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NONCOMPACTNESS PRINCIPLES IN NONLINEAR OPERATOR APPROXIMATION THEORY

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This paper, which builds on recent work of Anselone and Ansorge, is concerned with the approximate solution of nonlinear equations involving noncompact operators. Roughly speaking, the concepts developed (such as "measure of *d*-noncompactness", "generalized noncompact convergence", etc.) play the same role for approximation problems as the theory of condensing operators for existence problems.

1. Introduction. One of the fundamental concepts in nonlinear functional analysis is that of compactness. For example, the classical fixed point principles, degree theories, and bifurcation results are based on compactness arguments. Most of these results, however, involve nonconstructive existence proofs. The necessity of calculating (or at least approximating) solutions to integral or differential equations created a large literature on compact operator approximation. As an example, we mention the survey papers [**3**] and [**4**], which are essentially self-contained, and provide a large amount of methods, results, and examples, and were actually the motivation for the present paper: In fact, despite the importance of compactness principles, from the viewpoint of applications it seems worthwhile to extend the theory of operator approximation to a larger class of mappings. One of the simplest such classes is that of condensing mappings, i.e. those which diminish some measure of noncompactness. (A good recent survey on condensing operators is [**1**], on their application to functional-differential equations is [**2**].) It turns out that a suitable way of describing noncompact operator approximation is to introduce "*d*-condensing" mappings, i.e. those which diminish some measure of *d*-noncompactness (which means "lack of discrete compactness"). It is the purpose of this paper to develop this idea and to study its applicability to analytic problems, where noncompact operators occur.

It should be mentioned that there exists already some literature on noncompact approximation theory (e.g. on the links between condensing maps and Galerkin methods). In the present paper, however, we restrict ourselves to generalizing solely the concepts described in [**4**], but as systematically as possible. Unfortunately, this made it necessary to introduce a great number of numerical characteristics in §§2 and 3, all describing the "lack of some regularity" (compactness, d-compactness, convergence etc.). At least in part these characteristics will be justified by a model example in §4; we request the reader's indulgence until then.

2. Discrete noncompactness of sets. Let X be a real Banach space with norm $\| \|$. Let $[x_n]_N$ be any sequence of elements $x_n \in X$ (N being an infinite index set of natural numbers). We will always distinguish between sequences $[x_n]_N$ and their values $\{x_n\} = \{x_n : n \in N\}$. Let

$$[x_n]_N^* = \{x : x \in X, \exists N' \subseteq N \text{ such that } x_n \to x \ (n \in N')\}$$

denote the set of all accumulation points (limits of subsequences) of the sequence $[x_n]_N$. Similarly, for a sequence $[S_n]_N$ of subsets $S_n \subseteq X$ let

$$[S_n]_N^* = \bigcup \{[x_n]_{N'}^* : N' \subseteq N, x_n \in S_n \text{ for all } n \in N'\}.$$

Clearly, $[\{x_n\}]_N^* = [x_n]_N^*$.

In what follows, we shall sometimes omit the index N when the context is clear.

The following properties are not hard to verify:

(2.1) $S_n \subseteq T_n$ *for all* n *implies* $[S_n]^* \subseteq [T_n]^*$;

(2.2) $[S_n]^*$ *is a closed subset of* X;

(2.3) $[S_n]^* \subseteq \overline{\bigcup_n S_n}$.

Recall (see [**4**]) that a sequence $[S_n]_N$ is said to be *d-compact* (discretely compact) if $[S_n]_{N'}^* \neq \emptyset$ for any $N' \subseteq N$ for which $S_n \neq \emptyset$ for $n \in N'$. Thus, in particular, $[x_n]_N$ is d-compact if and only if $[x_n]_N^* \neq \emptyset$.

For any $S \subseteq X$ and $\varepsilon \geq 0$ let

$$\Omega_\varepsilon(S) = \bigcup_{x \in S} \overline{B}_\varepsilon(x),$$

where $\overline{B}_\varepsilon(x) = \{x' : x' \in X, \|x' - x\| \leq \varepsilon\}$; let $\Omega_\varepsilon(\emptyset) = \emptyset$.

Theorem 2.1 below contains two characterizations of the d-compactness of a sequence $[S_n]_N$. For proofs and further details we refer to [**4**].

THEOREM 2.1. *Let* $[S_n]_N$ *be a sequence of subsets* $S_n \subseteq X$. *Then*

(i) $[S_n]_N$ *is d-compact if and only if* $[S_n]_N^*$ *is compact and for any* $\varepsilon > 0$ *there exists* $n_\varepsilon \in N$ *such that* $S_n \subseteq \Omega_\varepsilon([S_n]_N^*)$ *for all* $n \geq n_\varepsilon$;

(ii) $\overline{\bigcup_n S_n}$ *is compact if and only if* $[S_n]_N$ *is d-compact and the sets* $\overline{S_n}$ *are compact for each* $n \in N$. □

For a given $S \subseteq X$, let us recall the *Hausdorff measure of noncompactness* (see e.g. [**11, 13**])

$$\alpha(S) = \inf\{\varepsilon : \varepsilon > 0, \text{there exists a finite } \varepsilon\text{-net for } S\}$$

with the properties:

(i) $\alpha(S) < \infty$ if and only if S is bounded;
(ii) $\alpha(S) = 0$ if and only if $\bar{S}$ is compact;
(iii) $S \subseteq T$ implies $\alpha(S) \leq \alpha(T)$;
(iv) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$;
(v) $\alpha(S \cap T) \leq \min\{\alpha(S), \alpha(T)\}$;
(vi) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$;
(vii) $\alpha(\lambda S) = |\lambda|\alpha(S)$ for $\lambda \in \mathbf{R}$;
(viii) $\alpha(\overline{\text{co}} S) = \alpha(S)$, where $\overline{\text{co}}$ denotes the closed convex hull.

After these preliminaries we are now in a position to introduce our measure of discrete noncompactness δ. Namely, for a given sequence $[S_n]_N$ of bounded sets $S_n \subset X$ we define

$$\delta([S_n]_N) = \alpha([S_n]_N^*) + \omega([S_n]_N),$$

where

$$\omega([S_n]_N) = \inf\{\varepsilon : \varepsilon > 0, \text{ there exists } n_\varepsilon \in N \text{ such that } S_n \subseteq \Omega_\varepsilon([S_n]_N^*) \text{ for all } n \geq n_\varepsilon\}.$$

In particular, for a given sequence $[x_n]_N$,

$$\delta([x_n]_N) = \alpha([x_n]_N^*) + \omega([x_n]_N),$$

where

$$\omega([x_n]_N) = \inf\{\varepsilon : \varepsilon > 0, \text{ there exists } n_\varepsilon \in N \text{ such that } \operatorname{dist}(x_n, [x_n]_N^*) \leq \varepsilon \text{ for all } n \geq n_\varepsilon\}.$$

EXAMPLE 2.1. (a) For any $n \in N$, let S_n be the unit ball $B_1(0)$. Then $\omega([S_n]) = 0$ and $\alpha([S_n]^*) = \alpha(\bar{B}_1(0)) = 1$, so $\delta([S_n]) = 1$.

