PACIFIC JOURNAL OF MATHEMATICS
Vol 115, No. 2, 1984

NONOSCILLATORY FUNCTIONAL DIFFERENTIAL EQUATIONS

G. LADAS, Y. G. SFICAS AND I. P. STAVROULAKIS

Our aim in this paper is to obtain sufficient conditions under which certain functional differential equations have a "large" number of non-oscillatory solutions. Using the characteristic equation of a "majorant" delay differential equation with constant coefficients and Schauder's fixed point theorem, we obtain conditions under which the functional differential equation in question has a nonoscillatory solution. Then a known comparison theorem is employed as a tool to demonstrate that if the functional differential equation has a nonoscillatory solution, then it really has a "large" number of such solutions.

Our aim in this paper is to obtain sufficient conditions under which the functional differential equation

$$x'(t)+\sum_{i=1}^{n}p_i(t)x(t-\tau_i(t))=0 \tag{1}$$

has a "large" number of nonoscillatory solutions. It is to be noted that the literature is scarce concerning conditions under which there exist nonoscillatory solutions. Using the characteristic equation of a "majorant" delay differential equation with constant coefficients and Schauder's fixed point theorem, we obtain conditions under which (1) has a nonoscillatory solution. Then we employ a known comparison theorem [see **1**, p. 224, also **4**, Ch. 6] as a tool to demonstrate that if (1) has a nonoscillatory solution then it really has a "large" number of such solutions.

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros. A differential equation is called oscillatory if all of its solutions oscillate; otherwise, it is called nonoscillatory. In this paper we restrict our attention to real valued solutions $x(t)$.

2. Non-oscillations.

THEOREM 1. *Consider the differential equation*

$$x'(t)+\sum_{i=1}^{n}p_i(t)x(t-\tau_i(t))=0 \tag{1}$$

where $p_i(t)$ *and* $\tau_i(t)$ *are continuous functions such that* $|p_i(t)| \leq P_i$, $|\tau_i(t)| \leq T_i$, $|p_i'(t)| \leq A_i$ *and* $|\tau_i'(t)| \leq B_i$, $i = 1, 2, \ldots, n$, *where* P_i, T_i, A_i *and* B_i *are positive constants. Assume that*

$$\lambda = \sum_{i=1}^{n} P_i e^{\lambda T_i} \tag{2}$$

has a positive root. Then equation (1) *has a nonoscillatory solution of the form*

$$x(t) = \exp\left(-\int_{t_0}^{t} \lambda(s)\, ds\right) \tag{3}$$

where $\lambda(t)$ *is a bounded continuous function.*

Proof. Suppose that λ_0 is a positive root of (2), i.e.,

$$\lambda_0 = \sum_{i=1}^{n} P_i e^{\lambda_0 T_i}.$$

We will prove that (1) has a nonoscillatory solution of the form (3). Substituuting (3) into (1) we obtain

$$\lambda(t) = \sum_{i=1}^{n} p_i(t) \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds\right). \tag{4}$$

It suffices to show that (4) has a bounded solution. We will employ Schauder's fixed point theorem. Define the sets

$$X = \{\lambda(t):\ \text{bounded continuous functions mapping } \mathbf{R} \text{ into } \mathbf{R}\}$$

with sup-norm, which is a Banach space, and

$$M = \{\lambda(t) \in X: \|\lambda(t)\| \leq \lambda_0\}$$

which is a closed and convex subset of X. Consider the mapping F on M given by

$$F\lambda(t) = \sum_{i=1}^{n} p_i(t) \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds\right).$$

Observe that

$$\|F\lambda(t)\| \leq \sum_{i=1}^{n} |p_i(t)| \exp\left(\left|\int_{t-\tau_i(t)}^{t} \|\lambda(s)\| ds\right|\right)$$
$$\leq \sum_{i=1}^{n} P_i e^{\lambda_0 T_i} = \lambda_0.$$

Hence F: $M \to M$.

To show that (4) has a solution it suffices to show that the mapping F has a fixed point. To this end it remains to show that F is continuous and that FM is a relatively compact subset of X.

