

TOPOLOGICAL PROPERTIES OF BANACH SPACES

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Let X be a Banach space, B_X its closed unit ball. We study several topological properties of B_X with its weak topology. In particular, we consider spaces X such that (B_X, weak) is a Polish topological space. If X has RNP and X^* is separable, then B_X is Polish; if B_X is Polish, then X is somewhat reflexive. We also consider spaces X such that every closed subset of (B_X, weak) is a Baire space. This is equivalent to property (PC), studied by Bourgain and Rosenthal.

1. Introduction. Let X be a Banach space, and B_X the closed unit ball. It is well-known that B_X is metrizable in the relative weak topology of X precisely when the dual space X^* is norm separable. Moreover, (B_X, weak) is compact metrizable precisely when X is separable and reflexive. Here we address an intermediate question: When is (B_X, weak) completely metrizable?

THEOREM A. *Let X be a separable Banach space. Then the following are equivalent: (1) (B_X, weak) is completely metrizable; (2) (B_X, weak) is a Polish space; (3) X has property (PC) and is an Asplund space; (4) (B_X, weak) is metrizable, and every closed subset of it is a Baire space.*

Property (PC) states that for every weakly closed bounded subset A of X , the identity map $(A, \text{weak}) \rightarrow (A, \text{norm})$ has at least one point of continuity. (The letters “PC” stand for “point of continuity”.) Property (PC) is a consequence of the Radon-Nikodym property.

Examples of spaces satisfying Theorem A include: spaces with separable second dual (such as separable quasi-reflexive spaces), the predual of the James Tree space [30], [38] (but not JT itself), and the James-Lindenstrauss spaces $JL(S)$ modelled on separable Banach spaces S [29], [36]. On the other hand, if X contains an isomorphic copy of c_0 or l^1 , then (B_X, weak) is not Polish. An example of Bourgain and Delbaen [5] satisfies Theorem A, but has dual space isomorphic to l^1 .

There is a natural non-metrizable version of the Polish property. A completely regular Hausdorff space T is said to be Čech complete iff it admits a complete sequence (\mathcal{U}_n) of open covers. Here completeness is defined as follows: if \mathcal{F} is a family of closed subsets of T such that \mathcal{F} has

the finite intersection property, and \mathcal{F} is \mathcal{U}_n -small for all n (that is, there exists $F_n \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ with $F_n \subseteq U_n$), then $\bigcap \mathcal{F} \neq \emptyset$. Completely metrizable spaces are Čech complete (take \mathcal{U}_n to be the family of open sets with diameter less than $1/n$). Locally compact Hausdorff spaces are also Čech complete (take each \mathcal{U}_n to be the family of relatively compact open sets). On the other hand, the usual proofs of the Baire Category Theorem can be easily adapted to show that every Čech complete space is a Baire space (the countable intersection of dense open sets is dense). A space T is Čech complete if and only if T is a G_δ set in some (every) compactification of T . A metrizable space is Čech complete if and only if it is completely metrizable. Čech completeness is preserved by closed subspaces, G_δ subspaces, and countable products. (See [18, p. 142], [19], [1].)

If X is an infinite-dimensional Banach space, then (X, weak) is not a Baire space, since each multiple of B_X is closed and nowhere dense. We shall be interested in spaces X such that (B_X, weak) is a Čech complete topological space. For this to occur, it is necessary and sufficient that B_X be a G_δ in $(B_{X^{**}}, \text{weak}^*)$, or that $B_{X^{**}} \setminus B_X$ be weak* σ -compact, or that X be a G_δ in (X^{**}, weak^*) . (Thus the property is preserved by linear homeomorphisms.) We will say that X has Polish ball, if (B_X, weak) is a complete separable metric space under some metric. Note that if (x_n^*) is a norm dense sequence in $B_{X^{**}}$, then

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n^*(x) - x_n^*(y)|$$

is a metric for (B_X, weak) . However, this metric is not complete unless X is reflexive; in fact, the completion of (B_X, d) is $B_{X^{**}}$ in a natural way.

If X is separable, then X has Čech complete ball if and only if X has Polish ball (see 3.1 and 3.2, below). In this case, X^* is also separable.

THEOREM B. *Let X be a Banach space. The following are equivalent: (1) X has Čech complete ball; (2) X is isomorphic to a direct sum $R \oplus S$, where R is reflexive and S has Polish ball.*

Theorems A and B yield structural information about spaces with Čech complete ball. Every space with Čech complete ball is weakly compactly generated (WCG), somewhat reflexive, and has property (PC). If X has Čech complete ball, then X^* is WCG and has the Radon-Nikodým property (RNP). Also, X and X^* have the same linear dimension and cardinality. If X has Čech complete ball, then (X^{**}, weak) is realcompact,

but X^{**} may fail the Pettis integral property. Moreover, $(B_{X^{**}}, \text{weak}^*)$ and (B_{X^*}, weak^*) are Eberlein compacts. The dual of a space with Čech complete ball cannot contain c_0 , but may contain l^1 . For spaces X with Čech complete ball, these conditions are equivalent: $X^* \not\supset l^1$, X^{**} has property (C), $(B_{X^{***}}, \text{weak}^*)$ is angelic.

Here is an indication of the organization of this paper. Section 2 reviews terminology and results from Banach space theory and general topology. Figure 1 may be helpful in relating the various concepts. Section 3 presents the proofs of Theorems A and B. Theorem A can be derived from results of Hurewicz ([28], [53, p. 333]) and Saint-Raymond [49] in a more general setting. We give a self-contained proof. Theorem B builds on results due to Amir and Lindenstrauss [2] and Valdivia [54]. Section 4 gives the basic structural results about spaces with Čech complete ball. It includes a discussion of properties stronger than Čech completeness. For example, the condition “ $B_{X^{**}} \setminus B_X$ is a countable union of weak* compact convex sets” implies both Čech complete ball and the Radon-Nikodym property. Section 5 considers property (PC) in detail. Some partial three-space properties are proved. In particular, if $Y \subseteq X$, Y has Polish ball, X/Y has property (PC), and Y^\perp is complemented in X^* , then X has property (PC). Section 6 contains a number of examples. Table 1 summarizes information about them.

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2. Preliminary results. Notation and terminology used here can be found, for example, in [39]. A good reference for the point-set topology we shall need is [18].

If X is a Banach space, we write X^* for the dual space and B_X for the closed unit ball of X . We use brackets $\langle \cdot, \cdot \rangle$ for the pairing between a Banach space and its dual: if $x \in X$ and $x^* \in X^*$, then $\langle x, x^* \rangle = x^*(x)$. The space X is identified with a subspace of X^{**} via the map $j: X \rightarrow X^{**}$ defined by

$$\langle x^*, j(x) \rangle = \langle x, x^* \rangle.$$

In most cases we omit mention of the map j . We write (X, weak) for the topological space obtained by equipping the Banach space X with its weak topology ([14, p. 419]). If A is a subset of a Banach space X , then its annihilator is

$$A^\perp = \{x^* \in X^*: \langle a, x^* \rangle = 0 \text{ for all } a \in A\}.$$

If B is a subset of X^* , then

$$B_{\perp} = \{x \in X : \langle x, b \rangle = 0 \text{ for all } b \in B\}.$$

Occasionally, we will need to refer to more than one topology in the same argument. We write $\text{wcl}(A)$ for the closure of A in the weak topology, and $\text{w}^*\text{cl}(A)$ for the closure in the weak* topology of X^{**} . If X is a Banach space, we write $X \leftarrow l^1$ if X has a closed subspace isomorphic to the space l^1 , and $X \not\leftarrow l^1$ if not.

In this paper we are primarily concerned with topological properties of the weak topology on a Banach space X or its subsets. Some classical results in this direction are:

(1) X is reflexive if and only if (B_X, weak) is compact, or (equivalently) (X, weak) is σ -compact ([14, V. 4.7])

(2) If A is a (closed) subset of (X, weak) , then A is sequentially compact if and only if A is compact. (Eberlein-Šmulian Theorem, see [14, V. 6.1])

(3) (X, weak) has countable tightness; that is, if $A \subseteq X$ and $x \in \text{wcl}(A)$, then there is a countable set $D \subseteq A$ with $x \in \text{wcl}(D)$. [56, p. 229]

(4) (B_X, weak) is metrizable if and only if X^* is separable [14, V. 5.2]. Also (B_{X^*}, weak^*) is metrizable if and only if X is separable [14, V. 5.1], so in this case (B_{X^*}, weak^*) is a compact metric space.

Corson [8] initiated the detailed study of (X, weak) . Godefroy [21, 22] used the Čech complete and Polish properties of (B_X, weak) in his study of unique isometric preduals. Some of the Banach space ideas used in this paper are discussed in [15, 16].

2.1. DEFINITIONS. Let X be a Banach space.

(1) The space X is *weakly compactly generated* (WCG) iff there is a weakly compact subset whose linear span is dense in X (see [37]).

(2) The space X is *somewhat reflexive* iff every infinite-dimensional closed subspace contains an infinite-dimensional reflexive subspace. If a space is somewhat reflexive, then it cannot contain c_0 or l^1 . The converse is an important open question.

(3) The space X is an *Asplund space* iff every separable subspace of X has a separable dual space. It is equivalent to require that X^* have the Radon-Nikodým property [50, 51]. Asplund spaces have a topological characterization: the space X is an Asplund space iff every separable subset of (B_X, weak) is metrizable, or iff every Radon measure on (B_X, weak) has metrizable support.

(4) The space X is a *Godefroy space* iff (a) X^* is separable; (b) $X^* \leftrightarrow l^1$; and (c) X^\perp is weak* separable in X^{***} . Note that $X^\perp = (X^{**}/X)^*$, and the weak* topology on X^\perp in this pairing coincides with the relative weak* topology of X^{***} .

(5) The space X is *separably distinguished* iff there is a (norm) separable subspace Y of X^* such that if $x^{**} \in X^{**}$, $x \in X$, $x^{**}|_Y = x|_Y$, then $x^{**} \in X$. It is equivalent to require that $Y^\perp \subseteq X^{**}$ be a subspace of X .

(6) The space X has property (PCA) iff X has property (PC) and X is an Asplund space.

(7) The space X has the *Radon-Nikodým property* (RNP) iff for every non-empty bounded set A , and every $\epsilon > 0$, there exists a linear functional $x^* \in X^*$ and scalar α such that $A \cap \{x: \langle x, x^* \rangle > \alpha\}$ is non-empty and has diameter less than ϵ . (Other equivalent definitions are discussed in [13].)

Some of the relationships among these properties are shown in Figure 1. Several examples and counterexamples are discussed in §6.

Some special properties of the spaces c_0 and l^1 will be useful.