(b) Let X be a Hilbert space with countable ON-basis $\{e_1, e_2, e_3, \ldots\}$. Take $[x_n] = [0, e_1, 0, e_2, \ldots]$. Then $\alpha([x_n]^*) = \alpha(\{0\}) = 0$ and $\delta([x_n]) = \omega([x_n]) = 1$. □

Let us point out that neither ω nor δ is monotone. In fact, let X be the Banach space of continuous functions over the interval $[0, 1]$ and let z_n be the characteristic function of the interval $[n^{-1}, (n-1)^{-1}]$. Clearly, for $n \neq m$ we have $\|z_n - z_m\| = 1$. Take $S_n = \{0, z_1, \ldots, z_n\}$, $T_n = \text{co}\, S_n$. Then $\alpha([S_n]^*) = 0$, $\omega([S_n]) = 1$, and, since $[T_n]^* = \bigcup_n T_n$, $\alpha([T_n]^*) = 1/2$, $\omega([T_n]) = 0$. Thus $\delta([S_n]) = 1 > 1/2 = \delta([T_n])$.

One might suspect that there is at least some constant $c > 0$ such that $\delta([S_n]) \leq c\delta([T_n])$ whenever $S_n \subseteq T_n$. The following example shows that this is not so:

Consider an infinite dimensional Banach space X and sequences $[y_n]$ and $[z_n]$ in X such that $\|y_n\| = 1$, $[y_n]^* = \varnothing$, $\|z_n - z_0\| = 1$ ($\|z_0\| > 2$), and $[z_n]^* = \varnothing$. Define $[x_n] = [0, y_1, z_1, 0, y_2, z_2, \ldots]$, $S_n = \{x_1, x_2, \ldots, x_{3n}\}$, and $T_n = B_1(z_0) \cup \overline{\mathrm{co}}\{0, y_1, \ldots, y_n\}$. Then $[S_n]^* = \{0\}$, $[T_n]^* = \overline{B}_1(z_0) \cup \overline{\mathrm{co}}\{0, y_1, y_2, \ldots\}$, so that $\alpha([S_n]^*) = \omega([T_n]) = 0$, $\omega([S_n]) \geq \|z_0\| - 1$, and $\alpha([T_n]^*) = 1$. Therefore, letting $\|z_0\| \to \infty$ we can make $\delta([S_n]) = \omega([S_n])$ arbitrarily large, while $\delta([T_n])$ remains equal to 1 independently of z_0.

Theorem 2.1, interpreted in terms of α, ω and δ, reads as follows:

(2.4) $[S_n]_N$ *is d-compact if and only if* $\delta([S_n]_N) = 0$;

(2.5) $\alpha(\bigcup_n S_n) = 0$ *if and only if* $\delta([S_n]_N) = 0$ *and* $\sup_n \alpha(S_n) = 0$.

In particular, we deduce that

(2.6) $\delta([S_n]_N) = 0$ *implies* $\alpha([S_n]_N^*) = 0$;

(2.7) $\alpha(\bigcup_n S_n) = 0$ *implies* $\omega([S_n]_N) = 0$.

It should be observed that the converse implications do not hold as Example 2.1 shows. Obviously, (2.6) can easily be derived from the inequality $\alpha([S_n]^*) \leq \delta([S_n])$, which holds by definition. On the other hand, a similar estimate for (2.7) (i.e. $\omega([S_n]) \leq c\alpha(\bigcup_n S_n)$) does not hold. In fact, let $[z_n]$ be a sequence in the closed ball $\overline{B}_r(x_0)$ ($\|x_0\| > r$) without accumulation points and let $[x_n] = [0, z_1, 0, z_2, \ldots]$. Then $\alpha(\{x_n\}) \leq r$ and $\|x_0\| - r \leq \omega([x_n]) \leq \|x_0\| + r$, which implies $\omega([x_n]) > c\alpha(\{x_n\})$ for $\|x_0\| \geq (c + 1)r$.

It is not hard to give a direct proof of (2.7). Taking into account this fact, a suitable extension of (2.5) turns out to be the following:

PROPOSITION 2.1. *The equivalence*

$$\alpha\Big(\bigcup_n S_n\Big) + \omega([S_n]_N) \sim \sup_n \alpha(S_n) + \delta([S_n]_N)$$

holds, i.e. there exists c, $C > 0$ *such that*

$$c\Big\{\alpha\Big(\bigcup_n S_n\Big) + \omega([S_n])\Big\} \leq \sup_n \alpha(S_n) + \delta([S_n])$$

$$\leq C\Big\{\alpha\Big(\bigcup_n S_n\Big) + \omega([S_n])\Big\}.$$

Proof. Since $S_n \subseteq \bigcup_n S_n$ and $[S_n]^* \subseteq \overline{\bigcup_n S_n}$, we get

$$\sup_n \alpha(S_n) + \delta([S_n]) = \sup_n \alpha(S_n) + \alpha([S_n]^*) + \omega([S_n])$$

$$\leq 2\alpha\Big(\bigcup_n S_n\Big) + \omega([S_n]),$$

that is $C = 2$. To show the left inequality, take $\varepsilon > \sup_n \alpha(S_n) + \delta([S_n])$ (clearly, if this sum is infinite, there is nothing to prove). Hence, $\varepsilon > \omega([S_n])$ so $[S_n]^* \neq \varnothing$ and there exists $n_\varepsilon \in N$ such that $S_n \subseteq \Omega_\varepsilon([S_n]^*)$ for $n \geq n_\varepsilon$. Therefore,

$$\bigcup_{n \geq n_\varepsilon} S_n \subseteq \Omega_\varepsilon([S_n]^*).$$

On the other hand, we also have $\varepsilon > \alpha([S_n]^*)$ which implies $\alpha(\Omega_\varepsilon([S_n]^*)) < 2\varepsilon$. Thus,

$$\alpha\Big(\bigcup_n S_n\Big) = \max\Big\{\alpha\Big(\bigcup_{n<n_\varepsilon} S_n\Big), \alpha\Big(\bigcup_{n\geq n_\varepsilon} S_n\Big)\Big\} < 2\varepsilon.$$

Consequently, $\alpha(\bigcup_n S_n) + \omega([S_n]) < 3\varepsilon$ and we can take $c = 1/3$. □

The idea of measuring the "lack of d-compactness" of a sequence is not new. For instance, in **[16]** a discrete noncompactness measure was introduced which in its simplest version has the form

$$\chi([x_n]_N) = \inf\{\varepsilon : \varepsilon > 0, \text{ for any } N' \subseteq N \text{ there exist } N'' \subseteq N' \text{ and } x \in X \text{ such that } \|x_n - x\| \leq \varepsilon \text{ for all } n \in N''\}.$$

We close this section by comparing our measure δ with χ.

