) \end{aligned}$$

by (4.2)–(4.3), we get (8.17). □

LEMMA 8.5. *Let j be a positive integer. Assume that $\{\mathbf{b}_I(x)\}_{I:\text{ dyadic}}$ and $\{\lambda_I\}_{I:\text{ dyadic}}$ satisfy* (8.4), (8.6),

$$\operatorname{supp} \mathbf{b}_I \subset 2^j I \tag{8.18}$$

and

$$\|\mathbf{b}_I\|_{\mathrm{Lip}\,1} \leq \left(2^j l(I)\right)^{-1}. \tag{8.19}$$

Let $\alpha > 0$. Set

$$\mathbf{f}(x) = \sum_{I:\, l(I)<\alpha} \lambda_I \mathbf{b}_I(x).$$

Then

$$\|\mathbf{f}\|_{\mathrm{BMO}\, w} \leq C 2^{jn}, \tag{8.20}$$

where C is independent of α.

Proof. Take any cube J (not necessarily dyadic). Let $2^{-k} \leq l(J) < 2^{-k+1}$. Set

$$\tilde{\mathbf{f}} = \sum_{I:\, l(I)\geq 2^{-j-k+1}} \lambda_I \mathbf{b}_I$$

and

$$\tilde{\tilde{\mathbf{f}}} = \sum_{\substack{I:\, l(I)<2^{-j-k+1},\\ I\cap 3J\neq\varnothing}} \lambda_I \mathbf{b}_I.$$

Then

$$\mathbf{f} = \tilde{\mathbf{f}} + \tilde{\tilde{\mathbf{f}}} \quad \text{on } J. \tag{8.21}$$

Note that

$$\begin{aligned} |\tilde{\mathbf{f}}(x) - \tilde{\mathbf{f}}(y)| &\leq \sum_{h=-\infty}^{j+k-1} \sum_{l(I)=2^{-h}} \lambda_I |\mathbf{b}_I(x) - \mathbf{b}_I(y)| \\ &\leq \sum \sum \lambda_I 2^{-j+h} |x-y| \chi_{2^{j+1}I}(x) \\ &\leq 2^k |x-y| \sum \sum \lambda_I 2^{-j-k+h} \chi_{2^{j+1}I}(x) \\ &\leq 2^k |x-y| 2^{jn} w(I(x, 2^{-k})) \end{aligned} \tag{8.22}$$

provided $|x-y| < 2^{-k}$. By (8.4), (8.6), (8.18), (8.19) and by Lemma 3.3 of **[32]** we get

$$\begin{aligned} \|\tilde{\tilde{\mathbf{f}}}\|_{L^2} &\leq C 2^{jn} \Bigg(\sum_{\substack{I:\, l(I) < 2^{-j-k+1}, \\ I \cap 3J \neq \varnothing}} \lambda_I^2 |I| \Bigg)^{1/2} \\ &\leq C 2^{jn} w(J) |J|^{1/2}. \end{aligned} \tag{8.23}$$

Thus by (8.21)–(8.23)

$$\begin{aligned} &\int_J |\mathbf{f}(x) - \tilde{\mathbf{f}}(x_J)| \, dx / m_w(J) \\ &\quad \leq C \left(\int_J |\mathbf{f}(x) - \tilde{\mathbf{f}}(x_J)|^2 \, dx / |J| \right)^{1/2} \Big/ w(J) \leq C 2^{jn}. \qquad \square \end{aligned}$$

9. Proof of the Main Lemma in §4. We may assume $t = 1$ in (4.6) and

$$w(I(0,1)) = 1. \tag{9.1}$$

In this section $C_{9.1}$ is a large constant depending only on $\theta_1, \dots, \theta_m$. Let M be a large integer depending only on $\theta_1, \dots, \theta_m$ and $C_{9.1}$. Let $c_0 > 0$ be small enough depending only on $\theta_1, \dots, \theta_m$, $C_{9.1}$ and M. In particular

$$C_{9.1} 2^{-M} < 1 \quad \text{and} \quad C_{9.1}^4 2^{M(n+2)} c_0 < 1. \tag{9.2}$$

First, we give a rough explanation of the procedure to construct $\mathbf{g}(x)$. We construct a sequence $\{\mathbf{g}_k\}_{k=M}^{\infty}$ such that

(i) $|\mathbf{g}_k(x)| \leq w_k(x) \chi_{I(0,4)}(x)$,

(ii) $\mathbf{f}_k - \mathbf{g}_k + (\text{small errors}) \in S$.

[For the definitions of w_k and $\mathbf{f}_k$, recall Lemmas 8.2 and 8.4.] Then by letting $k \to +\infty$, we get $\tilde{\mathbf{g}}$ such that

$$|\tilde{\mathbf{g}}(x)| \le w(x)\chi_{I(0,4)}(x),$$
$$\mathbf{f} - \tilde{\mathbf{g}} + (\text{small errors}) \in S.$$

Next we estimate the weighted BMO norms of the error terms and repeat the same procedure for them.

In order to meet the condition (i), we must adjust the length of the vector-valued function $\mathbf{g}_k$. We must do this adjustment under the restriction (ii). Here we use the property of the space S that was proved in Lemma 6.1.