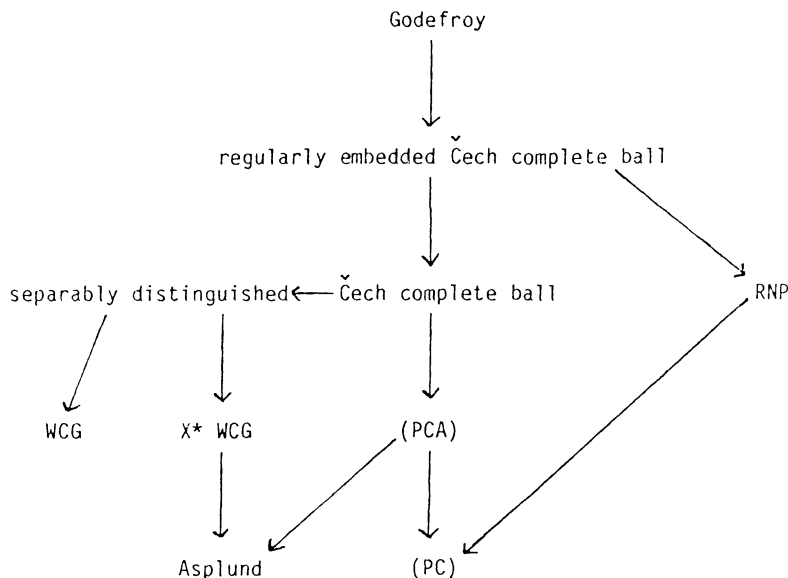


FIGURE 1

2.2. PROPOSITION. (1) If a Banach space X admits a continuous linear map onto $l^1(A)$, then $l^1(A)$ is isomorphic to a complemented subspace of X . (2) If $X^* \leftrightarrow l^1$, then $X \leftrightarrow l^1$. (3) if Y is a closed subspace of X such that $Y \leftrightarrow l^1$ and $X/Y \leftrightarrow l^1$, then $X \leftrightarrow l^1$. (4) If X is separable, then $X^* \leftrightarrow l^1$ if

and only if c_0 is isomorphic to a quotient space of X . (5) $X^* \leftrightarrow c_0$ if and only if l^1 is isomorphic to a complemented subspace of X , and then l^∞ is isomorphic to a complemented subspace of X^* .

For the proof of (1), see [39, p. 107]; (2) follows from (1). For (3), see [12, p. 42]; (4) is [39, 2.e.9]; (5) is [39, 2.e.8].

Čech completeness for a completely regular Hausdorff space T was defined in the Introduction. In the definition we may take each of the open covers \mathcal{U}_n to be a base for the topology of T and to be closed under finite unions [19, 20].

A pseudobase for a topological space T is a collection \mathcal{P} of non-empty open sets such that every non-empty open set contains a member of \mathcal{P} . The space T is *almost Čech complete* iff it admits a complete sequence of pseudobases. A space is almost Čech complete if and only if it has a dense G_δ subset that is Čech complete [1, p. 28]. Every almost Čech complete space is a Baire space [1].

A norm on a Banach space X is called a *Kadec norm* iff the weak and norm topologies coincide on $S_X = \{x: \|x\| = 1\}$. Now S_X is dense in (B_X, weak) if X is infinite-dimensional. If the norm is a Kadec norm, then (S_X, weak) is a Polish space, hence Čech complete, hence a G_δ in (B_X, weak) , so that (B_X, weak) is almost Čech complete. Every WCG space is locally uniformly convexifiable, hence admits a Kadec norm [11].

Kerstan [34] studied spaces that are Čech complete and Lindelöf. Such spaces are characterized among the completely regular spaces T by each of the following: (1) If S is any compactification of T , then $S \setminus T$ is a countable union of compact G_δ subsets of S ; (2) T admits a perfect map onto a Polish space; (3) T is homeomorphic to a closed subset of $\mathbf{R}^N \times K$, where K is compact. From Theorem B, the Čech complete ball of a Banach space satisfies (3) for an Eberlein compact K .

The proof of the following proposition is left to the reader. Note that (2) is clearly preserved by linear homeomorphisms.

2.3. PROPOSITION. *The following are equivalent conditions on a Banach space X : (1) (B_X, weak) is Čech complete; (2) X is a G_δ -set in (X^{**}, weak^*) ; (3) $B_{X^{**}} \setminus B_X$ is weak* σ -compact; (4) $X^{**} \setminus X$ is weak* σ -compact.*

2.4. DEFINITION. A Banach space X has *regularly embedded Čech complete ball* iff $B_{X^{**}} \setminus B_X = \bigcup_{n=1}^\infty K_n$, where each K_n is weak* compact convex and $d(K_n, X) > 0$.

Here $d(K_n, X)$ is the distance from K_n to X , defined by $\inf\{\|y - x\|; y \in K_n, x \in X\}$. This property is preserved by linear homeomorphisms. Indeed, if C_X is the closed unit ball for an equivalent norm on X , with $C_X \subseteq mB_X$, then

$$C_{X^{**}} \setminus C_X = C_{X^{**}} \cap (mB_{X^{**}} \setminus mB_X) = \bigcup_{n=1}^{\infty} (C_{X^{**}} \cap mK_n),$$

where each $C_{X^{**}} \cap mK_n$ is weak* compact convex, and has positive distance from X with respect to the new norm. We do not know if Definition 2.4 is equivalent to the same condition with “ $d(K_n, X) > 0$ ” deleted. Note that if x^{**} is a point in X^{**} whose distance r from X is not attained, then $\{y^{**}: \|y^{**} - x^{**}\| \leq r\}$ is a weak* compact convex set with zero distance from X .

2.5. DEFINITIONS. Let K be a compact Hausdorff space.

(1) K is an *angelic compact* (also called a compact Fréchet space) iff the closure of every subset A is the set of limits of sequences from A .

(2) K is an *Eberlein compact* iff K is homeomorphic to a weakly compact subset of a Banach space (see [37]).

(3) K is a *Rosenthal compact* [23] iff K is homeomorphic to a pointwise compact set of functions of first Baire class on a Polish space.

The Odell-Rosenthal Theorem [43] asserts that if X is a separable Banach space and $X \leftrightarrow l^1$, then $K = (B_{X^{**}}, \text{weak}^*)$ is a Rosenthal compact with respect to the Polish space (B_{X^*}, weak^*) . Every compact metric space is both Eberlein and Rosenthal compact, and each of these is angelic. The product of two Eberlein compacts is an Eberlein compact, and the product of two Rosenthal compacts is a Rosenthal compact. While the product of two angelic compacts need not be angelic, we do have the following result (cf. [24, Th. 1(d)]).

2.6. THEOREM. *If K is an Eberlein compact and L is a Rosenthal compact, then $K \times L$ is angelic.*

Proof. Let $A \subseteq K \times L$ and $p = (k_0, l_0) \in \bar{A}$. Let U be an arbitrary closed neighborhood of k_0 in K . Then

$$p \in \overline{A \cap (U \times L)}, \quad \text{so} \quad l_0 \in \overline{\pi_2(A \cap (U \times L))}.$$

Since L is angelic, there is a sequence (k_i^U, l_i^U) in $A \cap (U \times L)$ with $l_i^U \rightarrow l_0$. Since K is angelic, we may assume that $k_i^U \rightarrow z^U \in U$. As the closed neighborhood U of k_0 decreases, we have $z^U \rightarrow k_0$. Choose a

sequence (U_j) with $z^{U_j} \rightarrow k_0$. Now we have (k_0, l_0) in the closure of the countable set

$$B = \{(k_i^{U_j}, l_i^{U_j}) : i, j = 1, 2, \dots\} \subseteq A.$$

Thus $p \in \overline{B} \subseteq \overline{\pi_1(B)} \times L$. Now $\overline{\pi_1(B)}$ is a separable Eberlein compact, hence compact metrizable [37], hence a Rosenthal compact. Thus $\overline{\pi_1(B)} \times L$ is a Rosenthal compact, so angelic. Thus some sequence in B converges to p . \square

3. Polish ball and property (PC). We begin with a proof of Theorem B. This factorization of spaces with Čech complete ball reduces their study to the separable case, where the special properties of Polish spaces can be used to advantage.

3.1. LEMMA. *If X is a separable Banach space with Čech complete ball, then X^* is separable.*

Proof. By Proposition 2.3, we may write $X = \bigcap_{n=1}^{\infty} U_n$, where each U_n is a weak* open subset of X^{**} . Each U_n contains a basic neighborhood of 0, determined by a finite set F_n in X^* . Since X is separable, (X^*, weak^*) is also separable. Let D be a countable dense subset of (X^*, weak^*) , and let Y be the norm closed linear span of $D \cup \bigcup_{n=1}^{\infty} F_n$ in X^* . The annihilator $Y^\perp = \{x^{**} \in X^{**} : \langle y, x^{**} \rangle = 0 \text{ for all } y \in Y\}$ is a subset of each U_n , and contains no non-zero member of X . Thus $Y^\perp = \{0\}$, so $Y = X^*$, and X^* is separable. \square

Note that the proof actually shows that if X is a Banach space with Čech complete ball then X^* is norm separable if and only if it is weak* separable.

3.2. COROLLARY. *A Banach space has Polish ball if and only if it is separable and has Čech complete ball.*

Proof. The necessity is obvious. Since a separable metrizable Čech complete space is Polish [18, p. 190], the sufficiency follows from 3.1. \square

THEOREM B. *Let X be a Banach space. The following are equivalent: (1) X has Čech complete ball; (2) X is isomorphic to a direct sum $R \oplus S$, where R is reflexive and S has Polish ball.*

Proof. (2) \Rightarrow (1). Since (X, weak) is topologically isomorphic to $(R, \text{weak}) \times (S, \text{weak})$, the unit ball (B_X, weak) is homeomorphic to a closed subset of $(B_R, \text{weak}) \times (B_S, \text{weak})$. The topological property of Čech completeness is preserved by closed subsets and countable products [1, p. 14].

(1) \Rightarrow (2). Choose (U_n) and (F_n) as in 3.1, and let G be the closed linear span of $\bigcup_{n=1}^\infty F_n$ in X^* . Then G^\perp is a weak* closed subspace of X^{**} , and $G^\perp \subseteq \bigcap_{n=1}^\infty U_n = X$. Since the ball of G^\perp is $\sigma(X^{**}, X^*)$ -compact, this makes $R_1 = G^\perp$ a reflexive subspace of X . Since $(X/R_1)^* = G$ is separable, X/R_1 is separable. Let S_1 be a separable subspace of X that maps onto X/R_1 under the quotient map. Thus $R_1 + S_1 = X$. Now X is WCG, since X is the closed span of $B_{R_1} + A$, where A is a sequence converging to 0 in S_1 with span dense in S_1 . By a theorem of Amir and Lindenstrauss [2], there is a separable subspace S of X such that $S_1 \subseteq S$ and S is complemented in X . Hence X is isomorphic to $R \oplus S$, where $R = X/S = (X/S_1)/(S/S_1)$. But X/S_1 is a quotient of R_1 , so R is a quotient of R_1 , and thus R is reflexive. Finally, S is separable and has Čech complete ball, so it has Polish ball by 3.2. \square

Note that Theorem B shows that a space X with Čech complete ball is WCG. Also, $X^* = R^* \oplus S^*$, where R^* is reflexive and S^* is separable, so X^* is WCG.

3.3. EXAMPLE. The space c_0 has separable dual, but its ball is not Polish. Indeed, the ball of c_0 is not a Baire space in the weak topology: it can be expressed as $\bigcup_{n=1}^\infty A_n$, where each

$$A_n = \{x : |x_k| \leq 1/2 \text{ for all } k \geq n\}$$

is closed and nowhere dense. But if c_0 is renormed using Day's norm [11, p. 94], which is locally uniformly convex, then the unit sphere is a dense G_δ Polish subspace of the ball, so the ball is a Baire space.