PROPOSITION 2.2. *For any given sequence* $[x_n]_N$ *in* X *we have*
(i) $\chi([x_n]_N) \leq \delta([x_n]_N)$;
(ii) $\alpha([x_n]_N^*) \leq c\chi([x_n]_N)$ *for some* $c > 0$;
(iii) $\omega([x_n]_N) \not\leq C\chi([x_n]_N)$ *for any* $C > 0$.

Proof. (i) If $\delta([x_n])$ is infinite, the assertion is trivial. So, let $\varepsilon > \delta([x_n])$ and take $\alpha > \alpha([x_n]^*)$, $\omega > \omega([x_n])$ such that $\delta([x_n]) < \alpha + \omega \leq \varepsilon$. Hence $[x_n]^* \neq \varnothing$ and there exists $n_\omega \in N$ such that $x_n \in \Omega_\omega([x_n]^*)$ for $n \geq n_\omega$. Consequently, for any $n \geq n_\omega$, we have $\|x_n - y_n\| \leq \omega$ for some $y_n \in [x_n]^*$. On the other hand, there exists a finite α-net in X, say $\{z_1, \ldots, z_m\}$, such that $[x_n]^* \subseteq B_\alpha(z_1) \cup B_\alpha(z_2) \cup \cdots \cup B_\alpha(z_m)$. Thus, if $[x_n]_{N'}$ $(N' \subseteq N)$

is any subsequence of $[x_n]_{N'}$ one can find an index $j \in \{1,\ldots,m\}$ and an infinite subset N'' of N' such that $y_n \in B_\alpha(z_j)$ for $n \in N''$. So, for $n \in N''$,

$$\|x_n - z_j\| \leq \|x_n - y_n\| + \|y_n - z_j\| \leq \omega + \alpha \leq \varepsilon,$$

which implies $\chi([x_n]) \leq \varepsilon$.

(ii) As above, we may restrict ourselves to the case when $\chi([x_n])$ is finite. Let $\varepsilon > \chi([x_n])$ and suppose $\alpha([x_n]^*) > 3\varepsilon$. There exists a sequence $[y_m]$ in $[x_n]^*$ such that $\|y_p - y_q\| > 3\varepsilon$ for $p \neq q$. On the other hand, for any m, the sequence $[x_n]_N$ has a subsequence $[x_n^{(m)}]_{N'}$ $(N' \subseteq N)$ converging to y_m. Take the diagonal sequence $[x_n^{(n)}]_{N'}$. Clearly, $\|x_n^{(n)} - y_n\| \leq \varepsilon_n \to 0$ $(n \in N')$. Observe also that $[x_n^{(n)}]_{N'}$ is a subsequence of $[x_n]_N$. So, since $\varepsilon > \chi([x_n])$, there exist $N'' \subseteq N'$ and $x \in X$ such that $\|x_n^{(n)} - x\| \leq \varepsilon$ $(n \in N'')$. Thus, for p, q sufficiently large, we get

$$\begin{aligned} 3\varepsilon < \|y_p - y_q\| &\leq \left\|y_p - x_p^{(p)}\right\| + \left\|x_p^{(p)} - x\right\| + \left\|x - x_q^{(q)}\right\| + \left\|x_q^{(q)} - y_q\right\| \\ &\leq \varepsilon_p + 2\varepsilon + \varepsilon_q \leq 3\varepsilon, \end{aligned}$$

a contradiction. Therefore $\alpha([x_n]^*) \leq 3\varepsilon$ and assertion (ii) is proved with $c = 3$.

(iii) We give a counterexample. Let H be a hyperplane in an infinite dimensional space X, i.e. $X = H \oplus \mathbf{R}x_0$, $x_0 \neq 0$. Let $[z_n]$ be a sequence in H such that $\|z_n\| = 1$ and $[z_n]^* = \varnothing$. Consider $[x_n] = [z_1, \lambda x_0, z_2, \lambda x_0, z_3, \ldots]$, $\lambda > 0$. It is easily seen that $\chi([x_n]) = 1$. Moreover, since $[x_n]^* = \{\lambda x_0\}$, we have $\omega([x_n]) \geq \lambda\|x_0\|$. Therefore $\omega([x_n]) > C\chi([x_n])$ for $\lambda > C/\|x_0\|$. □

It is clear from the above result that χ and δ are not linearly equivalent. However, both are measures of d-noncompactness (in the sense that they are zero exactly on d-compact sequences), while δ seems to be "coarser" in some sense than χ (because of (iii)).

3. Discrete noncompactness of operators. Let K and K_n $(n \in N)$ be (not necessarily linear) operators acting between two Banach spaces X and Y. Recall the following definitions (see e.g. [**4**]):

K is called *compact* if $\overline{K(S)}$ is compact for any bounded $S \subset X$; the sequence $[K_n]_N$ is called *collectively compact* if $\overline{\bigcup_n K_n(S)}$ is compact, and *asymptotically compact* (or *d-compact*) if $[K_n(S)]_N$ is d-compact for any bounded $S \subset X$.

The following relation between these notions is shown in [**4**]:

THEOREM 3.1. *$[K_n]_N$ is collectively compact if and only if $[K_n]_N$ is asymptotically compact and K_n is compact for each $n \in N$.* □

Let us now introduce some further parameters (for the definition of $\alpha([K_n]_N)$ see also [**10, 17**]):

$$\alpha(K) = \inf\{\lambda : \lambda > 0, \alpha(K(S)) \leq \alpha(S), S \subset X \text{ bounded}\};$$

$$\alpha([K_n]_N) = \inf\{\lambda : \lambda > 0, \alpha(\cup_n K_n(S)) \leq \lambda\alpha(S), S \subset X \text{ bounded}\};$$

$$\omega([K_n]_N) = \inf\{\lambda : \lambda > 0, \omega([K_n(S)]_N) \leq \lambda\alpha(S), S \subset X \text{ bounded}\};$$

$$\delta([K_n]_N) = \inf\{\lambda : \lambda > 0, \delta([K_n(S)]_N) \leq \lambda\alpha(S), S \subset X \text{ bounded}\}.$$

According to the above definitions, we have:
(3.1) $\alpha(K) = 0$ *if and only if K is compact*;
(3.2) $\alpha([K_n]) = 0$ *if and only if* $[K_n]$ *is collectively compact*;
(3.3) $\delta([K_n]) = 0$ *if and only if* $[K_n]$ *is asymptotically compact*;
(3.4) $\alpha([K_n]) = 0$ *implies* $\omega([K_n]) = 0$.
Moreover, from Proposition 2.1 we can immediately deduce the following extension of Theorem 3.1:

PROPOSITION 3.1. *The equivalence*

$$\alpha([K_n]_N) + \omega([K_n]_N) \sim \sup_n \alpha(K_n) + \delta([K_n]_N)$$

holds, where $\sim$ *is defined as in Proposition* 2.1. □

The notions of compact, collectively compact, and asymptotically compact operators can be characterized also in terms of sequences of elements in X (see [**4**]). From this point of view, in our context we are led to

PROPOSITION 3.2. *Let* $[x_n]_N$ *be a bounded sequence in* X, *and let* $\pi: N \to N$ *be any bijection. Then*
(i) $\delta([Kx_n]_N) \leq \alpha(K)\alpha(\{x_n\}) + \omega([Kx_n]_N)$;
(ii) $\delta([K_{\pi(n)}x_n]_N) \leq \alpha([K_n]_N)\alpha(\{x_n\}) + \omega([K_{\pi(n)}x_n])$;
(iii) $\delta([K_nx_n]_N) \leq \delta([K_n]_N)\alpha(\{x_n\})$.
In particular,
(i)′ $\alpha(K) = 0$ *if and only if* $\delta([Kx_n]_N) = 0$ *for any bounded sequence* $[x_n]_N$;
(ii)′ $\alpha([K_n]_N) = 0$ *if and only if* $\delta([K_{\pi(n)}x_n]_N) = 0$ *for any bounded sequence* $[x_n]_N$ *and bijection* π;
(iii)′ $\delta([K_n]_N) = 0$ *if and only if* $\delta([K_nx_n]_N) = 0$ *for any bounded sequence* $[x_n]_N$.

Proof. It suffices to observe that for any $\varepsilon > 0$,

$$\alpha([Kx_n]^*) \le \alpha(\{Kx_n\}) \le (\alpha(K) + \varepsilon)\alpha(\{x_n\}),$$
$$\alpha([K_{\pi(n)}x_n]^*) \le \alpha(\{K_{\pi(n)}x_n\}) \le (\alpha([K_n]) + \varepsilon)\alpha(\{x_n\}),$$
$$\delta([K_nx_n]) \le (\delta([K_n]) + \varepsilon)\alpha(\{x_n\}),$$

and moreover, that

$$\alpha(K) = 0 \quad \text{implies} \quad \omega([Kx_n]) = 0,$$
$$\alpha([K_n]) = 0 \quad \text{implies} \quad \omega([K_{\pi(n)}x_n]) = 0. \qquad \square$$

We will now introduce various types of convergence for operator sequences.

$K_n \to K$ (*pointwise convergence*) means that $K_n x \to Kx$ as $n \to \infty$ for all $x \in X$;

$K_n \overset{c}{\to} K$ (*continuous convergence*) means that $x \to x_0$ implies $K_n x \to Kx_0$ $(n \to \infty)$, or equivalently $x_n \to x_0$ $(n \to \infty)$ implies $K_n x_n \to Kx_0$ $(n \to \infty)$, or equivalently $x_m \to x_0$ $(m \to \infty)$ implies $K_n x_m \to Kx_0$ $(m, n \to \infty)$.

Observe that $K_n \overset{c}{\to} K$ implies the continuity of K; furthermore, for continuous K_n the continuous convergence $K_n \overset{c}{\to} K$ is equivalent to the pointwise convergence $K_n \to K$ and to the equicontinuity of the family $[K_n]$. Moreover, for linear continuous K, K_n from the Banach-Steinhaus theorem it follows that pointwise convergence implies equicontinuity, and hence pointwise and continuous convergence coincide.

Furthermore, we will define two other types of convergence, generalizing those in [**4**]; given $\lambda \ge 0$,

$K_n \overset{\lambda\text{-cc}}{\to} K$ (λ-*collectively compact convergence*) means that each K_n is continuous, $K_n \overset{c}{\to} K$, and $\alpha([K_n]_N) + \omega([K_n]_N) \le \lambda$;

$K_n \overset{\lambda\text{-ac}}{\to} K$ (λ-*asymptotically compact convergence*) means that each K_n is continuous, $K_n \overset{c}{\to} K$, and $\delta([K_n]_N) \le \lambda$.

We remark that, for $\lambda = 0$, our 0-cc and 0-ac convergences are nothing else but the cc and ac convergences introduced in [**4**]. Hence, the next proposition generalizes corresponding results in [**4**].

PROPOSITION 3.3. *The following relations hold*:
(i) $K_n \underset{c}{\to} K$ *implies* $\alpha(K) \leq \alpha([K_n]_N)$;
(ii) $K_n \to K$ *implies* $\alpha(K) \leq \delta([K_n]_N)$;
(iii) $K_n \overset{\lambda\text{-cc}}{\to} K$ *implies* $K_n \overset{\lambda\text{-ac}}{\to} K$ *and* $\sup_n \alpha(K_n) \leq \lambda$;
(iv) $K_n \overset{\lambda\text{-ac}}{\to} K$ *and* $\sup_n \alpha(K_n) \leq \lambda$ *imply* $K_n \overset{C\lambda\text{-cc}}{\to} K$ *for some* $C > 0$.

Proof. To show (i), (ii) and (iii) it suffices to observe that, for any bounded subset $S \subset X$, we have

$$\overline{K(S)} \subseteq [K_n(S)]_N^* \quad \subseteq \overline{\bigcup_n K_n(S)}.$$

(iv) follows from Proposition 3.1. □

Let us now consider $X = Y$ and K, K_n as operators in X. Our aim is to obtain information about the nonemptiness of the set

$$\operatorname{Fix}(K, S) = \{x : x \in S \subseteq X, Kx = x\}$$

and especially of $\operatorname{Fix}(K) = \operatorname{Fix}(K, X)$. This will be carried out by means of the various kinds of convergence introduced above, more precisely by comparing the fixed point problem $Kx = x$ with the exact problem $K_n x = x$ or with the approximate problem $\|K_n x_n - x_n\| \to 0$.

Lemma 3.1 below will be used in proving our main results.

LEMMA 3.1. *The following relations hold*:
(i) *If* $\alpha(K) < 1$, *then* $\alpha(\operatorname{Fix}(K, S)) = 0$ *for any bounded* $S \subset X$.
(ii) *Let* $[S_n]_N$ *be any sequence of subsets in* X *such that* $\bigcup_n S_n$ *is bounded. Then* $[S_n]_N^* \neq \varnothing$ *if and only if* $\delta([S_n]_N) < \infty$.
(iii) *Let* $[S_n]_N$, $[T_n]_N$ *be sequences of subsets in* X. *Let* $\Delta(S, T)$ *be the Hausdorff distance of* S *and* T, *i.e. the infimum of all* $\varepsilon > 0$ *such that* $S \subseteq \Omega_\varepsilon(T)$ *and* $T \subseteq \Omega_\varepsilon(S)$. *Then* $\Delta(S_n, T_n) \to 0$ $(n \to \infty)$ *implies* $\delta([S_n]_N) = \delta([T_n]_N)$.

