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ON THE ORDERS OF AUTOMORPHISMS OF A CLOSED RIEMANN SURFACE

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Let S be a closed Riemann surface of genus g (≥ 2). It is known that the maximum value of the orders of automorphisms of S is $4g+2$. In this paper we determine the orders of automorphisms of S which are greater than or equal to $3g$, and characterize those Riemann surfaces having the corresponding automorphisms. Except for several cases, such Riemann surfaces are determined uniquely up to conformal equivalence.

THEOREM 1. *Let $N(S, h)$ be the order of an automorphism h of S. Then, $\max_{S,h} N(S, h) = 4g + 2$. The Riemann surface having the automorphism of maximum order $4g + 2$ is conformally equivalent to the Riemann surface defined by*

$$y^2 = x(x^{2g+1} - 1).$$

The automorphism h of order $4g+2$ is given by

$$h(x, y) = \left(e^{2\pi i/(2g+1)}x, e^{2\pi i/(4g+2)}y\right).$$

Although the existence of the Riemann surface with the automorphism of order $4g+2$ is well known, in the above theorem the uniqueness (up to conformal equivalence) is shown.

To simplify, we write Theorem 1 in the following form:

$$\max N = 4g + 2, \qquad S\colon y^2 = x(x^{2g+1} - 1),$$
$$h(x, y) = \left(e^{2\pi i/(2g+1)}x, e^{2\pi i/(4g+2)}y\right).$$

Under similar notation,

THEOREM 2.

$$\max_{N<4g+2} N = 4g, \quad S\colon y^2 = x(x^{2g} - 1), \quad h(x, y) = \left(e^{2\pi i/2g}x, e^{2\pi i/4g}y\right).$$

THEOREM 3. *If $g \equiv 0 \pmod 3$, for $g \neq 3$,*

$$\max_{N<4g} N = 3g + 3, \qquad S\colon y^3 = x^2(x^{g+1} - 1),$$
$$h(x, y) = \left(e^{2\pi i/(g+1)}x, e^{4\pi i/(3g+3)}y\right).$$

For $g = 3$, we have $4g = 3g + 3$. Then there exist two Riemann surfaces defined by

$$y^2 = x(x^6 - 1) \quad \text{and} \quad y^3 = x^2(x^4 - 1)$$

which have an automorphism of order 12. Furthermore,

$$\max_{N<3g+3} N = 3g, \quad S\colon y^3 = x(x^g - 1), \quad h(x, y) = \left(e^{2\pi i/g}x, e^{2\pi i/3g}y\right),$$

except for

$$\begin{aligned} S\colon y^{20} &= x^5(x-1)^4 \quad (g = 6, N = 20 = 3g + 2),\\ \colon y^{28} &= x^7(x-1)^4 \quad (g = 9, N = 28 = 3g + 1),\\ \colon y^{36} &= x^9(x-1)^4 \quad (g = 12, N = 36 = 3g). \end{aligned}$$

THEOREM 4. *If* $g \equiv 1 \pmod 3$,

$$\max_{N<4g} N = 3g + 3, \qquad S\colon y^3 = x(x^{g+1} - 1),$$

$$h(x, y) = \left(e^{2\pi i/(g+1)}x, e^{2\pi i/(3g+3)}y\right).$$

$$\max_{N<3g+3} N = 3g, \quad S\colon y^3 = x(x^g - 1), \quad h(x, y) = \left(e^{2\pi i/g}x, e^{2\pi i/3g}y\right),$$

except for

$$\begin{aligned} S\colon y^{12} &= x^3(x-1)^2 \quad (g = 4, N = 12 = 3g),\\ \colon y^{30} &= x^5(x-1)^6 \quad (g = 10, N = 30 = 3g). \end{aligned}$$

THEOREM 5. *If* $g \equiv 2 \pmod 3$,

$$\max_{N<4g} N = 3g, \quad S\colon y^3 = x^2(x^g - 1), \quad h(x, y) = \left(e^{2\pi i/g}x, e^{4\pi i/3g}y\right),$$

except for

$$S\colon y^6 = x^3(x-1)^3(x-\zeta)^2 \qquad (g = 2, N = 6 = 3g, \zeta \in \mathbf{C}, \zeta \neq 0, 1).$$