The following result is due to Godefroy.

3.4. THEOREM. *A Godefroy space has Polish ball.*

Proof. Recall that a Godefroy space X has these three properties: X^* is separable, $X^* \leftarrow l^1$, X^\perp is weak* separable. Let (y_n) be a countable dense subset of (X^\perp, weak^*) . Since X^* is separable and $X^* \leftarrow l_1$, by the Odell-Rosenthal Theorem [43], each y_n is the weak* limit of a sequence

(x_{nk}^*) in X^* . Then

$$\begin{aligned} X &= \{x^{**}: \langle x^{**}, y_n \rangle = 0 \text{ for all } n\} \\ &= \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \left\{x^{**}: |\langle x_{nk}^*, x^{**} \rangle| < \frac{1}{m}\right\} \end{aligned}$$

is a weak* G_δ set in X^{**} . By Corollary 3.2, X has Polish ball. □

In particular, every Banach space with separable second dual has Polish ball [22]. This holds for separable quasi-reflexive spaces and the separable long James spaces $J(\alpha)$, α countable [17].

An examination of this proof may suggest the following generalization.

3.5. DEFINITION. A Banach space has property (A) iff there exist norm one vectors y_{ni}^* in X^* ($1 \leq i \leq m_n, n \in \mathbf{N}$) such that if we define

$$p_n(x^{**}) = \max\{|\langle y_{ni}^*, x^{**} \rangle|: 1 \leq i \leq m_n\},$$

then (1) $\lim p_n(x) = 0$ for all $x \in X$, but (2) $\liminf p_n(x^{**}) > 0$ for all $x^{**} \in X^{**} \setminus X$.

Any Banach space with property (A) has Čech complete ball, since

$$X = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\{x^{**}: p_n(x^{**}) < \frac{1}{k}\right\}.$$

We do not know if every space with Čech complete ball has property (A). Theorem 4.14 is a step in this direction.

3.6. EXAMPLE. The predual B of the James Tree space JT satisfies property (A), hence has Polish ball. (Basic material on the space JT is in [30], [38].) Note that B is not a Godefroy space, since B^\perp is a nonseparable Hilbert space [38, Theorem 1], and not weak* separable. The y_{ni}^* can be taken to be the usual boundedly complete basis in JT, called $\{e_{ni}: 1 \leq i \leq 2^n, n \in \mathbf{N}\}$ in [38]. The bi-orthogonal functionals $\{f_{ni}\}$ form a shrinking basis for B , and a weak* basis for $\text{JT}^* = B^{**}$. If $x \in B$, then

$$\lim_n \max_{1 \leq i \leq 2^n} |\langle x, e_{ni} \rangle| = 0$$

(see Lemma 1 of [38]). If $x^{**} \in \text{JT}^* \setminus B$, then $\lim_n |\langle e_{ni}, x^{**} \rangle| \geq \delta > 0$ for some branch $\{(n, i_n): n \in \mathbf{N}\}$ of the binary tree, so

$$\liminf_n \max_{1 \leq i \leq 2^n} |\langle e_{ni}, x^{**} \rangle| \geq \delta > 0.$$

The space B has Polish ball, but fails the Radon-Nikodým property [38, Corollary 4], and is therefore not isomorphic to a dual space. The ball

of JT is not Polish, since JT has non-separable dual. The space JT^* does not have Čech complete ball, since it is not WCG. Since $B \subseteq JT^*$ and JT^*/B both have Čech complete ball, this shows that there is no three space property for Čech completeness of the ball. This proof that B has Polish ball is formally a strengthening of the fact that B has property (PC), which was asserted in [6].

Čech complete ball implies property (PC); the converse holds for spaces with separable dual. The proof centers on the notion of a *huskable set* (in French, *épluchable* [21], [22]). It is convenient to state the definition in some generality.

3.7. DEFINITION. Let X be a Banach space, and let τ be a locally convex topology on X that is coarser than the norm topology. A subset A of X is τ -*huskable* iff for every τ -open set U with $U \cap A \neq \emptyset$ and every $\varepsilon > 0$, there is a τ -open set V with $\emptyset \neq V \cap A \subseteq U \cap A$ and $\text{diam}(V \cap A) = \sup\{\|x - y\|: x, y \in V \cap A\} < \varepsilon$.

Another way to say this is: for each $\varepsilon > 0$, the collection of non-empty, relatively τ -open subsets of A with diameter less than ε is a pseudobase for (A, τ) . In practice, τ will be either the weak topology on X or the weak* topology on a dual space. Kenderov [33] showed that a dual space has the RNP if and only if each weak* closed bounded set is weak* huskable. The term “huskable”, unmodified, shall mean huskable in the weak topology.

3.8. PROPOSITION. *Let A be a subset of a Banach space. Then the following are equivalent. (1) A is huskable; (2) $wcl(A)$ is huskable; (3) $w^*cl(A)$ is weak* huskable in X^{**} .*

Proof. (1) \Rightarrow (3). Let $U \subseteq w^*cl(A)$ be a non-empty relatively weak* open set, and let $\varepsilon > 0$. Then A contains a non-empty, relatively weakly open set V with diameter less than $\varepsilon/2$ such that $V \subseteq U \cap A$. Choose a relatively weak* open set W in $w^*cl(A)$ with $W \subseteq U$ and $W \cap A = V$. Since V is weak* dense in W , $\text{diam}(W) \leq \varepsilon/2 < \varepsilon$. □

3.9. PROPOSITION. *Let A be a norm-closed subset of X . Then A is τ -huskable if and only if the identity map $(A, \tau) \rightarrow (A, \text{norm})$ is continuous at each point of a τ -dense τ - G_δ subset of A .*

Proof. Assume A is τ -huskable, and let $U_n = \bigcup\{U \subseteq A: U \text{ is relatively } \tau\text{-open in } A, \text{ and } \text{diam}(U) < 1/n\}$. Then $\bigcap_{n=1}^\infty U_n$ is exactly the set of

points of continuity of the identity. We need to show that this set is τ -dense in A . Let V be a non-empty, relatively τ -open subset of A . Choose recursively a sequence (W_n) of relatively τ -open subsets of V with $\text{diam}(W_n) < 1/n$ and the τ -closure of W_n in A contained in W_{n-1} for each n . Since A is norm closed, the unique point in $\bigcap W_n$ is a member of $V \cap (\bigcap U_n)$. This shows $\bigcap U_n$ is dense in A .

The converse is left to the reader. \square

3.10. PROPOSITION. *Let A be a weakly closed and bounded subset of X . Then (1) \Rightarrow (2a) \Leftrightarrow (2b) \Leftrightarrow (2c) \Rightarrow (3). If A is separable, then all five conditions are equivalent.*

- (1) A is huskable;
- (2a) (A, weak) is almost Čech complete;
- (2b) A contains a dense G_δ subset of $w^*\text{cl}(A)$;
- (2c) A is residual in $w^*\text{cl}(A)$;
- (3) A is a Baire space.

Proof. (1) \Rightarrow (2a): $\mathcal{U}_n = \{U \subseteq A : U \text{ is relatively open and } \text{diam}(U) < 1/n\}$ is a complete sequence of pseudobases of A . (2a) \Rightarrow (2b): A has a dense Čech complete subspace [20], which is necessarily a G_δ in its compactification $w^*\text{cl}(A)$. (2b) \Rightarrow (2c): Obvious. (Residual = comeager means that $w^*\text{cl}(A) \setminus A$ is first category in $w^*\text{cl}(A)$.) (2c) \Rightarrow (2a): A contains $B = \bigcap U_n$, where each U_n is a dense open subset of $w^*\text{cl}(A)$. Then B is a dense G_δ in $w^*\text{cl}(A)$, hence a dense G_δ Čech complete subset of A . (2a) \Rightarrow (3): This holds in general topology [1, p. 28].

Now assume A is separable in the weak topology. Then the norm-closed linear span Y of A is norm-separable, so every subset of (A, norm) has the Lindelöf property. Let U be a non-empty, relatively open subset of (A, weak) and let $\varepsilon > 0$. Then U is a countable union of norm-closed balls of radius less than ε , intersected with U . Since (U, weak) is a Baire space, one of these weakly closed subsets of U must have a weak interior point (for the relative topology of U of A). This proves that A is huskable. \square

The following is motivated by a similar result concerning dentability [40].

3.11. PROPOSITION. *A subset A of X is huskable if each countable subset is huskable.*

Proof. If A is not huskable, then there is a non-empty relatively weakly open subset U of A and $\varepsilon > 0$ such that each non-empty relatively

weakly open subset of U has diameter $\geq 2\varepsilon$. Choose any $x_0 \in U$. Then x_0 is in the weak closure of $\{x \in U: \|x - x_0\| \geq \varepsilon\}$. By the countable tightness of the weak topology [56, p. 229], there is a countable subset C_1 of $\{x \in U: \|x - x_0\| \geq \varepsilon\}$ with $x_0 \in \text{wcl}(C_1)$. Next, for each point $x_{1n} \in C_1$ there is a countable subset C_{2n} of $\{x \in U: \|x - x_{1n}\| \geq \varepsilon\}$ with $x_{1n} \in \text{wcl } C_{2n}$; let $C_2 = \cup C_{2n}$. Continue in this way, and let $C = \{x_0\} \cup \cup C_n$. Then C is a countable subset of A that is not huskable. \square

3.12. COROLLARY. *Let A be a weakly closed and bounded subset of X . Then A is huskable if every weakly closed separable subset is a Baire space.*

Proof. This follows from 3.8, 3.10, and 3.11. \square

The converses to 3.11 and 3.12 are not true. If X is c_0 with Day's norm (Example 3.3), then $A = B_X$ is huskable by 3.10, but no multiple of the usual ball U of c_0 is Baire, so some countable subset of U is not huskable.

3.13. THEOREM. *Let X be a Banach space. Then the following are equivalent:*

- (1) X has property (PC);
- (2) Every weakly closed bounded subset of X is huskable;
- (3) Every weakly closed bounded subset of X is almost Čech complete;
- (4) Every weakly closed bounded subset of X is a Baire space.

Moreover, it is sufficient that every separable subspace have (PC), or that every weakly closed bounded separable subset have property (2), (3), or (4).

Proof. The equivalence of (2), (3), (4) and their separable variants follows from 3.10 and 3.12. Proposition 3.9 shows that (2) \Rightarrow (1). Finally, the (separable) version of (1) implies the (separable) version of (2), since if $A \subseteq X$ has no relatively weakly open subsets with diameter $< \varepsilon$, then the identity $(A, \text{weak}) \rightarrow (A, \text{norm})$ has no points of continuity. \square

3.14. COROLLARY. *Banach spaces with Čech complete ball and Banach spaces with the RNP have property (PC).*

Proof. If X has the RNP, then every bounded subset is dentable [13, p. 136], and therefore every bounded subset is huskable. If X has Čech complete ball, then every weakly closed bounded subset is Čech complete, hence a Baire space. \square

The properties (PC) and RNP are “separably determined”. However, $J(\omega_1)$ and JT^* (see §6) do not have Čech complete balls but every separable subspace has Čech complete ball. The predual B of JT has Polish ball, but fails the RNP. The space l^1 has the RNP, so that the unit ball is huskable for every renorming, but the ball is not Čech complete (since the dual of l^1 is not separable, see Lemma 3.1). The space JT^* has (PC), but does not have Čech complete ball, and fails the RNP.