Proof. There exists $\lambda > 0$ such that $\alpha(K) < \lambda < 1$. On the other hand, since $\operatorname{Fix}(K, S) = K(\operatorname{Fix}(K, S))$, we have

$$\alpha(\operatorname{Fix}(K, S)) = \alpha(K(\operatorname{Fix}(K, S))) \leq \lambda\alpha(\operatorname{Fix}(K, S)).$$

Thus, $\alpha(\operatorname{Fix}(K, S)) = 0$.

To prove (ii), let $r > 0$ be such that $\alpha([S_n]^*) \leq \alpha(\bigcup_n S_n) \leq r$ and let $x \in [S_n]^*$. For any $n \in N$, $S_n \subseteq \overline{B}_r(0)$. Hence, $S_n \subseteq \Omega_\varepsilon([S_n]^*)$, for $\varepsilon > \|x\| + r$. Thus, also $\omega([S_n])$ is finite. Conversely, if $[S_n]^* = \varnothing$ then, by definition, $\omega([S_n]) = \infty$.

Under the assumptions of (iii), let us show first that $[S_n]^* = [T_n]^*$. Take $x \in [S_n]^*$ and, for each $n \in N$, $x_n \in S_n$ such that $x_n \to x$. Now, for any $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that $S_n \subseteq \Omega_\varepsilon(T_n)$, $T_n \subseteq \Omega_\varepsilon(S_n)$ for all $n \geq n_\varepsilon$. Therefore, for any $n \geq n_\varepsilon$ one can find $y_n \in T_n$ such that $\|x_n - y_n\| < \varepsilon$. Thus, $y_n \to x$ and, consequently, $x \in [T_n]^*$. Analogously, $[T_n]^* \subseteq [S_n]^*$. Observe now that the equality $[S_n]^* = [T_n]^*$ implies $\omega([S_n]) < \infty$ if and only if $\omega([T_n]) < \infty$. In fact, suppose $\omega([S_n]) < \infty$. Then there exists $\omega > \omega([S_n])$ and $n_\omega \in N$ such that, for all $n \geq n_\omega$,

$$S_n \subseteq \Omega_\omega\big([S_n]^*\big) = \Omega_\omega\big([T_n]^*\big).$$

Moreover, for any $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that, for $n \geq n_\varepsilon$, $T_n \subseteq \Omega_\varepsilon(S_n)$. Consequently, for $n \geq \max\{n_\omega, n_\varepsilon\}$, we have $T_n \subseteq \Omega_{\omega+\varepsilon}([T_n]^*)$, so $\omega([T_n]) \leq \omega + \varepsilon$. Since this inequality holds for all $\varepsilon > 0$, we have also proved that $\omega([T_n]) \leq \omega([S_n])$. By symmetry we clearly obtain the converse. Consequently, $\delta([S_n]) = \delta([T_n])$ as claimed. □

We state now some existence and convergence results which contain, as particular cases, Theorems 3.7 and 3.8 of **[4]**.

THEOREM 3.2. *Let K_n, K be operators in X. Suppose there exists a sequence $[x_n]_N$ in X such that $\|K_n x_n - x_n\| \to 0$. Then*

(i) *$K_n \overset{c}{\to} K$ implies*

$$\delta\big([K_n x_n]_N\big) = \delta\big([x_n]_N\big) \quad \textit{and} \quad [x_n]_N^* \subseteq \operatorname{Fix}(K);$$

(ii) *If $[x_n]_N$ is bounded, then*

$$\delta\big([x_n]_N\big) \leq \delta\big([K_n]_N\big)\alpha(\{x_n\})$$

and

$$\delta\big([x_n]_N\big) \leq \big(\alpha([K_n]_N) + \omega([K_n]_N)\big)\alpha(\{x_n\});$$

(iii) *If $K_n \overset{\lambda\text{-ac}}{\to} K$ and $[x_n]_N$ is bounded, then*

$$\delta\big([x_n]_N\big) \leq \lambda\alpha(\{x_n\}) \quad \textit{and} \quad \varnothing \neq [x_n]_N^* \subseteq \operatorname{Fix}(K).$$

Proof. (i) By (iii) of Lemma 3.1, and since $[K_n x_n - x_n]$ converges to zero, we immediately get $\delta([K_n x_n]) = \delta([x_n])$. Moreover, if $x \in [x_n]^*$, there exists $N' \subseteq N$ such that $x_n \to x$ $(n \in N')$. Therefore,

$$\|Kx - x\| \leq \|Kx - K_n x_n\| + \|K_n x_n - x_n\| + \|x_n - x\|$$

and the right-hand side converges to zero as $n \to \infty$.

(ii) As above, $\delta([K_n x_n]) = \delta([x_n])$. Thus, the assertion follows from Proposition 3.2 (iii) and (ii).

(iii) As above, $\delta([K_n x_n]) = \delta([x_n])$. Thus, $\delta([x_n]) \leq \delta([K_n])\alpha(\{x_n\}) \leq \lambda\alpha(\{x_n\})$. This implies, in particular, $\delta([x_n]) < \infty$. So, by (ii) of Lemma 3.1, we get $[x_n]^* \neq \varnothing$. The assertion follows now as in (i). □

Observe that in (i) above, $[x_n]^*$ may be empty even if $[x_n]$ is bounded. In other words, (iii) is an existence result for fixed points, while (i) is not.

To give an example, let $[x_n]$ be a sequence in an infinite dimensional space X such that $\|x_n\| = 1$ for each $n \in N$ and $[x_n]^* = \varnothing$. Define $K_n x = (1 + 1/n)x$, $Kx = x$. Then $[K_n x_n - x_n]$ converges to zero and, if $x \to x_0$, one has

$$\|K_n x - Kx_0\| = \left\|\left(1 + \frac{1}{n}\right)x - x_0\right\| \leq \|x - x_0\| + \frac{1}{n}\|x\| \to 0,$$

which means $K_n \xrightarrow{c} K$. Clearly, $K_n \overset{\lambda\text{-ac}}{\nrightarrow} K$ for any value of λ.