We introduce the following notation; $\langle h \rangle$ denotes the cyclic group generated by h of order N. $\tilde{S} = S/\langle h \rangle$ denotes the closed Riemann surface of genus $\tilde{g}$ obtained by identifying those points on S which are equivalent under the action of $\langle h \rangle$ on S. $\tilde{p}_1, \ldots, \tilde{p}_t \in \tilde{S}$ denote the projections of branch points of the covering map $\varphi\colon S \to \tilde{S}$. $\nu_1, \ldots, \nu_t$ denote the multiplicities of φ at the branch points over $\tilde{p}_1, \ldots, \tilde{p}_t$, respectively.

A Fuchsian group is said to be a $(\gamma;\ m_1,\dots,m_n)$ group if its signature is $(\gamma;\ m_1,\dots,m_n)$. If $n = 0$, it is said to be a surface group. A homomorphism from a Fuchsian group onto a finite group is said to be a surface kernel homomorphism if its kernel is a surface group.

LEMMA 1. (*Harvey* [**2**].) *Let* Γ *be a* $(\gamma;\ m_1,\dots,m_n)$ *group,* Z_N *the cyclic group of order* N, *and* $M = \operatorname{lcm}(m_1,\dots,m_n)$. *Then there exists a surface kernel homomorphism from* Γ *onto* Z_N *if and only if the signature* $(\gamma;\ m_1,\dots,m_n)$ *satisfies the following l.c.m. condition*;

(1) $M = \operatorname{lcm}(m_1,\dots,\check{m}_i,\dots,m_n)$ $(i = 1,\dots,n)$. *Here,* $\check{m}_i$ *denotes the omission of* m_i.

(2) $M|N$, *if* $\gamma = 0$ *then* $M = N$.

(3) $n \neq 1$, *if* $\gamma = 0$ *then* $n \geq 3$.

(4) *If* $2|M$, *the number of* m_i*'s which are divisible by the maximum power of* 2 *which divides* M *is even.*

LEMMA 2. (*Riemann-Hurwitz relation.*)

$$2g - 2 = N(2\tilde{g} - 2) + N\sum_{i=1}^{t}\left(1 - \frac{1}{\nu_i}\right).$$

LEMMA 3. *If* $\tilde{t} = 0$, *then* S *is conformally equivalent to the Riemann surface defined by*

$$y^N = f(x) \qquad (f(x) \text{ is a polynomial of } x).$$

LEMMA 4. $(\tilde{g};\ \nu_1,\dots,\nu_t)$ *satisfies the l.c.m. condition.*

Proof. Let D be the unit disk, K a Fuchsian surface group which uniformize S, and ψ the natural projection from D onto $S = D/K$. Let $D^* = D - (\varphi\circ\psi)^{-1}\{\tilde{p}_1,\dots,\tilde{p}_t\}$, $\tilde{S}^* = \tilde{S} - \{\tilde{p}_1,\dots,\tilde{p}_t\}$, and let Γ be the covering transformation group of the covering $\varphi\circ\psi\colon D^* \to S^*$. Then Γ is a $(\tilde{g};\ \nu_1,\dots,\nu_t)$ group and $\Gamma/K \simeq Z_N$. So from Lemma 1, we find that $(\tilde{g};\ \nu_1,\dots,\nu_t)$ satisfies the l.c.m. condition.

LEMMA 5. *If* $N > 2g - 2$, *then* $\tilde{g} = 0$, $t = 3, 4$.

Proof. From the Riemann-Hurwitz relation, if $\tilde{g} \geq 2$,

$$2g - 2 \geq N(2\tilde{g} - 2) \geq 2N.$$

This contradicts the hypothesis. If $\tilde{g} = 1$, from the l.c.m. condition, $t \geq 2$.

Then,

$$2g-2=N\sum_{i=1}^{t}\left(1-\frac{1}{\nu_i}\right)\geq tN/2\geq N.$$

This also contradicts the hypothesis. So $\tilde{g}=0$, and

$$2g-2=-2N+N\sum_{i=1}^{t}\left(1-\frac{1}{\nu_i}\right)\geq\frac{(t-4)N}{2}.$$

Thus $t=3$, 4 or 5. But if $t=5$,

$$2g-2=N\left(3-\sum_{i=1}^{5}\frac{1}{\nu_i}\right),$$

and from $N>2g-2$, we find that

$$2<\sum_{i=1}^{5}\frac{1}{\nu_i}<3.$$

The signatures which satisfy these inequalities are the following:

$$(0;2,2,2,2,*),\quad(0;2,2,2,3,3),\quad(0;2,2,2,3,4),\quad(0;2,2,2,3,5).$$

None of these satisfies the l.c.m. condition.