Now we go into details.

By Lemmas 8.3–8.4, we get

$$\mathbf{f}_M(x),\ \{\mathbf{b}_I(x)\}_{I:\text{ dyadic}} \quad \text{and} \quad \{\lambda_{f,I}\}_{I:\text{ dyadic}}$$
$$w_M(x),\ \{b_I(x)\}_{I:\text{ dyadic}} \quad \text{and} \quad \{\lambda_{w,I}\}_{I:\text{ dyadic}}$$

such that (8.9)–(8.10), (8.3)–(8.5), (8.6)′–(8.8)′ and (8.11)–(8.17) hold.

Set

$$\lambda_I = \lambda_{f,I} + \lambda_{w,I} \quad \text{if} \quad l(I) < 2^{-M}, \tag{9.3}$$
$$\lambda_I = c_0 w(I) \quad \text{if} \quad l(I) = 2^{-M}, \tag{9.4}$$
$$\lambda_I = 0 \quad \text{if} \quad l(I) > 2^{-M}. \tag{9.5}$$

Lemma 9.1.

$$\left\| \sum_I \lambda_I^2 |I| \delta_{(x_I,\, l(I))} \right\|_{c,w^2} \le Cc_0^2.$$

This is clear from (8.6)′ and (8.16).

From these $\{\lambda_I\}_I$, we define $\eta_k(x)$ and $\varepsilon_k(x)$ by Definition 7.2. Then by Lemmas 9.1 and 7.3 we get

Lemma 9.2. *If $x \in I$ and $l(I) = 2^{-k}$, then*

$$C\lambda_I \le \eta_k(x) \le \varepsilon_k(x) \le C'c_0 w(I).$$

Lemma 9.3.

$$\left\| \sum_{k=-\infty}^{\infty} \varepsilon_k(x)^2 \delta_{t=2^{-k}} \right\|_{c,w^2} \le Cc_0^2.$$

We inductively construct

$$\{\boldsymbol{\varphi}_k(x)\}_{k=M+1}^{\infty} \quad \text{and} \quad \{\boldsymbol{\beta}_{I,j}(x)\}_{I:\ \text{dyadic},\, l(I)<2^{-M};\ j=4,5,6,\ldots}$$

with the following properties (C.1)–(C.8). Put

$$(9.6)\qquad \mathbf{p}_{I,1}(x) = \sum_{j=4}^{M-1} 2^{-j(n+1)}\boldsymbol{\beta}_{I,j}(x),$$

$$(9.7)\qquad \mathbf{p}_{I,2}(x) = \sum_{j=M}^{\infty} 2^{-j(n+1)}\boldsymbol{\beta}_{I,j}(x),$$

$$(9.8)\qquad \mathbf{g}_M(x) = \mathbf{f}_M(x),$$

$$(9.9)\qquad \mathbf{g}_k(x) = \mathbf{f}_k(x) + \sum_{h=M+1}^{k} \sum_{I:\, l(I)=2^{-h}} \lambda_I \mathbf{p}_{I,1}(x) - \sum_{h=M+1}^{k} \boldsymbol{\varphi}_h(x)$$

for $k = M+1, M+2, \ldots$.

$$(\text{C.1})\qquad \operatorname{supp}\boldsymbol{\beta}_{I,j} \subset 2^j I,\quad \|\boldsymbol{\beta}_{I,j}\|_{\text{Lip}\,1} \le C_{9.1} 2^{-j} l(I)^{-1},\ \int \boldsymbol{\beta}_{I,j}\,dx = 0,$$

$$(\text{C.2})\qquad \boldsymbol{\beta}_{I,j}(x) \equiv 0 \quad \text{if} \quad I \cap I(0,4) = \varnothing,$$

$$(\text{C.3})\qquad \mathbf{p}_{I,1} + \mathbf{p}_{I,2} \in S,$$

$$(\text{C.4})\qquad |\boldsymbol{\varphi}_k(x)| \le C_{9.1}^2 \varepsilon_k(x)^2 2^{M(n+2)}/w(I(x,2^{-k})),$$

$$(\text{C.5})\qquad |\boldsymbol{\varphi}_k(x) - \boldsymbol{\varphi}_k(y)| \le C_{9.1}^4 c_0 2^{M(n+2)} \varepsilon_k(x) 2^k |x-y| \quad \text{provided } |x-y| < 2^{-k},$$

$$(\text{C.6})\qquad \operatorname{supp}\boldsymbol{\varphi}_k \subset \operatorname{supp}\mathbf{g}_k \subset I(0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-k+M}),$$

$$(\text{C.7})\qquad |\mathbf{g}_k(x)| \le w_k(x),$$

$$(\text{C.8})\qquad |\mathbf{g}_k(x) - \mathbf{g}_k(y)| \le C_{9.1}\varepsilon_k(x) 2^k |x-y| \quad \text{provided } |x-y| < 2^{-k}.$$