THEOREM 3.3. *Let K_n, K be operators in X with $K_n \xrightarrow{\lambda\text{-ac}} K$. Let $[S_n]_N$ be a sequence of subsets in X such that $\bigcup_n S_n$ is bounded, and let $0 < \varepsilon_n \to 0$. Define*

$$\tilde{S}_n = \{x : x \in S_n, \|x - K_n x\| \leq \varepsilon_n\}.$$

Then

(i) $\delta([\tilde{S}_n]_N) \leq \lambda \sup_n \alpha(\tilde{S}_n)$;

(ii) $\varnothing \neq [\tilde{S}_n]_N^* \subseteq \mathrm{Fix}(K, S)$ *for any bounded* $S \supseteq [S_n]_N^*$;

(iii) *for any* $\varepsilon > 0$ *there exists* $n_\varepsilon \in N$ *such that* $n \geq n_\varepsilon$ *implies* $\tilde{S}_n \subseteq \Omega_{\eta+\varepsilon}(\mathrm{Fix}(K, S))$ *for any bounded* $S \supseteq [S_n]_N^*$, *where* $\eta = \lambda \sup_n \alpha(\tilde{S}_n)$.

Proof. (i) Since $\Delta(\tilde{S}_n, K_n(\tilde{S}_n)) \leq \varepsilon_n \to 0$, from Lemma 3.1 (iii) we get $\delta([\tilde{S}_n]) = \delta([K_n(\tilde{S}_n)])$. Hence, because of λ-ac convergence, $\delta([\tilde{S}_n]) \leq \lambda\alpha(\tilde{S}_n)$ for all $n \in N$.

(ii) By (i), $\delta([\tilde{S}_n]) < \infty$. So, because of Lemma 3.1 (ii), $[\tilde{S}_n]^* \neq \varnothing$. Now apply Theorem 3.2 (i).

(iii) From (ii) we know that $[\tilde{S}_n]^* \subseteq \mathrm{Fix}(K, S)$. On the other hand, (i) implies that, given $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that $\tilde{S}_n \subseteq \Omega_{\eta+\varepsilon}([\tilde{S}_n]^*)$ for $n \geq n_\varepsilon$. □

4. Applications. In this section we shall illustrate Theorem 3.2 by means of two examples: First, we will study the significance of the assumptions of this theorem for a simple nonlinear operator, namely the Hammerstein operator

$$Kx(s) = \int_\Omega h(s,t)f(t,x(t))\,dt \tag{4.1}$$

in various function spaces. Second, we will indicate an application of Theorem 3.2 to an initial value problem for ordinary differential equations in a Banach space.

EXAMPLE 4.1. (a) Let Ω be a bounded domain in $\mathbf{R}^N$, and let f_n, f and h_n, h be real continuous functions on $\overline{\Omega}\times\mathbf{R}$ and $\overline{\Omega}\times\overline{\Omega}$, respectively. Consider the operators

$$H_n x(s) = \int_\Omega h_n(s,t)x(t)\,dt, \tag{4.2}$$

$$Hx(s) = \int_\Omega h(s,t)x(t)\,dt,$$

$$F_n x(t) = f_n(t,x(t)), \qquad Fx(t) = f(t,x(t)). \tag{4.3}$$

Obviously, all these operators act in the space $X = C(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$. Together with the operator (4.1) (i.e. $Kx = HFx$) we consider the approximations

$$K_n x(s) = H_n F_n x(s) = \int_\Omega h_n(s,t)f_n(t,x(t))\,dt. \tag{4.4}$$

Now, $K_n \overset{c}{\to} K$ means that uniform convergence (on $\overline{\Omega}$) of $[x_n]_N$ implies the same convergence of $[H_n F_n x_n]_N$. If this holds, then

$$[K_n(S)]^* = K(S) \tag{4.5}$$

for each bounded $S \subset X$. In fact, given $y \in [K_n(S)]^*$, there exist $N' \subseteq N$ and a sequence $[x_n]$ in S such that $K_n x_n \to y$ $(n \in N')$. But since also $K_n x_m \to Kx_m$ (m fixed), we must have $y = Kx_m = Kx_{m+1} = \cdots$ for some m, that is $y \in K(S)$. Conversely, given $y = Kx$ $(x \in S)$, take $y_n = K_n x$; obviously, $y_n \in K_n(S)$ and $y_n \to y$, so $y \in [K_n(S)]^*$.

As a consequence of (4.5), $\omega([K_n(S)]) = 0$. In other words, the condition $\delta([K_n]) \le \lambda$ (which is the condition $K_n \overset{\lambda\text{-ac}}{\to} K$ in Theorem 3.2 (iii)) means nothing else than

$$\alpha(K) \le \lambda. \tag{4.6}$$

Roughly speaking, this shows that for a c-convergent sequence $K_n \xrightarrow{c} K$ the discrete noncompactness of $[K_n]$ is essentially the noncompactness of K (and vice versa).

Clearly, $\alpha(K) \leq \|H\|\alpha(F)$; but $\|H\|$ can be evaluated in terms of the kernel function h, and $\alpha(F)$ is exactly the minimal Lipschitz constant for the function f with respect to the second argument (see [**5**, Theorem 1]).

(b) Consider now the same problem, but with f_n, f satisfying only a Carathéodory condition. More precisely, let F_n and F map the space $L_p(\Omega)$ into the space $L_q(\Omega)$ $(1 \leq q < p < \infty)$, while H_n and H vice versa. In this situation, $K_n \xrightarrow{c} K$ means that convergence in the mean (to the power p) of $[x_n]_N$ implies the same convergence of $[H_n F_n x_n]_N$. Again, we are led to the estimate (4.6). While $\|H\|$ can be evaluated in terms of the kernel function h as before, the number $\alpha(F)$ is now given by

$$\alpha(F) = c^{(p-q)/p}[q/(p-q)]^{-1/p}[p/(p-q)]^{1/q},$$

where $c = \sup_{\varepsilon>0}\|a_\varepsilon\|_q \varepsilon^{q/(p-q)} < \infty$ and f is supposed to satisfy a family of Lipschitz conditions

$$|f(s,u) - f(s,v)| \leq a_\varepsilon(s) + \varepsilon|u - v|^{p/q} \qquad \left(a_\varepsilon \in L_q(\Omega), \varepsilon > 0\right),$$

(see [**5**, Theorem 3]).

(c) Before considering the third variant of this example, we emphasize that all definitions and results obviously carry over to the case where the space X is a metric space rather than a Banach space. (In fact, all constructions in [**4**] are posed in metric space settings.) So, let $X = S(\Omega)$ be the space of all (Lebesgue) measurable functions on Ω, equipped with the metric

$$d(x, y) = \inf_{h>0} \left\{h + \operatorname{meas}\left\{s : s \in \Omega, |x(s) - y(s)| > h\right\}\right\};$$

convergence with respect to this metric coincides with convergence in measure. Given f_n and f on $\Omega \times \mathbf{R}$, it is known that (Lebesgue) measurability of these functions is neither sufficient ([**14**]) nor necessary ([**8**]) for the operators (4.3) to map the space X into itself. Apart from the classical Carathéodory condition, there are many other sufficient conditions guaranteeing $F(X) \subseteq X$ (see e.g. [**9**] and the bibliography therein). Moreover, a condition on f_n and f which ensures that $x_m \to x$ (in measure) implies $F_n x_m \to Fx$ (in measure) can be found in [**15**].