The next few results establish the equivalence of (PC) and Polish ball for spaces with separable dual. This leads directly to Theorem A.

3.15. DEFINITION. A Banach space X is *quasi-Čech complete* iff $X = \bigcap_{n=1}^{\infty} (F_n \cup U_n)$, where F_n is closed and U_n is open in (X^{**}, weak^*) .

Of course, if X^* is separable, then $(B_{X^{**}}, \text{weak}^*)$ is metrizable, so a set of the form $\bigcap_{n=1}^{\infty} (F_n \cup U_n)$ is a G_δ -set. Assertion (3) of the following shows that quasi-Čech completeness is preserved by linear homeomorphisms.

3.16. PROPOSITION. *The following conditions on a Banach space X are equivalent: (1) X is quasi-Čech complete; (2) $(X^{**} \setminus X, \text{weak}^*)$ is σ -locally compact; (3) If A is any weakly closed bounded subset of X , then $A = \bigcap_{n=1}^{\infty} (F_n \cup U_n)$, where F_n is compact and U_n is open in $w^*\text{cl}(A)$; (4) If A is any weakly closed bounded subset of X , then $w^*\text{cl}(A) \setminus A$ is weak* σ -locally compact.*

Proof. To see that (1) \Leftrightarrow (2), note that locally compact spaces are precisely the open subsets of compact spaces. Thus if X is quasi-Čech complete, then $X = \bigcap_{n=1}^{\infty} (F_n \cup U_n)$, so that

$$X^{**} \setminus X = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} ((X^{**} \setminus F_n) \cap (kB_{X^{**}} \setminus U_n))$$

is σ -locally compact. The argument is reversible. The proof that (3) \Leftrightarrow (4) is similar. For (2) \Rightarrow (4), observe that $w^*\text{cl}(A) \setminus A$ is a closed subset of $X^{**} \setminus X$. For (4) \Rightarrow (2), consider $A = B_X$. \square

3.17. PROPOSITION. *For Banach spaces, Čech complete ball \Rightarrow quasi-Čech complete \Rightarrow (PC).*

Proof. The first implication is clear. If X is quasi-Čech complete, and A is a weakly closed bounded subset of X , then (by 3.16 (3))

$$A = \bigcap_{n=1}^{\infty} (F_n \cup U_n).$$

By adjoining the interior of F_n to U_n , we may assume that each U_n is weak* dense in $w^*cl(A)$. Thus A is residual in $w^*cl(A)$. Now 3.10 and 3.13 show that X has property (PC). \square

The next result is the main step in the proof of Theorem A. Part (2) can be deduced from a theorem of Hurewicz [28] on analytic sets. We include here a self-contained proof.

3.18. THEOREM (1) *If X is separable and has (PC), then X is quasi-Čech complete;* (2) *If X^* is separable and X has (PC), then B_X is Čech complete.*

Proof. (1) Let $A \subseteq X$ be a non-empty, weakly closed, bounded set (such as B_X). Fix $\epsilon > 0$. We define recursively a transfinite sequence A_α of closed subsets of A , and a sequence U_α of open sets in $(X^{**}, weak^*)$. Let $A_0 = A$. Suppose A_α has been defined, and is non-empty. Let $\mathcal{U}_\alpha = \{V \subseteq X^{**}: V \text{ is weak}^* \text{ open, } V \cap A_\alpha \neq \emptyset, \text{ and } diam(V \cap A_\alpha) < \epsilon\}$. Since X has (PC), \mathcal{U}_α is a non-empty collection. Let $U_\alpha = \bigcup \mathcal{U}_\alpha$. Then $A_{\alpha+1} = A_\alpha \setminus U_\alpha$ is a closed proper subset of A_α . If λ is a limit ordinal, let $A_\lambda = \bigcap_{\alpha < \lambda} A_\alpha$.

We claim that $A_\gamma = \emptyset$ for some countable ordinal γ . If not, $(A_\alpha)_{\alpha < \omega_1}$ is a strictly decreasing sequence of weakly closed sets. Since X is separable, every subset of $(X, weak)$ is Lindelöf, so

$$A \setminus \bigcap_{\alpha < \omega_1} A_\alpha = \bigcup_{\alpha < \omega_1} (A \setminus A_\alpha) = \bigcup_{\alpha < \gamma} (A \setminus A_\gamma)$$

for some $\gamma < \omega_1$, which contradicts the strictly decreasing property of the A_α . Hence there is a least γ such that $A_\gamma = \emptyset$.

Let $B_\alpha = w^*cl(A_\alpha)$, $C_\alpha = B_\alpha \cup (\bigcup_{\beta < \alpha} U_\beta)$ for each $\alpha \leq \gamma$. Then

$$A \subseteq \bigcap_{\alpha \leq \gamma} C_\alpha \subseteq C_0 = w^*cl(A).$$

We claim that if $x^{**} \in \bigcap_{\alpha \leq \gamma} C_\alpha$, then $d(x^{**}, A) \leq \epsilon$. If $x^{**} \in B_\alpha \cap U_\alpha$ for some $\alpha \leq \gamma$, then certainly $d(x^{**}, A) \leq \epsilon$, since there is $V \in \mathcal{U}_\alpha$ with $x^{**} \in B_\alpha \cap V \subseteq w^*cl(A_\alpha \cap V)$, and $diam(A_\alpha \cap V) < \epsilon$. But if $x^{**} \in \bigcap_{\alpha \leq \gamma} C_\alpha$, then $x^{**} \in B_0$. But $x^{**} \notin B_\gamma$, so there is a least α_0 such that $x^{**} \notin B_{\alpha_0}$. Since $x^{**} \in C_{\alpha_0}$, we have $x^{**} \in U_\beta$ for some $\beta < \alpha_0$. Thus $x^{**} \in B_\beta \cap U_\beta$. So $d(x^{**}, A) \leq \epsilon$, as claimed.

Since A is weakly closed in X , it is norm closed in X^{**} . If we take $\epsilon = 1/n$ in the preceding discussion, we obtain $(C_{\alpha,n})_{\alpha < \gamma_n}$, and so

$$A = \bigcap_{n=1}^{\infty} \bigcap_{\alpha \leq \gamma_n} C_{\alpha,n}.$$

Each $C_{\alpha,n}$ is the union of a closed set and an open set in (X^{**}, weak^*) . So X is quasi-Čech complete.

(2) If X^* is separable, then X is separable, so the preceding construction can be carried out. But $(B_{X^{**}}, \text{weak}^*)$ is metrizable, so the weak* compact sets B_α are G_δ sets in $w^*\text{cl}(A)$. Thus each C_α is a G_δ , so A is a G_δ , and thus A is Čech complete. \square

The significance of quasi-Čech completeness lies in the fact that (for separable X) it gives a one-set criterion for property (PC), as opposed to conditions on all closed bounded sets.

The proof of 3.18 yields an open cover characterization of property (PC) that is valid even for nonseparable spaces.

3.19. COROLLARY. *A Banach space X has property (PC) if and only if there is a sequence $\{V_{\alpha,n}: \alpha \leq \gamma_n\}_{n \in \mathbb{N}}$ of transfinite open covers of (B_X, weak) such that*

$$\text{daim} \left(B_X \cap \left(V_{\alpha,n} \setminus \bigcup_{\beta < \alpha} V_{\beta,n} \right) \right) < \frac{1}{n}$$

for all $\alpha \leq \gamma_n$.

Proof. If X has (PC) and $\{V_{\beta,n}: \beta < \alpha\}$ fails to cover B_X , then there is a weakly open set $V_{\alpha,n}$ such that $\text{diam}(B_X \cap (V_{\alpha,n} \setminus \bigcup_{\beta < \alpha} V_{\beta,n})) < 1/n$. Conversely, if the covers exist, and A is a non-empty weakly closed subset of B_X , then for each n there is a least α such that $V_{\alpha,n} \cap A \neq \emptyset$. Then $\text{diam}(V_{\alpha,n} \cap A) < 1/n$. \square

THEOREM A. *Let X be a separable Banach space. The following are equivalent.*

- (1) (B_X, weak) is completely metrizable;
- (2) (B_X, weak) is a Polish space;
- (3) X has property (PC) and is an Asplund space;
- (4) (B_X, weak) is metrizable, and every closed subset of it is a Baire space;

Proof. For separable spaces, (1) and (2) are ways of saying that X has Čech complete ball, using 3.2. Also, (3) and (4) are ways of saying that X has (PC) and X^* is separable, using 3.13. Thus the equivalence of (1)–(4) follows from 3.18. \square

Property (3) is intersecting even in nonseparable Banach spaces. If X has property (PC) and X is an Asplund space, we say that X has property (PCA). Equivalently, X has (PCA) if and only if every separable subspace has Polish ball, or every separable subspace has Čech complete ball. Examples of spaces with (PCA) are $J(\omega_1)$ and JT^* (see §6).

Another consequence of Theorem A is that if X is separable, and both X and X^* have RNP, then X has Polish ball. The converse is false: the space B of Example 3.6 has Polish ball, but fails the RNP.

4. Čech complete ball. The results of the previous section produce structural information about spaces with Čech complete ball, their duals and their quotients. Here is a refinement of Theorem B.

4.1. THEOREM. *If X has property (PCA), then X is somewhat reflexive.*

Proof. Since every separable subspace of X has Čech complete ball, it suffices to show that every infinite-dimensional space X with Čech complete ball contains an infinite-dimensional reflexive subspace. First, $X^{**} \setminus X = \bigcup_{n=1}^\infty K_n$, where each K_n is weak* compact. For each n , let $\mathcal{U}_n = \{U \subseteq X: U \text{ is a weakly open set with } w^*\text{cl}(U) \cap K_n = \emptyset\}$. Each \mathcal{U}_n is an open cover of (X, weak) . Then $(\mathcal{U}_n)_{n=1}^\infty$ is a complete sequence of open covers for B_X . Indeed, if \mathcal{F} is a family of closed subsets of B_X with the finite intersection property that is \mathcal{U}_n small for each n , then $\bigcap_{F \in \mathcal{F}} w^*\text{cl}(F)$ is a non-empty subset of X^{**} disjoint from all K_n , so $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

We proceed with a recursive construction. Choose $x_1 \in X$, $x_1 \neq 0$, and let $P_1: X \rightarrow [x_1]$ be a projection. (We use brackets $[]$ for the span of a finite set.) Then $P_1(B_X)$ is compact, and covered by basic open sets from \mathcal{U}_1 , so there is a finite subcover \mathcal{U}'_1 . Choose a finite set A_1 of X^* that determines P_1 and the members of \mathcal{U}'_1 . Let $F_1 = [x_1] + [A_1]_\perp$; this is a closed subspace of X with finite codimension. Also, B_{F_1} is covered by \mathcal{U}'_1 , since $\langle x - P_1x, y \rangle = 0$ for each $x \in F_1$ and $y \in A_1$.