The construction of the above functions is explained at the end of this section. We accept this construction temporarily and prove the Main Lemma. By the same argument as **[32]**, we can show that $\tilde{\mathbf{g}} = \lim_{k\to\infty} \mathbf{g}_k$ exists in L^2. By (C.6)–(C.7), we get

$$(9.10)\qquad \operatorname{supp}\tilde{\mathbf{g}} \subset I(0,4)$$

and

$$(9.11)\qquad |\tilde{\mathbf{g}}(x)| \le w(x).$$

By (9.9)

$$(9.12)\qquad \tilde{\mathbf{g}}(x) = \mathbf{f}(x) + \sum_{I:\, l(I)<2^{-M}} \lambda_I \mathbf{p}_{I,1}(x) - \sum_{h=M+1}^{\infty} \boldsymbol{\varphi}_h(x).$$

Set

$$\mathbf{p}_2(x) = \sum_{I:\, l(I)<2^{-M}} \lambda_I \mathbf{p}_{I,2}(x), \tag{9.13}$$

$$\boldsymbol{\varphi}(x) = \sum_{h=M+1}^{\infty} \boldsymbol{\varphi}_h(x). \tag{9.14}$$

LEMMA 9.4.

$$\mathbf{f} - (\tilde{\mathbf{g}} + \mathbf{p}_2 + \boldsymbol{\varphi}) \in S.$$

Proof. Since

$$\tilde{\mathbf{g}} = \mathbf{f} + \sum_{I:\, l(I)<2^{-M}} \lambda_I(\mathbf{p}_{I,1} + \mathbf{p}_{I,2}) - \mathbf{p}_2 - \boldsymbol{\varphi}$$

by (9.12), the lemma follows from (C.3). □

LEMMA 9.5.

$$\operatorname{supp} \boldsymbol{\varphi} \subset I(0,4), \tag{9.15}$$

$$\|\boldsymbol{\varphi}\|_{\mathrm{BMO}\, w} \le C C_{9.1}^4 c_0^2 2^{M(n+2)}. \tag{9.16}$$

Proof. Condition (9.15) is clear from (C.6). Take any I (not necessarily dyadic). Then

$$\begin{aligned}
&\int_I \sum_{k:\, 2^{-k}<l(I)} |\boldsymbol{\varphi}_k(x)|\,dx/m_w(I) \\
&\quad \le C_{9.1}^2 2^{M(n+2)} \int_I \sum_{k:\, 2^{-k}<l(I)} \varepsilon_k(x)^2 w\big(I(x,2^{-k})\big)^{-1}\,dx/m_w(I) \\
&\hspace{20em} \text{by (C.4)} \\
&\quad \le C_{9.1}^2 2^{M(n+2)} \int_I \sum \varepsilon_k(x)^2 w\big(I(x,2^{-k})\big)^{-2}\,dx/|I| \\
&\quad \le C C_{9.1}^2 c_0^2 2^{M(n+2)}
\end{aligned}$$

by Lemmas 9.3 and 7.1. On the other hand,

$$\Big|\sum_{k:\, 2^{-k}\ge l(I)} (\boldsymbol{\varphi}_k(x) - \boldsymbol{\varphi}_k(y))\Big| \Big/ w(I) \le C C_{9.1}^4 c_0^2 2^{M(n+2)}$$

if x, $y \in I$ by (C.5). Thus

$$\int_I \left| \sum_{k=M+1}^{\infty} \varphi_k(x) - \sum_{2^{-k} \geq l(I)} \varphi_k(x_I) \right| dx/m_w(I) \leq CC_{9.1}^4 c_0^2 2^{M(n+2)}. \qquad \square$$

Set

$$\mathbf{f}_2 = \varphi,$$

$$\mathbf{f}_3 = \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I:\, l(I) \leq 2^{-j}} \lambda_I \boldsymbol{\beta}_{I,j},$$

$$\mathbf{f}_k = \sum_{\substack{j,I:\, l(I)=2^{k-j-3}, \\ l(I)<2^{-M}}} 2^{-j(n+1)} \lambda_I \boldsymbol{\beta}_{I,j}, \qquad k \geq 4.$$

LEMMA 9.6.

$$\sum_{k=2}^{\infty} \mathbf{f}_k = \varphi + \mathbf{p}_2.$$

LEMMA 9.7. *For* $k \geq 3$,

$$\operatorname{supp} \mathbf{f}_k \subset I(0, 2^k), \tag{9.17}$$

$$\int \mathbf{f}_k \, dx = 0, \tag{9.18}$$

$$\|\mathbf{f}_k\|_{\mathrm{BMO}\, w} \leq CC_{9.1} c_0 2^{-M} 2^{-k(n+1)}. \tag{9.19}$$

Proof. We show only (9.19). If $k \geq 4$, then

$$\begin{aligned} \|\mathbf{f}_k\|_{\mathrm{BMO}\, w} &\leq \sum_{j=M+k-2}^{\infty} \sum_{I:\, l(I)=2^{k-j-3}} 2^{-j(n+1)} \lambda_I \|\boldsymbol{\beta}_{I,j}\|_{\mathrm{BMO}\, w} \\ &\leq \sum_j \sum_I 2^{-j(n+1)} C c_0 \leq \sum_j 2^{-j(n+1)} C c_0 2^{(j-k)n} \\ &\leq C c_0 2^{-M} 2^{-k(n+1)}. \end{aligned}$$

If $k = 3$, then

$$\begin{aligned} \|\mathbf{f}_3\|_{\mathrm{BMO}\, w} &= \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \left\| \sum_{I:\, l(I) \leq 2^{-j}} \lambda_I \boldsymbol{\beta}_{I,j} \right\|_{\mathrm{BMO}\, w} \\ &\leq \sum_{j=M+1}^{\infty} 2^{-j} C c_0 \leq C c_0 2^{-M} \quad \text{by Lemmas 8.5 and 9.1.} \qquad \square \end{aligned}$$

From Lemmas 9.4–9.7, we obtain the following.