Concerning (4.6) in the space $X = S(\Omega)$, this condition follows certainly from a Lipschitz condition for f, but this is far from being necessary. Actually, we do not know conditions on the functions f and h which are both necessary and sufficient for (4.6).

EXAMPLE 4.2. Let us now consider a more concrete example which shows the applicability of Theorem 3.2. To this end, consider in a Banach space E the initial value problem for $t \in [0, T]$,

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0 \in E, \tag{4.7}$$

where f is supposed to be uniformly continuous and bounded on $[0, T] \times \overline{B}_R(x_0)$ with

$$M = \sup\left\{\|f(t, x)\|_E : 0 \le t \le T,\ \|x - x_0\|_E \le R\right\} \le R/T. \tag{4.8}$$

It is a well-known fact that these conditions, however, are not sufficient for local solvability of (4.7), but there must be some compactness condition in addition. As a typical existence theorem we mention the following result (see [**1**, p. 229], see also [**12**]):

THEOREM 4.1. *Let α_E be the Hausdorff measure of noncompactness in the space E. Suppose*

$$\alpha_E(f[t, B]) \le \phi(t, \alpha_E(B)) \tag{4.9}$$

for any $B \subseteq \overline{B}_R(x_0)$ and $t \in [0, T]$, where ϕ is a continuous function on $[0, T] \times \mathbf{R}$ with the property that the scalar problem

$$z'(t) = \phi(t, z(t)), \qquad z(0) = 0, \tag{4.10}$$

has only the trivial solution $z(t) \equiv 0$ on $[0, T]$. Then there exists $\tau = \tau(x_0) > 0$ such that (4.7) is solvable on $[0, \tau]$. □

In recent years, this theorem has been weakened in different ways, often by constructing special new measures of noncompactness (for a sample result see [**6**]). It is our purpose here to deduce (global) solvability by means of Theorem 3.2. Let $Z = C([0, T]; E)$ be the space of continuous abstract functions, with norm $\|x\| = \max_{0 \le t \le T} \|x(t)\|_E$. In this space, the Arzelà-Ascoli criterion reads as follows: A subset $S \subseteq Z$ is relatively compact if and only if it is equicontinuous and the sets $S(t) = \{x(t) : x \in S\}$ are relatively compact for all $t \in [0, T]$. Consequently (see [**12**]), by

$$\alpha(S) = \max_{0 \le t \le T} \alpha_E(S(t))$$

a measure of noncompactness is given on the system of all equicontinuous subsets $S \subset Z$. Now define X to be the subset of all $x \in Z$ such that $x(0) = x_0$ and $\|x(t) - x_0\|_E \le R$ for $t \in [0, T]$; then X is a bounded metric space with metric induced by the norm on Z. The initial value problem (4.7) is equivalent to the fixed point problem $Kx = x$, where

$$Kx(t) = x_0 + \int_0^t f(s, x(s))\, ds \qquad (t \in [0, T]),$$

and, because of (4.8), K maps X into itself.

Let $h = T/n$ $(n \in N)$, and $t_\nu = \nu h$ $(\nu = 0, 1, \ldots, n)$. As in [**4**], define K_n on X by

$$K_n x(t) = x_0 + h \sum_{\nu=0}^{k-1} f(t_\nu, x(t_\nu)) + (t - t_k) f(t_k, x(t_k)),$$

$$t_k \le t \le t_{k+1},\ k = 0, 1, \ldots, n-1.$$

Clearly $K_n x(0) = x_0$ and $K_n(X) \subseteq X$ since

$$\|K_n x(t) - x_0\|_E \le (k+1) hM \le nhM = TM \le R$$

$$(x \in X,\ t \in [0, T]).$$

Moreover, the continuity of f implies that $K_n \to K$; the uniform continuity of f together with the inequality

$$\|K_n x - K_n y\| \le T\|Fx - Fy\| \tag{4.11}$$

(F as in (4.3)) imply that $[K_n]$ is equicontinuous. Therefore, as previously observed, $K_n \xrightarrow{c} K$. Furthermore, $\bigcup_n K_n(X)$ is equibounded and, being

$$\|K_n x(s) - K_n x(t)\|_E \le 4M|s - t|,$$

also equicontinuous. Now assume there exists a continuous function γ on $[0, T]$ such that

$$\alpha_E(f[t, B]) \le \gamma(t) \alpha_E(B) \tag{4.12}$$

for any $B \subseteq \overline{B}_R(x_0)$ and $t \in [0, T]$ (compare this condition with (4.9)). We will show that $K_n \xrightarrow{\lambda\text{-cc}} K$ with $\lambda = T\|\gamma\|$. Equivalently, we will prove that

$$\alpha([K_n]) \le \lambda; \tag{4.13}$$

(the relation $\omega([K_n]) = 0$ follows again from the fact that $K_n \xrightarrow{c} K$). To this end, let S be any equicontinuous subset of X, and take $\eta > \alpha(S)$. Then, for any $t \in [0, T]$,

$$\alpha_E(F(S(t))) = \alpha_E(f[t, S(t)]) \le \gamma(t) \alpha_E(S(t)) \le \gamma(t)\eta.$$

Let $\{z_1(t), \ldots, z_m(t)\}$ be a $\gamma(t)\eta$-net for $F(S(t))$ in E, i.e. for any $x(t) \in S(t)$ there exists $z_j(t)$ such that $\|f(t, x(t)) - z_j(t)\|_E \le \gamma(t)\eta$.

Define

$$Z_j(t) = \int_0^t z_j(s)\, ds + x_0$$

and observe that $\{Z_1, \ldots, Z_m\}$ is a $\lambda\eta$-net ($\lambda = T\|\gamma\|$) for $K(S)$ in X, since

$$\|Kx(t) - Z_j(t)\|_E \leq \int_0^t \|f(s, x(s)) - z_j(s)\| ds \leq \eta T \|\gamma\|.$$

This means $\alpha(K(S)) \leq \lambda\alpha(S)$. Moreover, from (4.11) we also deduce $\alpha_E(K_n(S(t))) \leq T\gamma(t)\eta$ for all $t \in [0, T]$ and $n \in N$. On the other hand, since $K_n \overset{c}{\rightarrow} K$, for any $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that $K_n(S) \subseteq \Omega_\varepsilon(K(S))$ for $n \geq n_\varepsilon$. Thus

$$\begin{aligned} \alpha\Big(\bigcup_n K_n(S)\Big) &\leq \max\Big\{\alpha(K_1(S)), \ldots, \alpha\big(K_{n_\varepsilon}(S)\big), \alpha(K(S)) + \varepsilon\Big\} \\ &\leq \lambda\alpha(S) + \varepsilon \end{aligned}$$

for any $\varepsilon > 0$. This shows that $\alpha([K_n]) \leq \lambda$, hence $K_n \overset{\lambda\text{-cc}}{\rightarrow} K$.