Now choose $x_2 \in [A_1]_\perp \setminus [x_1]$; this is possible since X is infinite-dimensional. Let $P_2: X \rightarrow [x_1, x_2]$ be a projection. Then $P_2(B_X)$ is covered by a finite subfamily \mathcal{U}'_2 of the basic open sets in \mathcal{U}_2 . Choose a finite subset A_2 of X^* that determines P_2 and the members of \mathcal{U}'_2 . Let $F_2 = [x_1, x_2] + [A_1 \cup A_2]_\perp$. Then F_2 has finite codimension, and B_{F_2} is covered by \mathcal{U}'_2 .

Continuing in this way, we obtain a linearly independent sequence x_1, x_2, \dots in X and a decreasing sequence $F_1 \supseteq F_2 \supseteq \dots$ of subspaces of X such that $x_j \in F_k$ for $j \leq k$ and B_{F_k} is covered by finitely many basic

open sets from \mathcal{U}_k . Let $F = \bigcap_{k=1}^{\infty} F_k$. Then $x_j \in F$ for all j , so F is infinite-dimensional. Also, B_F is covered by finitely many sets from \mathcal{U}_k for all k . It follows that B_F is weakly compact: If \mathcal{F} is a filter of closed subsets of B_F , then \mathcal{F} is \mathcal{U}_n -small for each n , since \mathcal{U}_n is closed under finite unions. Hence $\bigcap \mathcal{F} \neq \emptyset$. Thus F is reflexive. \square

This theorem generalizes a result of Johnson and Rosenthal [31] stating that if Y embeds in a separable dual and Y^* is separable, then Y is somewhat reflexive.

Here is another result related to Theorem B.

4.2. PROPOSITION. *Let X be a Banach space and Y a reflexive subspace. Then X has Čech complete ball if and only if X/Y has Čech complete ball.*

Proof. We have $(X/Y)^{**} = X^{**}/Y^{\perp\perp} = X^{**}/Y$. If $\pi: X^{**} \rightarrow X^{**}/Y$ is the quotient map, then $(X/Y)^{**} \setminus (X/Y) = \pi(X^{**} \setminus X)$. The reflexivity of Y shows that π maps $B_{X^{**}}$ onto $B_{X^{**}/Y}$, so it maps $B_{X^{**}} \setminus B_X$ onto $B_{X^{**}/Y} \setminus B_{X/Y}$. The map π is $\sigma(X^{**}, X^*) - \sigma(X^{**}/Y, Y^{\perp})$ continuous. It follows that $B_{X^{**}} \setminus B_X$ is weak* σ -compact if and only if $B_{X^{**}/Y} \setminus B_{X/Y}$ is weak* σ -compact. By Theorem 2.3, B_X is Čech complete if and only if $B_{X/Y}$ is Čech complete. \square

This proposition and Corollary 4.7 are positive results about quotients of spaces with Čech complete ball. However, not all quotients preserve the Čech complete or Polish properties of the ball (see Proposition 4.8 and subsequent remarks). The space JT^* admits a separable subspace B with Polish ball such that JT^*/B is reflexive, but JT^* does not have Čech complete ball (since it is not WCG).

Spaces with Čech complete ball are separately distinguished, according to the next result.

4.3. THEOREM. *Let X be a Banach space. The following are equivalent: (1) X is separably distinguished; (2) X is isomorphic to $R \oplus S$, where R is reflexive and S^* is separable; (3) There is a non-empty G_{δ} set in (X^{**}, weak) that is contained in X .*

Proof. (1) \Rightarrow (2). Choose a separable subspace Y of X^* such that $Y^{\perp} = \{x^{**}: \langle y, x^{**} \rangle = 0 \text{ for all } y \in Y\}$ is a subspace of X . As in the proof of Theorem B, $R_1 = Y^{\perp}$ is reflexive, X/R_1 has separable dual, and X is WCG. Moreover, $Y = (X/R_1)^*$ is a separable dual space, so it has

RNP [13, p. 79]. Also, $X^*/Y = R_1^*$ has RNP, since it is reflexive [13, p. 76]. Thus X^* has RNP [15, Theorem 6.3], so X is an Asplund space [51]. The remainder of the argument in Theorem B shows that X is isomorphic to $R \oplus S$, where R is reflexive and S is separable. Since X is an Asplund space, S^* is separable.

(2) \Rightarrow (3). Let (f_n) be norm dense in S^* . Then $A = \bigcap_{n=1}^\infty f_n^{-1}(0)$ is a weak* G_δ in X^{**} . If $x^{**} \in A$, then $x^{**} \in R \oplus \{0\} \subseteq X$.

(3) \Rightarrow (1). Let U be a G_δ -set in (X^{**}, weak^*) with $x_0 \in U \subseteq X$. Then $V = U - x_0$ is a G_δ set and $0 \in V \subseteq X$. So there is a countable subset C of X^* such that $C^\perp \subseteq V \subseteq X$ (as in the proof of Theorem B). If Y is the closed span of C , then $Y^\perp \subseteq X$, so X is separably distinguished. \square

Note that (3) cannot be replaced by (3'): There is a non-empty G_δ set in $(B_{X^{**}}, \text{weak}^*)$ that is contained in B_X . Indeed, the usual norm on l^1 is a Kadec norm, so the unit sphere of l^1 is such a G_δ , but l^1 fails (2).

4.4. COROLLARY. *Suppose X is separably distinguished. Then X is WCG and an Asplund space, and X^* is WCG. The spaces (B_{X^*}, weak^*) and $(B_{X^{**}}, \text{weak}^*)$ are Eberlein compacts.*

4.5. THEOREM. *Let X be a Banach space. Then the following are equivalent: (1) X^* has Čech complete ball; (2) X^{**}/X is separable; (3) X is isomorphic to $R \oplus S$, where R is reflexive and S^{**} is separable.*

Proof. (1) \Rightarrow (2). Since X^* has Čech complete ball, the space

$$X^{***} \setminus X^* = \{x^* + x^\perp : x^* \in X^*, x^\perp \in X^\perp, x^\perp \neq 0\}$$

is weak* σ -compact. Now X^\perp is weak* closed, so $X^\perp \cap (X^{***} \setminus X^*) = X^\perp \setminus \{0\}$ is weak* σ -compact. Thus $\{0\}$ is a weak* G_δ in $X^\perp = (X^{**}/X)^*$, so there is a countable subset C of X^{**}/X with $C^\perp = \{0\}$. Thus the norm closed span of C is X^{**}/X , so X^{**}/X is separable.

(2) \Rightarrow (3) is a result of Valdivia [54]. Note that X^{**}/X and S^{**}/S can be identified.

(3) \Rightarrow (1). Since S^* is a separable dual, it has RNP [13, p. 79] so it has (PC). By 3.18, S^* has Čech complete ball, so $R^* \oplus S^*$ has Čech complete ball. \square

Note the relative strengths of the factorization properties in Theorems B, 4.3, and 4.5.

4.6. COROLLARY. *Suppose X is separable. Then the following are equivalent: (1) B_{X^*} is Čech complete; (2) B_{X^*} is Polish; (3) X^{**} is separable.*

4.7. COROLLARY. *If X^* has Čech complete ball, then X (and every quotient space of X) has Čech complete ball.*

Proof. Let $Z = X/Y$ be a quotient of X . Then Z^{**}/Z is separable since X^{**}/X is separable [41]. So by 4.5, $Z = R \oplus S$, where S^{**} is separable, so Z has Čech complete ball, by 3.4. \square

Thus a dual space with Čech complete ball must have the RNP, since it is the dual of an Asplund space (or, alternatively, since it is WCG). The predual B of James Tree has Polish ball, but fails the RNP [38], and its dual JT does not have Čech complete ball (since JT^* is not separable).

Clark [7] proved that if X^{**}/X is separable, then X and X^* are somewhat reflexive (see also [10], [31]). Here 4.5 and 4.7 show that the conclusion can be strengthened to: X and X^* have Čech complete ball. Kuo [35] proved that if X^{**}/X is separable, then X^* and X^{**} have the RNP. This follows from 4.5, which shows that such an X^* or X^{**} is a sum of a reflexive space and a separable dual space.

4.8. EXAMPLE. If S is a separable Banach space, let $X = \text{JL}(S)$ denote the associated James-Lindenstrauss space [29], [36]. In more detail, X^* is separable, and admits a quotient map π onto S . If $j: X \rightarrow X^{**}$ is the canonical embedding, then $X^{**} = j(X) \oplus \pi^*(S^*)$. The space $Y = \ker \pi$ is a predual of X , since $Y^* = X^{**}/(\ker \pi)^\perp$. Thus X always has Polish ball (whether or not S does). By 4.5, the space X^* has Polish ball if and only if S^* is separable.

4.9. PROPOSITION. *The quotients of spaces with Polish ball are precisely the spaces with separable dual.*

Proof. If X has Polish ball and π maps X onto Y , then $Y^* \subseteq X^*$ is separable. Conversely, if S^* is separable, then $\text{JL}(S)^*$ has Polish ball and has S as a quotient. \square

Since c_0 has separable dual, it can be the quotient of a space with Polish ball, so l^1 can be a subspace of a dual of a space with Polish ball. In fact, Example (2) in §6 has Polish ball and dual isomorphic to l^1 .

If X has Čech complete ball (or is even separably distinguished), then (X^{**}, weak) must be realcompact. Indeed, $X = R \oplus S$, where S^* is separable. Thus (S^{***}, weak^*) is separable, so (S^{**}, weak) is realcompact, using Corson’s criterion [8, Lemma 9]. Not much more than realcompactness can be expected for X^{**} where X has Čech complete ball. If $X = \text{JL}(l^1)$ then X has Polish ball, but l^∞ is a direct summand of X^{**} , so X^{**} fails the Pettis integral property. It may be helpful to consult Figure 1 of [16]; M. Talagrand has recently shown that property (C) implies the Pettis integral property.

We have no characterization of the remainders X^{**}/X , for X with Čech complete ball. Any such remainder is a quotient of the dual of a separable space, hence has at most the cardinal of the continuum. If $X^{**} = X \oplus Y$, where Y has Čech complete ball, then X has Čech complete ball if and only if Y is separable. Indeed, if Y is separable, then X^{**}/X is separable, so X^* has Čech complete ball (4.5), and hence X has Čech complete ball (4.7). Conversely, if X has Čech complete ball, then $X^{**} = X \oplus Y$ has Čech complete ball, so X^* has Čech complete ball (4.7), and hence $Y = X^{**}/X$ is separable (4.5).

As an example, let J be James’ original quasi-reflexive space of order 1. Then an uncountable l^2 sum of copies of J does not have Čech complete ball. If $X^* \not\leftarrow l^1$, then more can be said about X^{**} and X^{***} . A Banach space X has *property (C)* if any collection of closed, bounded, convex sets with empty intersection has a countable subcollection with empty intersection. See [45], [46].

4.10. THEOREM. *Let X be a separably distinguished Banach space. Then the following are equivalent:*

- (1) $(B_{X^{***}}, \text{weak}^*)$ is angelic;
- (2) $(B_{X^\perp}, \text{weak}^*)$ is angelic;
- (3) $(B_{X^{***}}, \text{weak}^*)$ is sequentially compact;
- (4) $(B_{X^\perp}, \text{weak}^*)$ is sequentially compact;
- (5) X^{**} has *property (C)*;
- (6) X^{**}/X has *property (C)*;
- (7) $X^{**} \not\leftarrow l^1(\mathfrak{N}_1)$;
- (8) $X^* \not\leftarrow l^1$.