LEMMA 9.8. *Assume the hypothesis of the Main Lemma. Then there exist* $\tilde{\mathbf{g}}(x)$ *and* $\{\mathbf{f}_j(x)\}_{j=2}^{\infty}$ *such that*

$$\mathbf{f} - \left(\tilde{\mathbf{g}} + \sum_{j=2}^{\infty} \mathbf{f}_j\right) \in S, \tag{9.20}$$

$$\operatorname{supp} \mathbf{f}_j \subset I(0, 2^j t), \tag{9.21}$$

$$\|\mathbf{f}_j\|_{\mathrm{BMO}\,w} \leq c_0 \alpha(M, c_0) 2^{-j(n+1)}, \tag{9.22}$$

$$|\tilde{\mathbf{g}}(x)| \leq w(x), \tag{9.23}$$

$$\operatorname{supp} \tilde{\mathbf{g}} \subset I(0, 4t), \tag{9.24}$$

where

$$\alpha(M, c_0) = C\left(C_{9.1} 2^{-M} + C_{9.1}^4 c_0 2^{M(n+2)}\right).$$

Since we have assumed $t = 1$ at the beginning of this section, we showed the above only for the case $t = 1$. But the general case follows easily from the case $t = 1$.

Proof of the Main Lemma. We continue to assume $t = 1$. Take M and c_0 so that

$$1 + \alpha(M, c_0) < 2^{1/4}. \tag{9.25}$$

Applying Lemma 9.8 to $\mathbf{f}$, we obtain $\tilde{\mathbf{g}}$ and $\{\mathbf{f}_j\}_{j=2}^{\infty}$ with (9.20)–(9.24). Next, applying Lemma 9.8 to each $\mathbf{f}_j$, we obtain $\tilde{\mathbf{g}}_j$ and $\{\mathbf{f}_{j,k}\}_{k=2}^{\infty}$. Repeating this process, we obtain $\{\tilde{\mathbf{g}}_{j_1,\dots,j_i}\}$ and $\{\mathbf{f}_{j_1,\dots,j_i}\}$ such that

$$\mathbf{f}_{j_1,\dots,j_i} - \left(\tilde{\mathbf{g}}_{j_1,\dots,j_i} + \sum_{k=2}^{\infty} \mathbf{f}_{j_1,\dots,j_i,k}\right) \in S,$$
$$\operatorname{supp} \mathbf{f}_{j_1,\dots,j_i,k} \subset I(0, 2^{j_1+\dots+j_i+k}),$$
$$\left\|\mathbf{f}_{j_1,\dots,j_i,k}\right\|_{\mathrm{BMO}\,w} \leq c_0 \alpha^{i+1} 2^{-(j_1+\dots+j_i+k)(n+1)},$$
$$\left|\tilde{\mathbf{g}}_{j_1,\dots,j_i}(x)\right| \leq \alpha^i 2^{-(j_1+\dots+j_i)(n+1)} w(x),$$
$$\operatorname{supp} \tilde{\mathbf{g}}_{j_1,\dots,j_i} \subset I(0, 4 \cdot 2^{j_1+\dots+j_i}).$$

Set

$$\mathbf{g}^i = \tilde{\mathbf{g}} + \sum_{s=1}^{i} \sum_{j_1,\dots,j_s} \tilde{\mathbf{g}}_{j_1,\dots,j_s}.$$

Then

$$\mathbf{f} - \left(\mathbf{g}^i + \sum_{j_1,\ldots,j_{i+1}} \mathbf{f}_{j_1,\ldots,j_{i+1}}\right) \in S.$$

Set

$$\mathbf{g} = \lim_{i\to\infty} \mathbf{g}^i.$$

Since $\sum_{j_1,\ldots,j_{i+1}} \mathbf{f}_{j_1,\ldots,j_{i+1}}$ tends to 0 in L^2 as $i \to \infty$, $\mathbf{g}$ satisfies (4.8). On the other hand,

$$\begin{aligned}\mathbf{g} &= \tilde{\mathbf{g}} + \sum_{k=1}^{\infty} \sum_{s:\ 1\le s\le k/2} \sum_{j_1,\ldots,j_s:\ j_1+\cdots+j_s=k} \tilde{\mathbf{g}}_{j_1,\ldots,j_s} \\ &= \tilde{\mathbf{g}} + \sum_{k=1}^{\infty} (9.26)_k\end{aligned}$$

and

$$\operatorname{supp}(9.26)_k \subset I(0, 4\cdot 2^k),$$

$$|(9.26)_k| \le 2^{-k(n+1)} w(x) \sum_{s:\ 1\le s\le k/2} \alpha^s \binom{k+s-1}{s-1} \le 2^{-k(n+1/2)} w(x)$$

by (9.25). Thus, (4.7) holds. □

Construction of $\{\boldsymbol{\beta}_{I,j}\}$ and $\{\boldsymbol{\varphi}_k\}$.