Observe now that the same argument as used in [**4**, p. 606] shows that, for each $n \in N$, the fixed point problem $K_n x = x$ has a unique solution $x_n \in X$. Therefore, from Theorem 3.2 (iii) (recall that $K_n \overset{\lambda\text{-cc}}{\rightarrow} K$ implies $K_n \overset{\lambda\text{-ac}}{\rightarrow} K$) we get that K has a fixed point $x \in [x_n]^*$ such that, for any $\varepsilon > 0$,

$$\operatorname{dist}(x_n, \operatorname{Fix}(K)) \leq \lambda\alpha(\{x_n\}) + \varepsilon \qquad (n \geq n_\varepsilon).$$

Let us give a more suggestive interpretation of this result by distinguishing various cases for the noncompactness of the sequence $[K_n]$:

(a) $\lambda = 0$. This corresponds to the ordinary ac convergence of $[K_n]$; hence, from [**4**, Theorem 3.7] we get existence of a fixed point, and convergence $x_n \to x$ if $[x_n]^* = \{x\}$.

(b) $0 < \lambda < 1$. In this case we have $\alpha(K) < 1$ according to Proposition 3.3 (i), and existence of fixed points follows from Sadovskii's theorem (see [**11, 13**]). Moreover, if $[x_n]^* = \{x\}$, then the difference $\|x_n - x\|$ is controlled by $\alpha(\{x_n\}) + \varepsilon$ ($n \geq n_\varepsilon$); but since $[x_n]^* = \{x\}$ implies compactness of $\{x_n\}$, we again have convergence $x_n \to x$.

(c) $\lambda \geq 1$. Now existence of fixed points does not follow any more from fixed point principles for condensing operators, but our Theorem 3.2 (iii) gives $\operatorname{Fix}(K) \neq \varnothing$. On the other hand, the distance of x_n to $\operatorname{Fix}(K)$ may now be very large: In fact, it will increase linearly with respect to the noncompactness of both $[K_n]$ and $[x_n]$.

5. Concluding remarks. Apart from fixed point problems, inhomogeneous problems are also treated in [4]; in this connection, the concept of regular operators and regular convergence plays an important role. In this final section, we shall briefly indicate a possible generalization.

First, let us recall that an operator A between two Banach spaces X and Y is called *regular* if for any bounded $S \subset X$, the compactness of $\overline{A(S)}$ implies the compactness of $\bar{S}$; a sequence $[A_n]_N$ of operators is called *asymptotically regular* if for any uniformly bounded sequence $[S_n]_N$ and all subsets $N' \subseteq N$, the d-compactness of $[A_n(S_n)]_{N'}$ implies the d-compactness of $[S_n]_{N'}$. A possible generalization of this is to introduce the numbers

$$\rho(A) = \inf\{\lambda : \lambda > 0, \alpha(S) \leq \lambda\alpha(A(S)), S \subset X \text{ bounded}\}$$

and

$$\rho([A_n]_N) = \sup\inf\{\lambda : \lambda > 0, \delta([S_n]_{N'}) \leq \lambda\delta([A_n(S_n)]_{N'}), \\ [S_n]_N \text{ uniformly bounded}\},$$

where the supremum is taken over all infinite subsets $N' \subseteq N$. Further, we introduce a type of convergence which generalizes the *regular convergence* in [4]; given $\lambda \geq 0$,

$$A_n \xrightarrow{\lambda\text{-r}} A \quad (\lambda\text{-}\textit{regular convergence}) \text{ means}$$

$$A_n \xrightarrow{c} A \quad \text{and} \quad \rho([A_n]_N) \leq \lambda.$$

As a motivation for this generalization, we give a result on λ-r convergent operator sequences which contains in part Theorem 4.10 in [4] and plays the same role for inhomogeneous equations $Ax = y$ as Theorem 3.3 for fixed point equations $Kx = x$.

THEOREM 5.1. *Let A_n, A be operators between X and Y and let $A_n \xrightarrow{\lambda\text{-r}} A$. Let $[S_n]_N$ be a sequence of subsets in X such that $\bigcup_n S_n$ is bounded, and let $[y_n]_N$ be a bounded sequence in Y. Define*

$$\tilde{S}_n = \{x : x \in S_n, A_n x = y_n\}.$$

Then

(i) $\delta([\tilde{S}_n]_N) \leq \delta([y_n]_N)$;

(ii) $[y_n]_N^* \neq \varnothing$ *implies* $[\tilde{S}_n]_N^* \neq \varnothing$ *and* $A([\tilde{S}_n]_N^*) \subseteq [y_n]_N^*$.

Proof. (i) From $A_n \overset{\lambda\text{-r}}{\to} A$ it follows that

$$\delta([\tilde{S}_n]) \leq \lambda\delta([A_n(\tilde{S}_n)]) = \lambda\delta([y_n])$$

by definition of $\tilde{S}_n$.

(ii) If $[y_n]^* \neq \varnothing$, then $\delta([y_n]) < \infty$, hence $\delta([\tilde{S}_n]) < \infty$, hence $[\tilde{S}_n]^* \neq \varnothing$. Moreover, given $x \in [\tilde{S}_n]^*$ and $x_n \in \tilde{S}_n$ such that $x_n \to x$ $(n \in N')$, we have $y_n = A_n x_n \to Ax$ $(n \in N')$ which means $Ax \in [y_n]^*$. □

Regular convergence has applications to boundary value problems for nonlinear differential equations (see [**4**, Example 4.2]). However, there seems to be another field of possible applications, namely nonlinear spectral theory. The reader can find some interesting topics on nonlinear spectral theory, for example, in [**7**]; in this paper the number

$$\beta(A) = \sup\{\lambda : \lambda > 0, \alpha(A(S)) \geq \lambda\alpha(S), S \subset X \text{ bounded}\}$$

plays a crucial role for the definition and description of the spectrum of a nonlinear operator A. It is not hard to see that $\beta(A) = 1/\rho(A)$; consequently, all results involving the number $\beta(A)$ can be expressed equivalently in terms of $\rho(A)$. (To give a simple example, a continuous *linear* operator A between two Banach spaces X and Y is Fredholm if and only if $\rho(A)$ and $\rho(A^*)$ are both finite, where A^* is the adjoint of A.)

Some connections between nonlinear spectral theory and regular operator convergence will be discussed in a forthcoming paper.

Note added in proof. After submitting the present paper, the authors became acquainted with G. M. Vainikko's work on measures of discrete noncompactness (see [**18–21**] and, in particular, [**22**]). Our measure δ, however, is defined in another way, and has different properties. We thank Professor Vainikko for pointing our attention to the papers [**18–22**].

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