Proof. First, let X be any Banach space. Then certainly $(1) \Rightarrow (2) \Rightarrow (4)$ and $(1) \Rightarrow (3) \Rightarrow (4)$. The results of Pol ([45] or [46]) show that $(1) \Rightarrow (5)$ and $(2) \Rightarrow (6)$. Since *property (C)* is preserved by continuous images, $(5) \Rightarrow (6)$. Since $l^1(\mathfrak{N}_1)$ fails *property (C)*, we have $(5) \Rightarrow (7)$. Also $(7) \Rightarrow (8)$,

for if $X^* \leftarrow l^1$, then l^∞ is a quotient of X^{**} . But $l^\infty \leftarrow l^1(\mathfrak{N}_1)$, so $l^1(\mathfrak{N}_1)$ is a quotient of some subspace of X^{**} , and thus (by 2.2(1)) we have $X^{**} \leftarrow l^1(\mathfrak{N}_1)$.

Thus it is enough to show (6) \Rightarrow (5), (4) \Rightarrow (8), (8) \Rightarrow (1) for separably distinguished spaces.

(6) \Rightarrow (5). Suppose X^{**}/X has property (C). Since X is separably distinguished, it is WCG (by 4.4), hence weakly Lindelöf, and thus X has property (C). Then X^{**} has (C) by the three-space property [45].

(4) \Rightarrow (8). Since X is separably distinguished, $X = R \oplus S$, where R is reflexive and S^* is separable (by 4.3). If $X^* \leftarrow l^1$, then $S^* \leftarrow l^1$ by 2.2 (3). Then by 2.2 (4), there is a continuous linear map π from X onto c_0 . Thus π^{**} maps X^{**} onto l^∞ and induces a continuous linear map of X^{**}/X onto l^∞/c_0 . The adjoint is a weak* continuous injection of $(l^\infty/c_0)^*$ into X^\perp . But $((l^\infty/c_0)^*, \text{weak}^*)$ contains a homeomorph of $\beta\mathbb{N} \setminus \mathbb{N}$, which is not sequentially compact.

(8) \Rightarrow (1). Suppose $X^* \not\leftarrow l^1$. As before, $X = R \oplus S$, where R is reflexive and S^* separable. So S^* is separable and does not contain l^1 . By the Odell-Rosenthal theorem [43], $(B_{S^{***}}, \text{weak}^*)$ is a Rosenthal compact. But (B_{R^*}, weak^*) is an Eberlein compact, so by Theorem 2.6, $(B_{X^{***}}, \text{weak}^*)$ is angelic. \square

We close this section by considering properties formally stronger than Čech complete ball. If X has Čech complete ball, then $X^{**} \setminus X = \bigcup_{n=1}^\infty K_n$, where each K_n is weak* compact. Additional requirements might be that each K_n be convex, or have positive distance from X (see Definition 2.4).

4.11. LEMMA. *Let X be a Banach space. Suppose X^* is separable and $X^* \not\leftarrow l^1$. Then X has regularly embedded Čech complete ball if and only if X^\perp is weak* separable.*

Proof. Assume X has regularly embedded Čech complete ball. Then $X^{**} \setminus X = \bigcup_{n=1}^\infty K_n$, where each K_n is weak* compact, convex, and $d(K_n, X) > \varepsilon_n > 0$. If π is the quotient map from X^{**} onto X^{**}/X , then $\pi(K_n)$ is convex, and does not contain 0 in its norm closure. Therefore there exists $f_n \in B_{X^\perp}$ such that $f_n(\pi(x^{**})) > \varepsilon_n$ for all $x^{**} \in K_n$. So $X = \{x^{**} \in X^{**}: f_n(\pi(x^{**})) = 0 \text{ for all } n\} = \{f_n: n \in \mathbb{N}\}^\perp$, so $X^\perp = \{f_n: n \in \mathbb{N}\}^{\perp\perp}$ is weak* separable.

Conversely, let (y_n) be a countable dense subset of (X^\perp, weak^*) . By the Odell-Rosenthal Theorem [43], for each n there is a sequence (x_{nk}^*) in X^* that converges to y_n in the weak* topology. Then $X^{**} \setminus X$ is a

countable union of sets of the form

$$\bigcap_{k=m}^{\infty} \left\{ x^{**} : |\langle x_{nk}^*, x^{**} \rangle| \geq \frac{1}{m} \right\},$$

$$\bigcap_{k=m}^{\infty} \left\{ x^{**} : |\langle x_{nk}^*, x^{**} \rangle| \leq -\frac{1}{m} \right\}$$

(see the proof of 3.4). These sets are weak* closed, convex, and have distance at least $1/(m \|y_n\|)$ from X . Indeed, if x^{**} belongs to either of these sets, and $x \in X$, then

$$\begin{aligned} \|x^{**} - x\| \|y_n\| &\geq |\langle y_n, x^{**} \rangle - \langle y_n, x \rangle| = |\langle y_n, x^{**} \rangle| \\ &= \lim_k |\langle x_{nk}^*, x^{**} \rangle| \geq 1/m. \end{aligned}$$

Intersecting these sets with balls $nB_{X^{**}}$ yields compact sets. □

4.12. THEOREM. *If either (1) X is a Godefroy space or (2) X^* has Čech complete ball, then X has regularly embedded Čech complete ball.*

Proof. If (1) holds, then X has regularly embedded Čech complete ball by 4.11. If (2) holds, then X is isomorphic to $R \oplus S$, where R is reflexive and S^{**} is separable (by 4.5). Then S is a Godefroy space, so $S^{**} \setminus S = \bigcup_{n=1}^{\infty} L_n$, where each L_n is weak* compact, convex, and $d(L_n, S) > 0$. Then

$$X^{**} \setminus X = R + (S^{**} \setminus S) = \bigcup_m \bigcup_n (mB_R + L_n),$$

which is the required decomposition. □

4.13. THEOREM. *Suppose $X^{**} \setminus X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is weak* compact and convex. Then X has the Radon-Nikodym property.*

Proof. First suppose X is separable. Let (Ω, Σ, μ) be a probability space, and let $F: \Sigma \rightarrow X$ be a vector measure with $\|F(E)\| \leq \mu(E)$ for all $E \in \Sigma$. This measure has a weak* density in X^{**} (see [13, p. 84]), that is, a weak* measurable function $f: \Omega \rightarrow X^{**}$ such that $\langle F(E), x^* \rangle = \int_E \langle x^*, f(\omega) \rangle d\mu(\omega)$ for all $E \in \Sigma$ and $x^* \in X^*$. We will show that f has almost all of its values in X . Since X has Polish ball, X^* is separable, so K_n is a weak* Baire set in X^{**} , so $D_n = f^{-1}(K_n)$ belongs to Σ [15, Theorem 2.3]. If $\mu(D_n) > 0$ for some n , then $F(D_n)/\mu(D_n)$ is in X , so not in K_n , so by the Hahn-Banach theorem there is $x^* \in X^*$ and α such that

$\langle F(D_n)/\mu(D_n), x^* \rangle > \alpha > \langle x^*, y^{**} \rangle$ for all $y^{**} \in K_n$. But then

$$\int_{D_n} \langle x^*, f(\omega) \rangle d\mu(\omega) \leq \alpha\mu(D_n) < \langle F(D_n), x^* \rangle,$$

a contradiction. Thus $\mu(D_n) = 0$ for all n , so $f(\omega) \in X$ for almost all ω . Then f (redefined to be 0 on a μ -null set) is weakly measurable, and is the Pettis derivative of F . Since X is separable, f is Bochner integrable. This shows that X has the RNP.

Now consider the nonseparable case. By Theorem B, $X = R \oplus S$, where R is reflexive and S is separable. Then S^{**} can be identified with $S^{\perp\perp}$ in X^{**} , and $S^{\perp\perp}$ is closed in (X^{**}, weak^*) . If $X^{**} \setminus X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is weak* compact and convex, then

$$S^{**} \setminus S = \bigcup_{n=1}^{\infty} (S^{\perp\perp} \cap K_n),$$

where each $S^{**} \cap K_n$ is also weak* compact and convex. By the first part of the argument, S has RNP. Therefore $X = R \oplus S$ also has RNP. \square

The predual B of the James Tree space has Čech complete ball, but fails the RNP. The weak* compact sets K_n in the decomposition of $B^{**} \setminus B$ can be chosen to have positive distance from B , but not to be convex.

Property (A) is defined in 3.5.

4.14. THEOREM. *Let X be a Banach space. Then X has property (A) if and only if there exist weak* compact sets K_n in X^{**} with $d(K_n, X) > 0$ such that $X^{**} \setminus X = \bigcup_{n=1}^{\infty} K_n$.*

Proof. Assume X has property (A), that is, there exist norm-one vectors $y_{ni}^* \in X^*$ ($1 \leq i \leq m_n, n \in \mathbf{N}$) such that if

$$p_n(x^{**}) = \max\{|\langle y_{ni}^*, x^{**} \rangle| : 1 \leq i \leq m_n\},$$

then $\lim p_n(x) = 0$ for all $x \in X$ but $\liminf p_n(x^{**}) > 0$ for all $x^{**} \in X^{**} \setminus X$. Let

$$L_{km} = mB_{X^{**}} \cap \{x^{**} : p_n(x^{**}) \geq 1/k \text{ for all } n \geq k\}.$$

Then $X^{**} \setminus X = \bigcup_{k,m} L_{km}$ and $d(L_{km}, X) \geq 1/k$.

For the converse, suppose first that X is separable. Let (x_j) be dense in X and let $X^{**} \setminus X = \bigcup_{n=1}^{\infty} K_n$, where $d(K_n, X) > \varepsilon_n > 0$. So if $x^{**} \in K_n$, then $d(x^{**}, X) > \varepsilon_n$, so there exists $g_{x^{**}} \in X^\perp \subseteq X^{***}$ with $\|g_{x^{**}}\| = 1$

and $\langle x^{**}, g_{x^{**}} \rangle > \varepsilon_n$. We can find, for $k \in \mathbf{N}$, functionals $g_{x^{**},k} \in B_{X^*}$ with $\langle x_j, g_{x^{**},k} \rangle < 1/k$ for $1 \leq j \leq k$ and $\langle x^{**}, g_{x^{**},k} \rangle > \varepsilon_n$. This shows that, for each k ,

$$K_n \subseteq \bigcup_{x^{**} \in K_n} \{y^{**} : \langle y^{**}, g_{x^{**},k} \rangle > \varepsilon_n\},$$

so (by compactness of K_n) there is a finite subcover, say

$$K_n \subseteq \bigcup_i \{y^{**} : \langle y^{**}, f_{nik} \rangle > \varepsilon_n\},$$

where i runs over a finite set $\{1, 2, \dots, m_{nk}\}$, and $f_{nik} \in B_{X^*}$, $\langle x_j, f_{nik} \rangle < 1/k$ for $1 \leq j \leq k$, and if $x^{**} \in K_n$, then $\max_{1 \leq i \leq m_{nk}} \langle x^{**}, f_{nik} \rangle > \varepsilon_n$ for all k . Define

$$P_k(x^{**}) = \max\{|\langle x^{**}, f_{nik} \rangle| : 1 \leq n \leq k, 1 \leq i \leq m_{nk}\}.$$

Now if $x^{**} \notin X$ then $x^{**} \in K_n$ for some n , so that $P_k(x^{**}) > \varepsilon_n$ for all $k \geq n$. On the other hand, if $x \in X$ and $\|x\| = 1$, then given $\varepsilon > 0$ there exists j with $\|x - x_j\| < \varepsilon/2$. If $k > \max\{j, 2/\varepsilon\}$, then for $n \leq k$ and $1 \leq i \leq m_{nk}$, we have

$$|\langle x, f_{nik} \rangle| \leq \|x - x_j\| \|f_{nik}\| + |\langle x_j, f_{nik} \rangle| \leq \varepsilon/2 + 1/k < \varepsilon.$$

Thus $p_k(x) \leq \varepsilon$. This shows that X has property (A).