We construct these functions inductively. We define $\mathbf{g}_M$ by (9.8). Then

$$\operatorname{supp} \mathbf{g}_M \subset I(0,3) \tag{9.27}$$

by (8.7)′. Since

$$w_M(x) \ge C2^{-Cc_0M} w(I(0,1))$$

by (4.3) and (8.17) and since

$$|\mathbf{f}_M(x)| \le Cc_0 M,$$

we get

$$|\mathbf{g}_M(x)| \le w_M(x) \tag{9.28}$$

if $c_0 > 0$ is small enough depending on M. By (8.8)′ and (9.4)

$$|\mathbf{g}_M(x) - \mathbf{g}_M(y)| \le C\varepsilon_M(x) 2^M |x-y| \tag{9.29}$$

provided $|x-y| < 2^{-M}$. [Recall that $\varepsilon_k(x)$ is defined by Definition 7.2 from $\{\lambda_I\}$ defined by (9.3)–(9.5).]

Let $k > M$. Suppose that

$$\{\boldsymbol{\beta}_{I,j}\}_{2^{-M}>l(I)>2^{-k},j=4,5,6,\ldots} \quad \text{and} \quad \{\boldsymbol{\varphi}_h\}_{h=M+1,\ldots,k-1}$$

have been constructed and that $\mathbf{g}_{k-1}$ defined by (9.8)–(9.9) satisfies

(C.6)′ $\operatorname{supp}\mathbf{g}_{k-1} \subset I(0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-(k-1-M)})$,

(C.7)′ $|\mathbf{g}_{k-1}(x)| \leq w_{k-1}(x)$,

(C.8)′ $|\mathbf{g}_{k-1}(x) - \mathbf{g}_{k-1}(y)| \leq C_{9.1}\varepsilon_{k-1}(x)2^{k-1}|x-y|$

provided $|x-y| < 2^{-k+1}$.

Notice that by (9.27)–(9.29) $\mathbf{g}_M$ satisfies the above (C.6)–(C.8).

LEMMA 9.9. *If* $|x-y| \leq 2^{M-k}$, *then*

$$|\mathbf{g}_{k-1}(x) - \mathbf{g}_{k-1}(y)| \leq C_{9.1}2^{M(n+1)}\varepsilon_{k-1}(x)2^k|x-y|.$$

This follows from (C.8)′ and (7.3).

Set

$$\tilde{w}_k(x) = |\mathbf{g}_{k-1}(x)| + \sum_{I:\,(9.31)} \lambda_{w,I} b_I(x) \tag{9.30}$$

where Σ is taken over all dyadic cubes I such that

$$\begin{aligned} &l(I) = 2^{-k} \quad \text{and} \\ &I \cap I(0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-(k-1-M)}) \neq \varnothing. \end{aligned} \tag{9.31}$$

[Recall that $\{b_I\}$ and $\{\mathbf{b}_I\}$ are defined by Lemmas 8.3–8.4.]

LEMMA 9.10. *If* $|x-y| < 2^{-k}$, *then*

$$|\tilde{w}_k(x) - \tilde{w}_k(y)| \leq CC_{9.1}\varepsilon_k(x)2^k|x-y|.$$

This follows from (C.8)′, the first two inequalities in Lemma 9.2 and (8.13)–(8.15).

From now we explain how to construct

$$\{\boldsymbol{\beta}_{I,j}\}_{l(I)=2^{-k},\, j=4,5,6,\ldots} \text{ and } \boldsymbol{\varphi}_k.$$

For each I with (9.31) we apply Lemma 6.1 to

$$\boldsymbol{\nu} = U(\mathbf{g}_{k-1}(x_I)),$$

$$b(x) = \lambda'_{w,I} b_I(x) - \langle \lambda'_{f,I}\mathbf{b}_I(x), \boldsymbol{\nu}\rangle,$$

where

$$\lambda'_{w,I} = \lambda_{w,I}/\lambda_I,$$
$$\lambda'_{f,I} = \lambda_{f,I}/\lambda_I.$$

[For the sake of convenience, we define $U(\mathbf{0}) = (1, 0, \dots, 0)$ and $0/0 = 0$.] Then we get $\mathbf{p}_I(x)$ satisfying (6.4)–(6.5), (6.7)–(6.8) and

$$(6.6)' \quad \langle \mathbf{p}_I(x), U(\mathbf{g}_{k-1}(x_I)) \rangle = \lambda'_{w,I} b_I(x) - \left\langle \lambda'_{f,I} \mathbf{b}_I(x), U(\mathbf{g}_{k-1}(x_I)) \right\rangle.$$

Applying Lemma 6.2 to $\mathbf{p}_I(x)$, we get $\{\boldsymbol{\beta}_{I,j}\}_{j=4}^{\infty}$. Define $\mathbf{p}_{I,1}(x)$ and $\mathbf{p}_{I,2}(x)$ by (9.6)–(9.7). Then (C.1)–(C.3) are clear.