We will show that F is continuous by showing that each of the mappings

$$F_i\lambda(t) = \exp\left(\int_{t-\tau_i}^{t} \lambda(s)\, ds\right), \qquad i = 1, 2, \ldots, n,$$

is continuous. Let $\lambda_n \to \lambda$ where $\lambda_n, \lambda \in M$. Then

$$|F_i\lambda(t) - F_i\lambda(t)| = F_i\lambda(t)\left|\frac{F_i\lambda_n(t)}{F_i\lambda(t)} - 1\right|$$

$$= F_i\lambda(t)\left|\exp\left(\int_{t-\tau_i}^{t} [\lambda_n(s) - \lambda(s)]\, ds\right) - 1\right|.$$

But

$$\left|\int_{t-\tau_i(t)}^{t} [\lambda_n(s) - \lambda(s)]\, ds\right| \leq \|\lambda_n - \lambda\| \cdot T_i \to 0 \quad \text{as } n \to \infty.$$

and because $F_i\lambda(t)$ is bounded, it follows that F_i is continuous.

To prove that FM is a relatively compact subset of X it suffices to prove that if K is a positive constant and λ is a function in X such that $\|\lambda\| \leq K$, then $(F\lambda(t))'$ is uniformly bounded. We have

$$(F\lambda(t))' = \sum_{i=1}^{n} p_i'(t) \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds\right)$$

$$+ \sum_{i=1}^{n} p_i(t)[\lambda(t) - \lambda(t - \tau_i(t))(1 - \tau_i'(t))]$$

$$\cdot \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds\right)$$

and therefore

$$|(F\lambda(t))'| \leq \sum_{i=1}^{n} A_i e^{KT_i} + \sum_{i=1}^{n} P_i K B_i e^{KT_i}.$$

Therefore Schauder's fixed point theorem applies and the proof is complete.

Note that the r.h.s. of (2) is a positive convex function of λ and so (2) has either two real roots, one real root, or no real root. Except in the case that all the P_i are zero, the roots are always positive. Thus (2) really just means $T_1, \ldots, T_n, P_1, \ldots, P_i$n are fairly small.

For the delay differential equation

$$(1)' \qquad x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$$

whose coefficients and delays are positive constants, it has been proved [**5**], see also [**3**], that every solution oscillates if and only if the characteristic equation

$$(2)' \qquad \lambda + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} = 0$$

has no real roots. This is equivalent to saying that (1)′ has a nonoscillatory solution if and only if (2)′ has a real root.

The following are immediate corollaries of Theorem 1.

COROLLARY 1. *Equation* (1) *is nonoscillatory provided that the "majorant" delay differential equation*

$$(5) \qquad x'(t) + \sum_{i=1}^{n} P_i x(t - T_i) = 0,$$

where P_i and T_i are as defined in Theorem 1, *is nonocillatory.*

COROLLARY 2. *The functional differential equation with constant coefficients and constant arguments*

$$(6) \qquad x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$$

is nonoscillatory provided that the delay differential equation

$$(7) \qquad x'(t) + \sum_{i=1}^{n} |p_i| x(t - |\tau_i|) = 0$$

is nonoscillatory.

3. A comparison theorem and its applications. Next we will demonstrate how the following comparison result [see **1**, p. 224, also 4, Ch. 6] may be used as a tool to establish that if a functional differential equation has a nonoscillatory solution then it has a "large" number of such solutions in a sense that will be made clear below.

THEOREM 2. (*Comparison Theorem.*) *Consider the delay differential equation*

$$(*)\qquad x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i) = 0, \qquad t \geq 0, n \geq 1,$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = \tau$ *are constants,* $p_0, p_1, \ldots, p_n$ *are continuous functions and* $p_1(t), p_2(t), \ldots, p_n(t)$ *positive on* $[0, \infty)$. *Let* θ, $\tilde{\theta}$: $[-\tau, 0) \to \mathbf{R}$ *be continuous and such that*

$$(8)\qquad \theta(t) < \tilde{\theta}(t) \quad on\ [-\tau, 0) \qquad and \qquad \theta(0) = \tilde{\theta}(0) > 0.$$