For the nonseparable case, observe that X has Čech complete ball, so $X = R \oplus S$, where R is reflexive and S has Polish ball. Thus $S^{**} = S^{\perp\perp}$ is weak* closed in (X^{**}, weak^*) , so $S^{**} \setminus S = \bigcup_{n=1}^\infty (S^{\perp\perp} \cap K_n)$. By the separable case, S has property (A), and it follows that X does too. \square

5. A factorization theorem for property (PC). The main theorem of this section employs a version of the Principle of Local Reflexivity. We record it here for convenience. A proof is sketched in [32].

5.1. PROPOSITION. *Let X be a Banach space, U and F finite-dimensional subspaces of X^{**} and X^* respectively, $\varepsilon > 0$. Then there is a one-to-one linear operator $T: U \rightarrow X$ such that (1) $Tx = x$ for all $x \in X \cap U$; (2) $\langle T(x^{**}), x^* \rangle = \langle x^*, x^{**} \rangle$ for all $x^* \in F, x^{**} \in U$; and (3) $\|T\| \|T^{-1}\| < 1 + \varepsilon$.*

Note that if $X \cap U \neq \{0\}$, then (1) implies $\|T\| \geq 1$ and $\|T^{-1}\| \geq 1$, so by (3) we have $\|T\| < 1 + \varepsilon, \|T^{-1}\| < 1 + \varepsilon$.

5.2. THEOREM. *Let X be a Banach space, Y a closed subspace. Suppose (1) Y has Polish ball; (2) X/Y has property (PC); and (3) Y^\perp is a complemented subspace of X^* . Then X has (PC).*

Proof. Let $i: Y \rightarrow X$ be the inclusion and $\pi: X \rightarrow X/Y$ the quotient map. Let $X^* = Y^\perp \oplus F$, where F is isomorphic to Y^* . Then there is a linear isomorphism $u: Y^* \rightarrow X^*$ of Y^* onto the closed subspace F of X^* , so that $i^* \circ u$ is the identity on Y^* . The space X^{**} is isomorphic to $(X^{**}/Y^{\perp\perp}) \oplus Y^{**}$, with projections π^{**} and u^* . There is an $\alpha < \infty$ such that

$$\|x^{**}\| \leq \alpha(\|u^*(x^{**})\| + \|\pi^{**}(x^{**})\|) \quad \text{for all } x^{**} \in X^{**}.$$

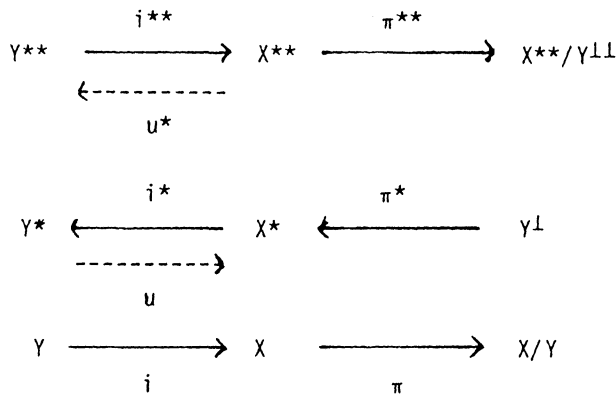


FIGURE 2

Let A be a nonempty bounded subset of X . We will show that A contains nonempty relatively weakly open subsets of arbitrarily small diameter. (This will show that X has (PC), by 3.13.) If not, we may assume that $\text{diam}(U \cap A) > 1$ for each weakly open set U with $U \cap A \neq \emptyset$. Let $M = 1 + \text{diam}(A)$, $B = \{y \in Y: \|y\| \leq M\}$. Since B is Polish, there is a complete sequence (\mathcal{Q}_n) of open covers of B closed under finite unions. Choose n_0 such that $2^{n_0} > \max(2\alpha, 2, 1 + 2\|u^*\|)$.

We will use the following notation for the binary tree $T = \bigcup_{n=0}^\infty \{0, 1\}^n$, and the Cantor set $\Delta = \{0, 1\}^\mathbb{N}$. If $t \in \{0, 1\}^n$, its length is n , $|t| = n$. If $k \leq n$, the restriction $t|_k$ is the element of $\{0, 1\}^k$ that agrees with t in the first k terms. Similarly, if $\tau \in \Delta$, then $\tau|_k \in \{0, 1\}^k$ is an initial segment. For $t \in \{0, 1\}^n$, its two successors in $\{0, 1\}^{n+1}$ are written $t0$ and $t1$. We write \emptyset for the element of $\{0, 1\}^0$.

For each $t \in T$, we will choose vectors $x_t \in A$, $y_t \in B$, a basic neighborhood V_t of 0 in (Y, weak) , and a weakly open subset W_t of X/Y such that $\pi(x_t) \in W_t$. We do this recursively on T .

First, $\pi(A)$ is bounded in X/Y , so there is a weakly open set W_\emptyset in X/Y with $\pi(A) \cap W_\emptyset \neq \emptyset$ and $\text{diam}(\pi(A) \cap W_\emptyset) < 2^{-n_0}$. Choose $x_\emptyset \in A \cap \pi^{-1}(W_\emptyset)$, let $y_\emptyset = 0$, and let V_\emptyset be a basic neighborhood of 0 in Y such that $V_\emptyset \cap B$ is contained in an element of \mathcal{Q}_0 .

Now suppose, for some $t \in T$, that x_t, y_t, V_t, W_t have been constructed. Let $n = |t|$. Now V_t is a basic neighborhood of 0 in Y , so there is a finite subset $\{f_j\}_{j=1}^m$ of Y^* such that $V_t = \{y: |\langle y, f_j \rangle| < 1, 1 \leq j \leq m\}$. Let $f'_j = u(f_j) \in X^*$, and let $U_t = \{x: |\langle x, f'_j \rangle| < 1, 1 \leq j \leq m\}$, a neighborhood of 0 in X . Now X/Y has (PC), so W_t contains a weakly open set $W_{t0} = W_{t1}$ such that $\pi((U_t + x_t) \cap A) \cap W_{t0}$ is nonempty and has diameter less than $2^{-(n_0+n+1)}$. Then $((U_t + x_t) \cap \pi^{-1}(W_{t0})) \cap A$ is nonempty, so it has diameter greater than 1. Choose x_{t0}, x_{t1} in this set with $\|x_{t0} - x_{t1}\| > 1$. Note that $\|\pi(x_{t0}) - \pi(x_{t1})\|, \|\pi(x_{ti}) - \pi(x_t)\|, i = 0, 1$, are less than $2^{-(n_0+n+1)}$. Thus

$$\begin{aligned} \|u^*(x_{t0}) - u^*(x_{t1})\| &> \frac{1}{\alpha} \|x_{t0} - x_{t1}\| - \|\pi^{**}(x_{t0} - x_{t1})\| \\ &> \frac{1}{\alpha} - 2^{-(n_0+n+1)} > \frac{1}{2\alpha}. \end{aligned}$$

Choose w_0 and w_1 in Y such that $\|x_{ti} - x_t - w_i\| < 2^{-(n_0+n)}$ for $i = 0, 1$. Let U be the span in Y^{**} of the 6 vectors $y_t, u^*(x_t), u^*(x_{t0}), u^*(x_{t1}), w_0, w_1$; let F be the span in Y^* of the functionals f_1, \dots, f_m . By local reflexivity, there is a one-to-one linear operator $T: U \rightarrow Y$ such that $T(y_t) = y_t, T(w_i) = w_i, i = 0, 1; \langle T(y^{**}), f_j \rangle = \langle f_j, y^{**} \rangle$ for $y^{**} \in U$ and $1 \leq j \leq m$; and $\|T\| < 2, \|T^{-1}\| < 2$. Then

$$\langle f_j, Tu^*(x_{ti} - x_t) \rangle = \langle f_j, u^*(x_{ti} - x_t) \rangle = \langle x_{ti} - x_t, f'_j \rangle < 1$$

for $1 \leq j \leq m$, since $x_{ti} - x_t \in U_t$. Thus $Tu^*(x_{ti} - x_t) \in V_t, i = 0, 1$. Let $y_{ti} = Tu^*(x_{ti} - x_t) + y_t = Tu^*(x_{ti} - x_t + y_t)$. Then

$$\begin{aligned} \|y_{ti} - y_t - w_i\| &= \|Tu^*(x_{ti} - x_t) - T(w_i)\| \leq 2\|u^*(x_{ti} - x_t - w_i)\| \\ &\leq 2\|u^*\|2^{-(n_0+n)} = \|u^*\|2^{-(n_0+n-1)}. \end{aligned}$$

On the other hand,

$$\|y_{t0} - y_{t1}\| = \|Tu^*(x_{t1} - x_{t0})\| \geq \frac{1}{2}\|u^*(x_{t1} - x_{t0})\| > 1/4\alpha.$$

Now y_{t0} and y_{t1} belong to $V_t + y_t$. Choose a basic open neighborhood $V_{t0} = V_{t1}$ of 0 in Y so that (1) $wcl(y_{ti} + V_{ti}) \subseteq y_t + V_t, i = 0, 1$; (2) if $z_i \in y_{ti} + V_{ti}$, then $\|z_0 - z_1\| > 1/4\alpha$; and (3) $(y_{ti} + V_{ti}) \cap B$ is contained in an element of \mathcal{Q}_{n+1} . To complete the construction, we need to verify that $y_t \in B$ for each t . If $t \in \{0, 1\}^n$, then

$$\begin{aligned} \|y_t\| &= \left\| \sum_{k=1}^n (y_{t|k} - y_{t|k-1}) \right\| \leq \left\| \sum_{k=1}^n (x_{t|k} - x_{t|k-1}) \right\| \\ &\quad + \left\| \sum_{k=1}^n ((y_{t|k} - y_{t|k-1}) - (x_{t|k} - x_{t|k-1})) \right\|. \end{aligned}$$

The first norm is $\|x_t - x_\emptyset\| \leq \text{diam}(A)$. A term in the second norm is of the form $(y_{si} - y_s) - (x_{si} - x_s)$ where $s \in \{0, 1\}^k$ and $i = 0, 1$. But if w_i is as above, then

$$\begin{aligned} \|(y_{si} - y_s) - (x_{si} - x_s)\| &\leq \|y_{si} - y_s - w_i\| + \|x_{si} - x_s - w_i\| \\ &\leq \|u^*\| 2^{-(n_0+k-1)} + 2^{-(n_0+k)}. \end{aligned}$$

Thus the norm of the sum is at most

$$2^{-n_0} \left(\sum_{k=1}^n (\|u^*\| 2^{-(k-1)} + 2^{-k}) \right) \leq 2^{-n_0} (2\|u^*\| + 1) < 1,$$

by the choice of n_0 . Thus $\|y_t\| \leq \text{diam}(A) + 1 = M$, so $y_t \in B$. This completes the recursive construction.