Set

$$\mathbf{q}_I(x) = \mathbf{p}_{I,1}(x) + \lambda'_{f,I} \mathbf{b}_I(x), \tag{9.32}$$

$$\mathbf{h}(x) = \sum_{I:\,(9.31)} \lambda_I \mathbf{q}_I(x), \tag{9.33}$$

$$\mathbf{k}(x) = \mathbf{g}_{k-1}(x) + \mathbf{h}(x). \tag{9.34}$$

Then

$$\operatorname{supp} \mathbf{q}_I \subset 2^{M-1} I, \tag{9.35}$$

$$|\mathbf{q}_I(x)| \le C\left(1 + 2^k |x - x_I|\right)^{-n-1}, \tag{9.36}$$

$$|\mathbf{q}_I(x) - \mathbf{q}_I(y)| \le C 2^k |x - y| \left(1 + 2^k |x - x_I|\right)^{-n-2} \quad \text{provided that } |x - y| < 2^{-k}, \tag{9.37}$$

$$|\mathbf{h}(x)| \le \sum_{I:\,(9.31)} \lambda_I |\mathbf{q}_I(x)| \le C \eta_k(x) \quad \text{by (9.36)}, \tag{9.38}$$

$$|\mathbf{h}(x) - \mathbf{h}(y)| \le \sum \lambda_I |\mathbf{q}_I(x) - \mathbf{q}_I(y)| \le C \eta_k(x) 2^k |x - y| \quad \text{provided } |x - y| < 2^{-k} \text{ by (9.37)}, \tag{9.39}$$

$$\operatorname{supp} \mathbf{k} \subset I\left(0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-k+M}\right) \quad \text{by (C.6)}' \text{ and (9.35)}, \tag{9.40}$$

$$\begin{aligned} |\mathbf{k}(x) - \mathbf{k}(y)| &\le |\mathbf{g}_{k-1}(x) - \mathbf{g}_{k-1}(y)| + |\mathbf{h}(x) - \mathbf{h}(y)| \\ &\le (C_{9.1} \varepsilon_{k-1}(x)/2 + C \eta_k(x)) 2^k |x - y| \\ &\le \tfrac{3}{4} C_{9.1} \varepsilon_k(x) 2^k |x - y| \\ &\qquad \text{provided } |x - y| < 2^{-k} \text{ by (C.8)}' \text{ and (9.39)} \end{aligned} \tag{9.41}$$

since $C_{9.1}$ is large enough.

Set

$$\mathbf{k}_1(x) = \langle \mathbf{k}(x), U(\mathbf{g}_{k-1}(x)) \rangle U(\mathbf{g}_{k-1}(x)) \tag{9.42}$$

and

$$\begin{aligned} \mathbf{k}_2(x) &= \mathbf{k}(x) - \mathbf{k}_1(x) \\ &= \mathbf{h}(x) - \langle \mathbf{h}(x), U(\mathbf{g}_{k-1}(x)) \rangle U(\mathbf{g}_{k-1}(x)). \end{aligned} \tag{9.43}$$

Then $\mathbf{k}_1$ and $\mathbf{k}_2$ are orthogonal. Set

$$v_I(x) = \langle \mathbf{q}_I(x), U(\mathbf{g}_{k-1}(x)) - U(\mathbf{g}_{k-1}(x_I)) \rangle. \tag{9.44}$$

Then

$$\begin{aligned} \langle \mathbf{q}_I(x), U(\mathbf{g}_{k-1}(x)) \rangle &= \langle \mathbf{q}_I(x), U(\mathbf{g}_{k-1}(x_I)) \rangle + v_I(x) \\ &= \langle \mathbf{p}_{I,1}(x), U(\mathbf{g}_{k-1}(x_I)) \rangle \\ &\quad + \langle \lambda'_{f,I} \mathbf{b}_I(x), U(\mathbf{g}_{k-1}(x_I)) \rangle + v_I(x) \\ &= \langle \mathbf{p}_I(x), U(\mathbf{g}_{k-1}(x_I)) \rangle + \cdots + \cdots \\ &= \lambda'_{w,I} b_I(x) + v_I(x) \end{aligned} \tag{9.45}$$

by (9.32) and (6.6)′. Thus

$$\begin{aligned} \mathbf{k}_1(x) &= \mathbf{g}_{k-1}(x) + \langle \mathbf{h}(x), U(\mathbf{g}_{k-1}(x)) \rangle U(\mathbf{g}_{k-1}(x)) \\ &= \tilde{w}_k(x) U(\mathbf{g}_{k-1}(x)) + \sum_{I:\,(9.31)} \lambda_I v_I(x) U(\mathbf{g}_{k-1}(x)) \end{aligned} \tag{9.46}$$

by (9.33), (9.45) and (9.30).

Take any dyadic cube J with $l(J) = 2^{-k}$.

LEMMA 9.11. (i) *If*

$$|\mathbf{g}_{k-1}(x_J)| \le 3w_{k-1}(x_J)/4, \tag{9.47}$$

then

$$|\mathbf{k}(x)| \le 7w_k(x)/8 \quad \text{on } J.$$

(ii) *If*

$$|\mathbf{g}_{k-1}(x_J)| \ge w_{k-1}(x_J)/2, \tag{9.48}$$

then

$$|\mathbf{g}_{k-1}(x)| \ge w(J)/4 \quad \text{on } 2^M J.$$

Proof. By (8.17) and Lemma 9.2,

$$w_k(x) > (1 - Cc_0)w_{k-1}(x).$$

By (9.38) and Lemma 9.2,

$$|\mathbf{h}(x)| \le Cc_0 w_k(x).$$

Thus (i) holds since c_0 is small enough.