LEMMA 6. *If* $N>2g+2$, *then* $t=3$.

Proof. From Lemma 5, $\tilde{g}=0$, $t=3$, 4. If $t=4$, from the Riemann-Hurwitz relation, we find that

$$1<\sum_{i=1}^{4}\frac{1}{\nu_i}<2.$$

The signatures which satisfy these inequalities and the l.c.m. condition are the following (N on the right side is given by $N=M=\operatorname{lcm}(\nu_1,\nu_2,\nu_3,\nu_4)$, g is calculated from $\tilde{g}$, ν_1, ν_2, ν_3, ν_4, N by the Riemann-Hurwitz relation):

$(0;2,2,m,m)$ $(m\neq 2)$	if $2\mid m$,	$g=m/2$, $N=m=2g$,
	if $2\nmid m$,	$g=m-1$, $N=2m=2g+2$,
$(0;2,3,3,6)$		$g=3$, $N=6=2g$,
$(0;2,3,4,12)$		$g=6$, $N=12=2g$,
$(0;2,3,5,30)$		$g=15$, $N=30=2g$,
$(0;3,3,3,3)$		$g=2$, $N=3=2g-1$,
$(0;3,3,4,4)$		$g=6$, $N=12=2g$,
$(0;3,3,5,5)$		$g=8$, $N=15=2g-1$.

None of these satisfies $N>2g+2$.

Proof of theorems. If we assume $N \geq 3g$ ($\geq 2g+2$), from Lemma 3, $\tilde{g}=0$, $t=3$ or exceptionally (I) $\tilde{g}=0$, $t=4$, $(\tilde{g};\ \nu_1, \nu_2, \nu_3, \nu_4) = (0;\ 2,2,3,3)$, $g=2$, $N=6$. When $\tilde{g}=0$, $t=3$, from the Riemann-Hurwitz relation, we find that

$$\frac{1}{3} < \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} < 1.$$

The signatures which satisfy these inequalities and the l.c.m. condition are the following;

	$(0;2,m,m)$	$(2 \mid m$ then $4 \mid m\ (m \neq 4))$	$g = m/4$,	$N = m = 4g$,
	$(0;2,m,2m)$	$(2 \nmid m\ (m \neq 3))$	$g = (m-1)/2$,	$N = 2m = 4g+2$,
	$(0;3,m,m)$	$(3 \mid m\ (m \neq 3))$	$g = m/3$,	$N = m = 3g$,
	$(0;3,m,3m)$	$(3 \nmid m)$	$g = m-1$,	$N = 3m = 3g+3$,
(II)	$(0;4,5,20)$		$g = 6$,	$N = 20 = 3g+2$,
	$(0;4,6,12)$		$g = 4$,	$N = 12 = 3g$,
	$(0;4,7,28)$		$g = 9$,	$N = 28 = 3g+1$,
	$(0;4,9,36)$		$g = 12$,	$N = 36 = 3g$,
	$(0;5,6,30)$		$g = 10$,	$N = 30 = 3g$.

So if we exclude the exceptional cases (I) and (II), the signatures $(\tilde{g};\ \nu_1, \nu_2, \nu_3)$ are listed as following;

If $N = 4g+2$, $\quad (0;\ 2, 2g+1, 4g+2)$.
If $N = 4g$, $\quad (0;\ 2, 4g, 4g)$.
If $N = 3g+3$, $\quad (0;\ 3, g+1, 3g+3)$.
(In this case, $3 \nmid m$ and $g = m-1$ imply $g \equiv 0, 1 \pmod 3$.)
If $N = 3g$, $\quad (0;\ 3, 3g, 3g)$.

