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ON THE ATOMIC DECOMPOSITION FOR HARDY SPACES

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We give an extremely easy proof of the atomic decomposition for distributions in $H^p(\mathbf{R}_+^{n+1})$, $0 < p \le 1$. Our proof uses only properties of the nontangential maximal function u^*. We then adapt our argument to give a "direct" proof of the Chang-Fefferman decomposition for $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$.

I. Introduction. Let $\mathbf{R}_+^{n+1} = \{(x, y)\colon x \in \mathbf{R}^n,\ y > 0\}$. For $u(x, y)$ harmonic on $\mathbf{R}_+^{n+1}$ and $A > 0$ define

$$u_A^*(x) = \sup_{|x-t|<Ay} |u(t, y)|.$$

We say that $u \in H^p$ if $u_A^* \in L^p$, for any A, and set $\|u\|_{H^p} = \|u_1^*\|_{L^p}$. If $u \in H^p$, $0 < p < \infty$, then $f = \lim_{y \to 0} u(\cdot, y)$ exists (in $\mathcal{S}'$) and is said to be in H^p. We set $\|f\|_{H^p} = \|u\|_{H^p}$ (see **[6]**).

For $0 < p \le 1$, a *p-atom* is a function $a(x) \in L^2(\mathbf{R}^n)$ satisfying:

(α) supp $a \subset Q$, Q a cube.

(β) $\|a\|_2 \le |Q|^{1/2-1/p}$ ($|Q|$ = the volume of Q).

(γ) $\int a(x) x^\alpha\, dx = 0$ for all monomials x^α with $|\alpha| \le [n(p^{-1} - 1)]$.

The following theorem is well known **[4] [7] [10]**:

THEOREM A. *Let $f \in H^p$, $0 < p \le 1$. There exist p-atoms a_k and numbers λ_k such that*

$$f = \sum \lambda_k a_k \quad \text{in } \mathcal{S}'. \tag{1}$$

The λ_k satisfy $\sum |\lambda_k|^p \le C(p, n) \|f\|_{H^p}^p$. Conversely, every sum (1) satisfies

$$\|f\|_{H^p}^p \le C(p, n) \sum |\lambda_k|^p.$$

Now let u be biharmonic on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$. Define

$$u_A^*(x_1, x_2) = \sup_{\substack{|x_i - t_i| < Ay_i \\ i=1,2}} |u(t_1, y_1, t_2, y_2)|.$$

As before, we say that $u \in H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ if $u^*_A \in L^p(\mathbf{R}^2)$, and we set $\|u\|_{H^p} = \|u^*_1\|_{L^p}$. Such u give rise to boundary distributions f, which are said to be in H^p. (See **[2]**.)

For $0 < p \leq 1$, a *Chang-Fefferman p-atom* is a function $a \in L^2(\mathbf{R}^2)$ satisfying:

(α') supp $a \subset \Omega$, Ω open, $|\Omega| < \infty$.

(β') $\|a\|_2 \leq |\Omega|^{1/2-1/p}$.

(γ') $a = \sum_R \lambda_R a_R$, where λ_R are numbers and the a_R are functions (called "elementary particles") satisfying:

(i) supp $a_R \subset \tilde{R} \subset \Omega$ where $R = I \times J$, I, J dyadic intervals, and $\tilde{R}$ denotes the triple of R.

(ii)

$$\left\| \frac{\partial^L a_R}{\partial x_1^L} \right\|_\infty \leq \frac{1}{\sqrt{|R|}\,|I|^L} \quad \text{and} \quad \left\| \frac{\partial^L a_R}{\partial x_2^L} \right\|_\infty \leq \frac{1}{\sqrt{|R|}\,|J|^L}$$

for all $L \leq [2/p - 1/2]$

(iii)

$$\int a(\tilde{x}_1, x_2) x_2^k \, dx_2 = 0 \quad \text{and} \quad \int a(x_1, \tilde{x}_2) x_1^k \, dx_1 = 0$$

for all $(\tilde{x}_1, \tilde{x}_2) \in \mathbf{R}^2$ and all $k \leq [2/p - 3/2]$. And

$$\left(\sum_R \lambda_R^2 \right)^{1/2} \leq |\Omega|^{1/2-1/p}.$$

If the "atoms" are Chang-Fefferman atoms, then Theorem A is true for $f \in H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ **[2]** **[3]**.

Until now, proofs of the atomic decomposition have relied on showing that $u^* \in L^p$ implies that some auxiliary function (such as the "grand" maximal function or the S-function) is in L^p. In this paper, we give proofs which get the atoms directly from "$u^* \in L^p$".