Now let

$$D = \bigcap_{n=0}^{\infty} \bigcup_{|t|=n} (y_t + V_t).$$

Since $\text{wcl}(y_{ti} + V_{ti}) \subseteq y_t + V_t$ for all t , the set D is weakly closed in Y . For each n , D is covered by finitely many sets from \mathcal{U}_n , and therefore by a single set from \mathcal{U}_n . If \mathcal{F} is a filter of closed sets in D , then \mathcal{F} is \mathcal{U}_n -small for all n , so $\bigcap \mathcal{F} \neq \emptyset$, by the completeness of the sequence (\mathcal{U}_n) . Therefore D is compact. For each $\tau \in \{0, 1\}^{\mathbb{N}}$, the set

$$D_\tau = \bigcap_{n=0}^{\infty} (y_{\tau|n} + V_{\tau|n})$$

is a nonempty subset of D . Moreover, if $\tau \neq \sigma$, $z_1 \in D_\tau$, $z_2 \in D_\sigma$, then $\|z_1 - z_2\| > 1/4\alpha$. Thus Y is nonseparable, contradicting the fact that B_Y is Polish. \square

5.3. COROLLARY. *If X has Čech complete ball, and X^{**}/X has property (PC), then X^{**} has property (PC).*

Proof. If X has Polish ball, then this follows from 5.2, since X^\perp is always complemented in X^{***} . The general case follows from Theorem B: since X has Čech complete ball, $X = R \oplus S$ where R is reflexive and S has Polish ball. But $S^{**}/S = X^{**}/X$ has (PC), so S^{**} has (PC). The reflexive space R has (PC), so $X = R \oplus S$ has (PC). \square

As an illustration, consider the dual JT^* of the James Tree space JT . In Example 3.6, we showed that the predual B of JT has Polish ball. But JT^*/B is a nonseparable Hilbert space [38], so it has (PC). Therefore JT^*

has (PC). It is useful to note that JT^* is a nonseparable dual of a separable space, so it fails the RNP [50].

We do not know if hypothesis (3) is necessary for 5.2, nor if (1) can be weakened to “ Y has Čech complete ball”, or even to “ Y has (PC)”.

Bourgain [3] has considered a convex version of (PC). We will say that a Banach space X has property (*) iff for each weakly closed bounded convex subset A of X , the identity map $(A, \text{weak}) \rightarrow (A, \text{norm})$ has at least one point of continuity. We do not know if this is equivalent to (PC). An argument similar to 3.11 shows that a Banach space has (*) if and only if every separable subspace has (*). An argument similar to [9] can be given to show that X has (*) if and only if (B_X, weak) is huskable for every equivalent norm on X . If X is separable, this is equivalent to saying that B_X is a Baire space for every equivalent norm on X . We omit the proofs.

The paper [55] investigated the conditions under which (B_X, weak) satisfies the countable chain condition (every pairwise disjoint collection of nonempty open sets is countable). Theorem 18 of that paper is easily modified to show the following.

5.4. PROPOSITION. *Let X be a Banach space with (PC). Then (B_X, weak) satisfies the countable chain condition if and only if X is separable.*

Rosenthal [48] proved that for an Eberlein compact, the three conditions (1) separable, (2) metrizable, (3) countable chain condition, are equivalent. Here is a related noncompact theorem.

5.5. PROPOSITION. *Let X be a Banach space with Čech complete ball, A a weakly closed bounded subset. Then the following conditions on (A, weak) are equivalent: (1) separable, (2) metrizable, (3) countable chain condition.*

Proof. (1) \Rightarrow (3) is true for all topological spaces. For (3) \Rightarrow (1), modify Theorem 18 of [55].

(1) \Rightarrow (2). Let Y be the closed linear span of A . Then Y has Polish ball, so A is metrizable.

(2) \Rightarrow (1). Let d be a metric for (A, weak) . If A is not separable, then there is an uncountable set $\{a_i: i \in I\}$ and $\epsilon > 0$ such that $d(a_i, a_j) > \epsilon$ for $i \neq j$. Now $X = R \oplus S$, where R is reflexive and S is separable. Let $a_i = r_i + s_i$ for $i \in I$, where $r_i \in R$, $s_i \in S$. Since S is separable, there is a net (s_{i_λ}) that converges in norm (perhaps s_{i_λ} is constant). Since R is reflexive, by taking a subnet we see that (a_{i_λ}) has a weak cluster point in A , a contradiction. □

Here is another application of Čech completeness. A topological space T is said to have a G_δ diagonal iff $\{(t, t) : t \in T\}$ is a G_δ set in $T \times T$. A paracompact Čech complete topological space with G_δ diagonal is metrizable [1, p. 37].

5.6. PROPOSITION. *Let A be a subset of (X, weak) . Then the following are equivalent:*

- (1) A is Polish;
- (2) A is Lindelöf and Čech complete, and there is a countable subset of X^* that separates points of A .

Proof. (1) \Rightarrow (2). If Y is the closed span of A , then Y^* is weak* separable.

(2) \Rightarrow (1). The final condition gives A a G_δ diagonal. Thus A is metrizable and Čech complete, so it is Polish. □

5.7. EXAMPLE. Since $(l^\infty)^*$ is weak* separable, every Lindelöf Čech complete subset of (l^∞, weak) is Polish. This is a generalization of the fact that every weakly compact subset of l^∞ is separable. Under the Continuum Hypothesis, there is a closed linear subspace Y of l^∞ such that every subset is Lindelöf in the weak topology, but Y is not separable [46, Remark 2].

TABLE 1

	Godefroy	X^* Čech complete ball	Čech complete ball	RNP	PC	somewhat reflexive	separably distinguished	Asplund
(1) B	N	N	Y	N	Y	Y	Y	Y
(2) BD	N	N	Y	Y	Y	Y	Y	Y
(3) c_0	N	N	N	N	N	N	Y	Y
(4) $J(\alpha), \alpha < \omega_1$	Y	Y	Y	Y	Y	Y	Y	Y
(5) $J(\omega_1)$	N	N	N	Y	Y	Y	N	Y
(6) JH	N	N	N	N	N	N	N	N
(7) JT	N	N	N	Y	Y	Y	N	N
(8) JT^*	N	N	N	N	Y	Y	N	Y
(9) l^1	N	N	N	Y	Y	N	N	N
(10) $L^1[0, 1]$	N	N	N	N	N	N	N	N
(11) $JL(c_0)$	Y	Y	Y	Y	Y	Y	Y	Y
(12) $JL(l^1)$	N	N	Y	Y	Y	Y	Y	Y
(13) $JL(JT)$	Y	N	Y	Y	Y	Y	Y	Y

6. Examples. Table 1 displays the properties of several Banach spaces. We include here some brief explanations.

(1) $X = B$, the predual of the James Tree (Example 3.6, [30], [38]). It has property (A), but $B_{X^{**}} \setminus B_X$ is not a countable union of weak* compact convex sets. It has Polish ball but is not isomorphic to a dual

space, nor complemented in its second dual. It has property (PC), but not RNP. The dual space $B^* = JT$ does not have Čech complete ball.

(2) $X = BD$, a space constructed by Bourgain and Delbaen called Y in [5], [4]. It is a separable \mathcal{L}^∞ space, has Polish ball, and has the Dunford-Pettis and Radon-Nikodým properties. The dual X^* is isomorphic to l^1 . The space X has no infinite-dimensional reflexive quotients.

(3) $X = c_0$ (Example 3.3) is separably distinguished, but does not have Čech complete ball. For the usual norm, (B_X, weak) is first category in itself. For Day's norm, (B_X, weak) is a Baire space.

(4) $X = J(\alpha)$, the long James space [17]. If α is a countable ordinal, then X^{**} is separable, so X has regularly embedded Čech complete ball.

(5) $X = J(\omega_1)$. This nonseparable space has property (PCA), so every separable subspace has Čech complete ball, and X is somewhat reflexive. But $J(\omega_1)$ is not WCG, so its ball is not Čech complete.

(6) $X = JH$, and James Hagler space [25]. This space X is hereditarily c_0 , and X^* is a Schur space. Hence no infinite-dimensional subspace of X or X^* has Čech complete ball. The separable subspace F of JH^* [6] has (PC) but fails the RNP.

(7) $X = JT$, the James Tree space (Example 3.6, [30], [38]). Now X is separable, X^{**}/X is reflexive, X is somewhat reflexive, but B_X is not Čech complete.

(8) $X = JT^*$, the dual of the James Tree space (Example 3.6, [38], [38]). This space is an Asplund space and has (PC), but not the RNP. Thus every closed bounded subset of (JT^*, weak) is huskable, but this is not true for (JT^*, weak^*) . There is a subspace $B \subseteq X$ with Polish ball such that X/B is reflexive, but X does not have Čech complete ball.

(9) $X = l^1$. This space X has the RNP, but X^{**} fails (PC), since it contains $L^1[0, 1]$. For any equivalent norm (B_X, weak) is almost Čech complete, but no nonempty G_δ set of (X^{**}, weak^*) is entirely contained in X .

(10) $X = L^1[0, 1]$. For the usual norm, (B_X, weak) is of first category in itself: If (r_n) is the sequence of Rademacher functions, then B_X is the union of the closed nowhere dense sets

$$A_k = \left\{ f \in B_X : \left| \int f(t)r_n(t) dt \right| \leq 1/4 \text{ for all } n \geq k \right\}.$$

For subspaces of L^1 , the properties (PC) and RNP are equivalent. Every subspace of X with separable dual has Polish ball (in fact, reflexive), but X does not have Polish ball.

(11) $X = \text{JL}(c_0)$, a James-Lindenstrauss space (Example 4.8). If S is separable, the James-Lindenstrauss space $\text{JL}(S)$ can be constructed so as to have property (A). The spaces $\text{JL}(S)$ are separable duals, so they have the RNP. For $X = \text{JL}(c_0)$, the bidual X^{**} is separable, but X^{**}/X does not have Čech complete ball. Also, B_{X^*} is Čech complete, but X^* admits c_0 as a quotient.

(12) $X = \text{JL}(l^1)$. This space X has Čech complete ball, but X^* contains a complemented copy of l^1 , and X^{**}/X fails the Pettis Integral Property. Also, X is a dual space that has Čech complete ball with a quotient that is not somewhat reflexive. The annihilator X^\perp is weak* separable, but B_{X^\perp} is not weak* sequentially compact.

(13) $X = \text{JL}(\text{JT})$. Now B_X is Čech complete, $X^* \leftarrow l^1$, but neither X^{**} nor X^{**}/X is measure-compact. Also, X is a Godefroy space, but B_{X^*} is not Čech complete.

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