Now S branches over three points of the Riemann sphere $\overline{\mathbf{C}}$, and the branching orders are given as above, so if we assume that the projections of branch points are 0, 1 and ∞, from Lemma 3, S is conformally equivalent to the Riemann surface defined by

$$y^N = x^a (x-1)^b,$$

where a, b are given by the following conditions;

$$1 \leq a, b < N, \quad N/(N,a) = \nu_1, \quad N/(N,b) = \nu_2, \quad N/(N,a+b) = \nu_3.$$

((N, a) denotes the g.c.m. of N and a.)

Then if $N = 4g+2$, S is defined by

(1) $\quad y^{4g+2} = x^{2g+1}(x-1)^{2k} \qquad ((2g+1, k) = 1, 1 \leq k < 2g+1).$

This surface is conformally equivalent to the Riemann surface defined by

$$Y^2 = X(X^{2g+1} - 1)$$

under the birational transformation

$$\begin{cases} y = \dfrac{Y}{X^{g+1+k}}, \\ x = -\dfrac{1}{X^{2g+1}} + 1, \end{cases} \qquad \begin{cases} Y = e^{(g+1)\pi i/(2g+1)} \dfrac{x^a(x-1)^b y^{(2g+1)c}}{\left(x^p(x-1)^q y^{2r}\right)^{g+1}}, \\ X = e^{\pi i/(2g+1)} \dfrac{1}{x^p(x-1)^q y^{2r}}, \end{cases}$$

where (a, b, c), (p, q, r) are the solutions of the indeterminate equations

$$\begin{cases} 2a + (2g+1)c = 1, \\ b + kc = 0, \end{cases} \qquad \begin{cases} p + r = 0, \\ (2g+1)q + 2kr = 1. \end{cases}$$

If $N = 4g$, S is defined by

$$(2) \quad y^{4g} = x^{2g}(x-1)^k \qquad ((4g, k) = (4g, 2g-k) = 1, 1 \le k < 4g).$$

This surface is conformally equivalent to the Riemann surface defined by

$$Y^2 = X(X^{2g} - 1),$$

under the birational transformation

$$\begin{cases} y = e^{\pi i/4g} X^{(k-1)/2} Y, \\ x = -X^{2g} + 1, \end{cases} \qquad \begin{cases} Y = e^{\pi i/4g} x^a (x-1)^b y^c, \\ X = e^{\pi i/2g} x^p (x-1)^q y^r, \end{cases}$$

where (a, b, c), (p, q, r) are the solutions of the indeterminate equations

$$\begin{cases} 2a + c = 1, \\ 4gb + kc = 1, \end{cases} \qquad \begin{cases} p + r = 0, \\ 2gq + kr = 1. \end{cases}$$

If $N = 3g + 3$, S is defined by

$$(3) \qquad y^{3g+3} = x^{j(g+1)}(x-1)^{3k}$$

$$((g+1, k) = (3g+3, (3-j)(g+1) - 3k) = 1,$$

$$j = 1, 2, 1 \le k < g+1).$$

When $g \equiv 0 \pmod 3$, (3) is conformally equivalent to the Riemann surface defined by

$$Y^3 = X^2(X^{g+1} - 1),$$

under the birational transformation

$$\begin{cases} y = e^{k\pi i/(g+1)} \dfrac{Y^j}{X^{k+j(g/3+1)}}, \\ x = -\dfrac{1}{X^{g+1}} + 1, \end{cases}$$

$$\begin{cases} Y = e^{(g+3)\pi i/(3g+3)} \dfrac{x^a(x-1)^b y^{(g+1)c}}{\left(x^p(x-1)^q y^{3r}\right)^{g/3+1}}, \\ X = e^{\pi i/(g+1)} \dfrac{1}{x^p(x-1)^q y^{3r}}, \end{cases}$$

where (a, b, c), (p, q, r) are the solutions of the indeterminate equations

$$\begin{cases} 3a + j(g+1)c = 1, \\ b + kc = 0, \end{cases} \qquad \begin{cases} p + jr = 0, \\ (g+1)q + 3kr = 1. \end{cases}$$