Let x, $y \in 2^M J$. Then, by Lemmas 9.9 and 9.2

$$|\mathbf{g}_{k-1}(x) - \mathbf{g}_{k-1}(y)| \le CC_{9.1}2^{M(n+2)}c_0 w(J).$$

Since c_0 is small enough depending on M and $C_{9.1}$, (ii) follows from (8.17). □

LEMMA 9.12. *If* (9.48) *holds and if* $|x - x_J|, |y - x_J| \le 2^{M-k}$, *then*

$$(9.49)\qquad |U(\mathbf{g}_{k-1}(x)) - U(\mathbf{g}_{k-1}(y))| \le CC_{9.1}2^{M(n+1)}\varepsilon_{k-1}(x)2^{k-1}|x-y|/w(J),$$

$$(9.50)\qquad |U(\mathbf{k}(x)) - U(\mathbf{k}(y))| \le CC_{9.1}2^{M(n+1)}\varepsilon_k(x)2^k|x-y|/w(J).$$

The first inequality follows from Lemma 9.9 and part (ii) of Lemma 9.11. The second inequality follows from (9.41) and part (ii) of Lemma 9.11.

LEMMA 9.13. *If* (9.48) *holds, then*

$$(9.51)\qquad |v_I(x)| \le CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)\left(1 + 2^k|x - x_I|\right)^{-n-1}/w(J) \quad \text{on } J,$$

$$(9.52)\qquad |v_I(x) - v_I(y)| \le CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)2^k|x-y|\left(1 + 2^k|x - x_I|\right)^{-n-1}/w(J) \quad \text{on } J,$$

$$(9.53)\qquad \left|\sum_{I:\,(9.31)} \lambda_I v_I(x)\right| \le CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)\eta_k(x)/w(J) \quad \text{on } J,$$

$$(9.54)\qquad \left|\sum_{I:\,(9.31)} \lambda_I v_I(x) - \sum_{I:\,(9.31)} \lambda_I v_I(y)\right| \le CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)\eta_k(x)2^k|x-y|/w(J) \quad \text{on } J.$$

Proof. (9.51) follows from (9.35)–(9.36) and (9.49). Note that

$$v_I(x) - v_I(y) = \langle \mathbf{q}_I(x) - \mathbf{q}_I(y), U(\mathbf{g}_{k-1}(x)) - U(\mathbf{g}_{k-1}(x_I))\rangle + \langle \mathbf{q}_I(y), U(\mathbf{g}_{k-1}(x)) - U(\mathbf{g}_{k-1}(y))\rangle.$$

Condition (9.35), (9.37) and (9.49) take care of the first term and conditions (9.36) and (9.49) take care of the second term. Thus, (9.52) holds. Conditions (9.53)–(9.54) follow from (9.51)–(9.52). □

LEMMA 9.14. *If* (9.48) *holds, then*

$$|\mathbf{k}_2(x)| \le C\eta_k(x) \quad on\ J, \tag{9.55}$$

$$|\mathbf{k}_2(x) - \mathbf{k}_2(y)| \le C\eta_k(x)2^k|x-y| \quad on\ J. \tag{9.56}$$

Proof. (9.55) follows from the last formula of (9.43) and (9.38). Note that

$$|U(\mathbf{g}_{k-1}(x)) - U(\mathbf{g}_{k-1}(y))| \le C2^{k-1}|x-y| \tag{9.57}$$

by (9.49), Lemma 9.2 and (9.2). So, (9.56) follows from the last formula of (9.43), (9.39), (9.38) and (9.57). □

LEMMA 9.15. *If* (9.48) *holds, then*

$$\big|\,|\mathbf{k}(x)| - \tilde{w}_k(x)\big| \le CC_{9.1}2^{M(n+2)}\varepsilon_k(x)^2/w(J) \quad on\ J, \tag{9.58}$$

$$\begin{aligned}\big|\big(|\mathbf{k}(x)| - \tilde{w}_k(x)\big) - \big(|\mathbf{k}(y)| - \tilde{w}_k(y)\big)\big| \\ \le CC_{9.1}^2 2^{M(n+2)}\varepsilon_k(x)^2 2^k|x-y|/w(J) \quad on\ J.\end{aligned} \tag{9.59}$$

Proof. Set $r_1(t) = (1+t)^{1/2} - 1$. Then

$$\begin{aligned}|\mathbf{k}(x)| - \tilde{w}_k(x) &= \left(|\mathbf{k}_1(x)|^2 + |\mathbf{k}_2(x)|^2\right)^{1/2} - \tilde{w}_k(x) \\ &= \left\{\left(\tilde{w}_k(x) + \sum_{I:\,(9.31)} \lambda_I v_I(x)\right)^2 + |\mathbf{k}_2(x)|^2\right\}^{1/2} - \tilde{w}_k(x) \quad \text{by (9.46)} \\ &= \tilde{w}_k(x) r_1\Big(2\textstyle\sum \lambda_I v_I(x)/\tilde{w}_k(x) \\ &\qquad + \left(\textstyle\sum \lambda_I v_I(x)/\tilde{w}_k(x)\right)^2 + \left(|\mathbf{k}_2(x)|/\tilde{w}_k(x)\right)^2\Big) \\ &= \tilde{w}_k(x) r_2(x).\end{aligned} \tag{9.60}$$

Then by (9.53) and (9.55)

$$|r_2(x)| \le CC_{9.1}2^{M(n+2)}\varepsilon_k(x)^2/w(J)^2. \tag{9.61}$$

So, (9.58) holds.