REMARK. Our argument is somewhat like that of A. P. Calderón in **[1]**. Calderón's "u^*" is the sum of two real-variable maximal functions. He writes his reproducing formula (see below) in terms of one kernel, and uses the other kernel to control the L^∞ size of his atoms. Our proof uses Green's Theorem to get L^2 bounds. This approach lets us adapt our proof to the bidisc setting, where L^∞ atoms do not seem to be the "right" ones.

II. The case $H^p(\mathbf{R}^2_+)$. Let $\psi \in C^\infty(\mathbf{R})$ be real, radial, supp $\psi \subset \{|x| \leq 1\}$, ψ has the cancellation property γ), and

$$\int_0^\infty e^{-\theta} \hat{\psi}(\theta) \, d\theta = -1.$$

For $y > 0$, set $y^{-1}\psi(t/y) = \psi_y(t)$.

Take $f \in L^2 \cap H^p$, f real-valued, $u = P_y * f$ (the Poisson integral of f). By Fourier transforms

$$f = \int_{\mathbf{R}^2_+} \frac{\partial u}{\partial y}(t, y)\psi_y(x - t)\,dt\,dy \quad \text{in } \mathscr{S}'.$$

(This trick is due to A. P. Calderón.) For $k = 0, \pm 1, \pm 2, \ldots$, define

$$E^k = \{u_2^* > 2^k\} = \bigcup_{j=1}^{\infty} I_j^k$$

where the I_j^k are component intervals. For I an interval, let

$$\hat{I} = \{(t, y) \in \mathbf{R}^2_+ : (t - y, t + y) \subset I\}$$

be the "tent" region. Define $\hat{E}^k = \bigcup \hat{I}_j^k$, $T_j^k = \hat{I}_j^k \setminus \hat{E}^{k+1}$. Then

$$f = \sum_{k,j} \int_{T_j^k} \frac{\partial u}{\partial y}(t, y)\psi_y(x - t)\,dt\,dy = \sum_{k,j} g_j^k = \sum_{k,j} \lambda_j^k a_j^k,$$

where $\lambda_j^k = C2^k|I_j^k|^{1/p}$ and the a_j^k (we claim) are atoms. The a_j^k inherit γ from ψ, and obviously supp $a_j^k \subset \tilde{I}_j^k$. Note also that

$$\sum \left(\lambda_j^k\right)^p \le C \int (u_2^*)^p\,dx \le C\|u\|_{H^p}^p.$$

Thus, we are done if we can show

$$\left\|g_j^k\right\|_2 \le C2^k\left|I_j^k\right|^{1/2}.$$

We do this by duality. Let $h \in L^2(\mathbf{R})$, $\|h\|_2 = 1$. Then

$$\left|\int h(x) g_j^k(x)\,dx\right| = \left|\int_{T^k} \frac{\partial u}{\partial y}(t, y)\big(h * \psi_y(t)\big)\,dt\,dy\right|$$

$$\le \left(\int_{T_j^k} y|\nabla u|^2\,dt\,dy\right)^{1/2} \left(\int_{\mathbf{R}^2_+} |h * \psi_y(t)|^2 \frac{dt\,dy}{y}\right)^{1/2}$$

(Plancherel)
$$\le C\left(\int_{T_j^k} y|\nabla u|^2\,dt\,dy\right)^{1/2}$$

We estimate the last integral by Green's Theorem. It is bounded by

$$\left(\int_{\partial T_j^k} \left(|u|y\left|\frac{\partial u}{\partial \nu}\right| + \frac{1}{2}u^2\left|\frac{\partial y}{\partial \nu}\right|\right) ds\right)$$

($\partial/\partial\nu$ is outward normal; ∂T_j^k is just smooth enough to let us use Green's Theorem). Because of the "2" (in u_2^*), both $|u|$ and $y|\nabla u|$ are bounded by $C2^k$ on ∂T_j^k. Since $|\partial y/\partial \nu| \le 1$ and $|\partial T_j^k| \le C|I_j^k|$, the last term is no larger than $C2^k|I_j^k|^{1/2}$. □

III. The case $H^p(\mathbf{R}^{n+1}_+)$. Let ψ be as in II, except now $\psi \in C^\infty(\mathbf{R}^n)$. Let $f \in H^p \cap L^2$ and u be as before. Define

$$E^k = \{u^*_{10^n} > 2^k\} = \bigcup_{j=1}^{\infty} \Omega^k_j;$$

where the Ω^k_j are Whitney cubes (for the definition see [9], p. 167). For Ω a cube in $\mathbf{R}^n$, define

$$\hat{\Omega} = \{(t, y) : t \in \Omega, 0 < y < l(\Omega)\}$$

where $l(\Omega)$ = sidelength of Ω. Define

$$\hat{E}^k = \bigcup \hat{\Omega}^k_j, \qquad T^k_j = \hat{\Omega}^k_j \setminus \hat{E}^{k+1}.$$

With these modifications, the preceding argument goes over practically verbatim; the details are left to the reader.