When $g \equiv 1 \pmod 3$, (3) is conformally equivalent to the Riemann surface defined by

$$Y^3 = X(X^{g+1} - 1),$$

under the birational transformation

$$\begin{cases} y = e^{k\pi i/(g+1)} \dfrac{Y^j}{X^{k+j(g+2)/3}}, \\ x = -\dfrac{1}{X^{g+1}} + 1, \end{cases}$$

$$\begin{cases} Y = e^{(g+2)\pi i/(3g+3)} \dfrac{x^a(x-1)^b y^{(g+1)c}}{\left(x^p(x-1)^q y^{3r}\right)^{(g+2)/3}}, \\ X = e^{\pi i/(g+1)} \dfrac{1}{x^p(x-1)^q y^{3r}}, \end{cases}$$

where (a, b, c), (p, q, r) are the solutions of the indeterminate equations

$$\begin{cases} 3a + j(g+1)c = 1, \\ b + kc = 0, \end{cases} \qquad \begin{cases} p + jr = 0, \\ gp + kr = 1. \end{cases}$$

If $N = 3g$, S is defined by

$$y^{3g} = x^{jg}(x-1)^k \tag{4}$$

$$((3g, k) = (3g, (3-j)g - k) = 1, j = 1, 2, 1 \le k < g).$$

Then we notice that $k \equiv j \pmod 3$ or $k \equiv 2j \pmod 3$. In the case $k \equiv j \pmod 3$, (4) is comformally equivalent to the Riemann surface defined by

$$Y^3 = X(X^g - 1),$$

under the birational transformation

$$\begin{cases} y = e^{((k+jg)\pi i/3g)} X^{(k-j)/3} Y^j, \\ x = -X^g + 1, \end{cases} \qquad \begin{cases} Y = e^{((g+1)\pi i/3g)} x^a (x-1)^b y^c, \\ X = e^{\pi i/g} x^p (x-1)^q y^{3r}, \end{cases}$$

where (a, b, c), (p, q, r) are the solutions of the indeterminate equations

$$\begin{cases} 3a + jc = 1, \\ 3gb + kc = 1, \end{cases} \qquad \begin{cases} p + jr = 0, \\ gq + kr = 1. \end{cases}$$

In the case $k \equiv 2j \pmod 3$, (4) is conformally equivalent to the Riemann surface defined by

$$Y^3 = X^2(X^g - 1),$$

under the birational transformation

$$\begin{cases} y = e^{((k+jg)\pi i/3g)} X^{(k-2j)/3} Y^j, \\ x = -X^g + 1, \end{cases} \qquad \begin{cases} Y = e^{\pi i/3} x^a (x-1)^b y^c, \\ X = e^{\pi i/3} x^p (x-1)^q y^{3r}, \end{cases}$$

where (a, b, c), (p, q, r) are the solutions of the indeterminate equations

$$\begin{cases} 3a + jc = 1, \\ 3gb + kc = 2, \end{cases} \qquad \begin{cases} p + jr = 0, \\ gq + kr = 1. \end{cases}$$

Finally, if $g \equiv 0 \pmod 3$, two Riemann surfaces

$$y^3 = x(x^g - 1) \quad \text{and} \quad Y^3 = X^2(X^g - 1)$$

are conformally equivalent under the birational transformation

$$\begin{cases} y = -X^{g/3+1} Y, \\ x = X^{-1}, \end{cases} \qquad \begin{cases} Y = -x^{g/3+1} y, \\ X = x^{-1}. \end{cases}$$

For a surface in (4), if $g \equiv 1 \pmod 3$, we obtain $k \equiv j \pmod 3$, while if $g \equiv 2 \pmod 3$, $k \equiv 2j \pmod 3$.

In the exceptional case (I), the surfaces are conformally equivalent to the Riemann surface defined by

$$y^6 = x^3(x-1)^3(x-\zeta)^2 \qquad (\zeta \in \mathbf{C}, \zeta \neq 0, 1).$$

In the case (II), the surfaces which have the same signature are conformally equivalent to each other. Thus we have the following forms of S:

$$
\begin{aligned}
y^{20} &= x^5(x-1)^4, \quad (0; 4,5,20),\\
y^{28} &= x^7(x-1)^4, \quad (0; 4,7,28),\\
y^{12} &= x^3(x-1)^2, \quad (0; 4,6,12),\\
y^{36} &= x^9(x-1)^4, \quad (0; 4,9,36),\\
y^{30} &= x^6(x-1)^5, \quad (0; 5,6,30).
\end{aligned}
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