Let x *and* $\tilde{x}$ *be the unique solutions of* $(*)$ *with initial functions* θ *and* $\tilde{\theta}$ *respectively. Assume that*

$$(9)\qquad \tilde{x}(t) > 0 \quad on\ [0, \infty).$$

Then

$$(10)\qquad x(t) > \tilde{x}(t) \quad on\ (0, \infty).$$

REMARK 1. If we denote by $x(t, t_0, \theta)$ the unique solution of $(*)$ with initial function θ at $t = t_0$, then $x(t, t_0, -\theta) = -x(t, t_0, \theta)$. From this observation we obtain a dual to the above theorem by simply reversing the signs of the inequalities in (8), (9), and (10). That is, under the hypotheses of Theorem 2 we have, on $(0, \infty)$,

$$x(t, 0, \theta) > \tilde{x}(t, 0, \tilde{\theta}) > 0 \quad \text{and} \quad x(t, 0, -\theta) < \tilde{x}(t, 0 - \tilde{\theta}) < 0.$$

Finally a close look at the proof of the comparison theorem [see **1**, p. 224] shows that the functional arguments in $(*)$ do not have to be constants. The results is true if we assume tha $\tau_i(t)$ are continuous function satisfiying the following condition

$$(11)\qquad \begin{cases} \text{(i) } \tau_0(t) \equiv 0 \quad \text{and} \quad \tau_j(t) \not\equiv 0 \quad \text{for } j = 1, 2, \ldots, n; \\ \text{(ii) } \exists \tau > 0 \quad \text{such that } 0 \leq \tau_j(t) \leq \tau,\ j = 1, 2, \ldots, n. \end{cases}$$

First we apply the comparison theorem to the delay differential equation

$$(1)'\qquad x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$$

where p_i and τ are positive constants. As discussed above $(1)'$ has a nonoscillatory solution provided that the characteristic equation

$$(2)'\qquad f(\lambda) \equiv \lambda + \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} = 0$$

has a real root. The condition, for example,

$$\left(\sum_{i=1}^{n} p_i\right)\rho \le \frac{1}{e} \quad \text{where } \tau = \max\{\tau_1, \tau_2, \ldots, \tau_n\} \tag{12}$$

implies that $f(0)f(-1/\tau) \le 0$ and therefore $(2)'$ has a real (negative) root in the interval $(-1/\tau, 0)$.

Now assume that $(2)'$ has a real root λ_0. Then $(1)'$ has the nonoscillatory solution

$$\mu e^{\lambda_0 t} \quad \text{for any } \mu \in \mathbf{R},\ \mu \neq 0.$$

But then, by the comparison theorem, any solution of $(1)'$ with initial function $\phi(t)$ satisfying

$$\phi(t) < \phi(0)e^{\lambda_0 t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \phi(0) > 0$$

and any solution of $(1)'$ with initial function $\psi(t)$ satisfying

$$\psi(t) > \psi(0)e^{\lambda_0 t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \psi(0) < 0$$

is nonoscillatory. In particular (and also when λ is not known) we have the following result.

COROLLARY 3. *Assume that* $(2)'$ *has a real root. Then any solution of* $(1)'$ *with initial function* ϕ *or* ψ *satisfying*

$$\phi(t) < \phi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \phi(0) > 0$$

or

$$\psi(t) > \psi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \psi(0) < 0$$

is nonoscillatory.

EXAMPLE 1. For the delay differential equation

$$x'(t) + \tfrac{1}{2}e^{-1/3}x\left(t - \tfrac{1}{3}\right) + \tfrac{1}{2}e^{-1/2}x\left(t - \tfrac{1}{2}\right) = 0 \tag{13}$$

condition (12) is satisfied. Therefore its characteristic equation

$$\lambda + \tfrac{1}{2}e^{-1/3-\lambda/3} + \tfrac{1}{2}e^{-1/2-\lambda/2} = 0 \tag{14}$$

has a real (negative) root in the interval $(-2, \infty)$. Observe that $\lambda = -1$ is a root of (14). Thus (13) has the nonoscillatory solution μe^{-t} for any $\mu \in \mathbf{R}$, $\mu \neq 0$. Also, using the comparison theorem, any solution of (13) with initial function ϕ or ψ satisfying

$$\phi(t) < \phi(0)e^{-t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \phi(0) > 0$$

or

$$\psi(t) > \psi(0)e^{-t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \psi(0) < 0$$

is nonoscillatory.