By (9.60), the left-hand side of (9.59)

$$\begin{aligned}
&\le |\tilde{w}_k(x) - \tilde{w}_k(y)| r_2(x) + \tilde{w}_k(y)|r_2(x) - r_2(y)| \\
&\le |\tilde{w}_k(x) - \tilde{w}_k(y)| r_2(x) \\
&\quad + C\tilde{w}_k(y)\Big\{\big|\textstyle\sum \lambda_I v_I(x) - \sum \lambda_I v_I(y)\big| / \tilde{w}_k(x) \\
&\qquad + \big|\textstyle\sum \lambda_I v_I(y)\big|\, \big|\tilde{w}_k(x)^{-1} - \tilde{w}_k(y)^{-1}\big| \\
&\qquad + \big|\, |\mathbf{k}_2(x)|^2 - |\mathbf{k}_2(y)|^2 \big| / \tilde{w}_k(x)^2 \\
&\qquad\qquad + |\mathbf{k}_2(y)|^2 \big|\tilde{w}_k(x)^{-2} - \tilde{w}_k(y)^{-2}\big|\Big\}.
\end{aligned}$$

Lemma 9.10 and (9.61) take care of the first term. Conditions (9.54), (9.53), Lemma 9.10, (9.56) and (9.55) take care of the second term. □

Let $t_k(x) \ge 0$ be such that

$$t_k(x) = 0 \quad \text{if } |\mathbf{g}_{k-1}(x)| \le w_{k-1}(x)/2, \tag{9.62}$$

$$t_k(x) = 1 \quad \text{if } |\mathbf{g}_{k-1}(x)| \ge 3w_{k-1}(x)/4, \tag{9.63}$$

$$|t_k(x) - t_k(y)| \le 2^k |x - y|. \tag{9.64}$$

Set

$$\begin{aligned}
\boldsymbol{\varphi}_k(x) &= t_k(x)\big(|\mathbf{k}(x)| - \tilde{w}_k(x)\big) U(\mathbf{k}(x)), \\
\mathbf{g}_k(x) &= \mathbf{k}(x) - \boldsymbol{\varphi}_k(x).
\end{aligned}$$

By (9.32)–(9.34) this definition of $\mathbf{g}_k$ coincides with (9.9).

Condition (C.4) follows from (9.62) and (9.58). Condition (C.5) follows from the inequality

$$\begin{aligned}
|\boldsymbol{\varphi}_k(x) - \boldsymbol{\varphi}_k(y)| \le\ & |t_k(x) - t_k(y)|\, \big|\, |\mathbf{k}(x)| - \tilde{w}_k(x)\big| \\
& + t_k(y)\big|\big(|\mathbf{k}(x)| - \tilde{w}_k(x)\big) - \big(|\mathbf{k}(y)| - \tilde{w}_k(y)\big)\big| \\
& + |\boldsymbol{\varphi}_k(y)|\, |U(\mathbf{k}(x)) - U(\mathbf{k}(y))|,
\end{aligned}$$

when combined with (9.62), (9.64), (9.58), (9.59), (C.4), (9.50), Lemma 9.2 and $c_0 2^{M(n+1)} < 1$. Condition (C.6) follows from (C.6)′, (9.62) and (9.40). Condition (C.7) is clear from the definition of $\boldsymbol{\varphi}_k(x)$, part (i) of Lemma 9.11 and (9.58). Condition (C.8) follows from (9.41) and (C.5) if c_0 is small enough depending on M and $C_{9.1}$.

10. Proof of Remark 3. In the proof of Main Lemma, if $\mathbf{f}$ is $\mathbf{R}^m$-valued, then $\mathbf{f}_M$ and $\{\mathbf{b}_I\}$ are $\mathbf{R}^m$-valued. By Remark 2.2 of **[32]**, if $\boldsymbol{\nu} \in \mathbf{R}^m \cap \Sigma_{2m-1}$ and $\theta_j(\xi) = \bar{\theta}_j(-\xi)$ for $j = 1, \ldots, m$, then we can take $\mathbf{p}(x)$ in Lemma 6.1 to be $\mathbf{R}^m$-valued. Thus, if $\mathbf{f}$ is $\mathbf{R}^m$-valued, then we can take $\tilde{\mathbf{g}}$ and $\{\mathbf{f}_j\}$ in Lemma 9.8 to be $\mathbf{R}^m$-valued. Thus we can take $\mathbf{g}$ in Main Lemma to be $\mathbf{R}^m$-valued.

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