In view of Theorems 1 and 2 and Remark 1, we obtain the following result equation (1).

COROLLARY 4. *Consider the differential equation* (1) *subject to the hypotheses of Theorem* 1 *and in addition assume that* $p_i(t) > 0$, $i = 1, 2, \dots, n$, *and condition* (11) *is satisfied. Then, any solution of* (1) *with initial function* ϕ *or* ψ *satisfying*

$$\phi(t) < \phi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \phi(0) > 0$$

or

$$\psi(t) > \psi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \psi(0) < 0$$

is nonoscillatory.

Finally we apply the comparison theorem to the delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \ge t_0, \tag{15}$$

where τ is a positive constant and $p(t)$ is a τ-periodic continuous function with

$$K \equiv \int_{t-\tau}^{t} p(s)\,ds \le \frac{1}{e}. \tag{16}$$

With these hypotheses equation (15) has a nonoscillatory solution of the form

$$x(t) = \exp\left(\lambda \int_{t_0}^{t} p(s)\,ds\right) \tag{17}$$

with $\lambda < 0$. In fact, substituting (17) into (15), we obtain

$$g(\lambda) \equiv \lambda e^{K\lambda} + 1 = 0.$$

It suffices to show that $g(\lambda)$ has a negative root.

Case 1. $K < 0$. Then $g(-\infty) = -\infty$ and $g(0) = 1$. Therefore $g(\lambda)$ has a root in $(-\infty, 0)$.

Case 2. $K = 0$. Then $\lambda = -1$ is a root

Case 3. $K > 0$. Then $g(-1/K) = (Ke - 1)/Ke \le 0$ and $g(0) = 1$. Therefore $g(\lambda)$ has a root in $[-1/K, 0)$.

Thus in each case (15) has a nonoscillatory solution of the form given by (17). If in addition to (16) we assume that $p(t) > 0$ then the comparison theorem applies and we have the following result.

COROLLARY 5. *Consider the differential equation* (15) *under the assumptions that* $p(t) > 0$ *and* (16) *holds. Then the solution of* (15) *with initial function* ϕ *and* ψ *satisfying*

$$\phi(t) < \phi(t_0), \qquad t_0 - \tau \le t < t_0 \quad \text{and} \quad \phi(t_0) > 0$$

or

$$\psi(t) > \psi(t_0), \qquad t_0 - \tau \le t < t_0 \quad \text{and} \quad \psi(t_0) < 0$$

is nonoscillatory.

EXAMPLE 2. Consider the differential equation

$$x'(t) + (\sin t)x(t - 2\pi) = 0, \qquad t \ge 0.$$

Observe that $\sin t$ is a 2π-periodic function and condition (15) is satisfied, with $K = 0$. Note that $e^{\cos t}$ is a multiple of the nonoscillatory solution given by (17).

REMARK 2. When $p(t) > 0$ the condition $K > 1/e$ implies, see [**2**], that every solution of (15) oscillates. This is our motivation for the following

Conjecture. If $K > 1/e$ then (15) is oscillatory.

REFERENCES

[1] R. D. Driver, *Introduction to Ordinary Differential Equations*, Harper and Row, New York, 1978.

[2] G. Ladas, *Sharp conditions for oscillations caused by delays*, Applicable Anal., **9** (1979), 93–98.

[3] G. Ladas, Y. G. Sficas and I. P. Stavroulakis, *Necessary and sufficient conditions for oscillations*, Amer. Math. Monthly, **90** (1983), 637–640.

[4] A. D. Myškis, *Linear Differential Equations with Retarded Argument* (Russian), Izdatel'stvo "Nauka", 2nd Edition, Moscow, 1972.

[5] M. I. Tramov, *Conditions for oscillatory solutions of first order differential equations with a delayed argument*, Izv. Vysš. Učebn. Zaved. Mat., **19**, No. 3 (1975), 92–96.

Received April 6, 1983 and in revised form January 31, 1984.

UNIVERSITY OF RHODE ISLAND
KINGSTON, R. I. 02881

AND

UNIVERSITY OF IOANNINA
IOANNINA, GREECE