IV. The case $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$. We first show that the proof in II yields a Chang-Fefferman decomposition for $\mathbf{R}^2_+$. For $I \subset \mathbf{R}$ a dyadic interval, let

$$I^+ = \{(t, y) : t \in I, |I|/2 < y \leq |I|\}.$$

Define

$$\mathscr{I}^k_j = \{Q = I^+ \cap T^k_j\},$$

$$g_Q = \int_Q \frac{\partial u}{\partial y}(t, y)\psi_y(x - t)\, dt\, dy = \lambda^k_j \lambda_Q a_Q \quad \text{for } Q \in \mathscr{I}^k_j,$$

where we set

$$\lambda_Q = C(\lambda^k_j)^{-1}\left(\int_Q y|\nabla u|^2\, dt\, dy\right)^{1/2}.$$

Then it is easily verified that the a_Q have the right cancellation, support and smoothness properties for elementary particles. And obviously

$$a^k_j = \sum_{Q \in \mathscr{I}^k_j} \lambda_Q a_Q,$$

$$\left(\sum_{Q \in \mathscr{I}^k_j} \lambda^2_Q\right)^{1/2} \leq |\tilde{I}^k_j|^{1/2 - 1/p}.$$

In order to do our proof in $\mathbf{R}^2_+ \times \mathbf{R}^2_+$, we need tents, and we need a way to do Green's Theorem. For these, we need some notation.

For $(t, y) = (t_1, y_1, t_2, y_2) \in (\mathbf{R}^2_+)^2$, let $R_{t,y}$ be the rectangle with sides parallel to the coordinate axes, centered at $(t_1, t_2) \in \mathbf{R}^2$, and with dimensions $2y_1 \times 2y_2$.

Take $f \in L^2 \cap H^p$, $u = P_{y_1} \cdot P_{y_2} * f$ (the double Poisson integral of f). Let ψ be as in II but with cancellation corresponding to (iii). Then

$$f = \int_{(\mathbf{R}^2_+)^2} \frac{\partial^2 u}{\partial y_1 \partial y_2}(t, y)\psi_{y_1}(x_1 - t_1)\psi_{y_2}(x_2 - t_2)\, dt\, dy \quad \text{in } \mathscr{S}'.$$

Let M be the strong maximal function. Let $\varepsilon > 0$ be small, to be chosen later. Define

$$E^k = \{u^*_{100} > 2^k\}, \qquad F^k = \{M\chi_{E^k} > \varepsilon\}.$$

It is a fact that $|F^k| \le C_\varepsilon |E^k|$. Set

$$\hat{F}^k = \{(t, y) : R_{t,y} \subset F^k\},$$

$$T^k = \hat{F}^k \setminus \hat{F}^{k+1},$$

$$g^k = \int_{T^k} \frac{\partial^2 u}{\partial y_1 \partial y_2}(t, y)\psi_{y_1}(x_1 - t_1)\psi_{y_2}(x_2 - t_2)\, dt\, dy = \lambda_k a_k,$$

where we set $\lambda_k = C2^k|E^k|^{1/p}$.

For $R = I \times J$, I, J dyadic intervals, let $R^+ = I^+ \times J^+ \subset R^2_+ \times R^2_+$. Set

$$\mathscr{I}_k = \{Q = R^+ \cap T^k\},$$

$$g_Q = \int_Q \frac{\partial^2 u}{\partial y_1 \partial y_2}(t, y)\psi_{y_1}(x_1 - t_1)\psi_{y_2}(x_2 - t_2)\, dt\, dy$$

$$= \lambda_k \lambda_Q a_Q \qquad (Q \in \mathscr{I}^k),$$

where we set

$$\lambda_Q = C(\lambda_k^{-1})\left(\int_Q y_1 y_2 |\nabla_1 \nabla_2 u|^2\, dt\, dy\right)^{1/2}$$

with

$$|\nabla_1 \nabla_2 u|^2 = \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial x_1 \partial y_2}\right|^2 + \left|\frac{\partial^2 u}{\partial y_1, \partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial y_1 \partial y_2}\right|^2.$$

Then, in exact analogy to case II, everything will be done once we show

(2) $$\int_{T^k} y_1 y_2 |\nabla_1 \nabla_2 u|^2\, dt\, dy \le C2^{2k}|E^k|.$$

For this we need a lemma of Merryfield. The lemma requires a little more notation.

Let $\eta \in C^{\infty}(\mathbf{R})$, $\eta \geq 0$, $\operatorname{supp} \eta \subset [-1,1]$, $\eta \geq \frac{1}{2}$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\int \eta = 1$. Define

$$\Phi_{y_1 y_2}(t_1, t_2) = \eta_{y_1}(t_1) \cdot \eta_{y_2}(t_2).$$

For $E \subset \mathbf{R}^2$, set

$$V_E(t, y) = \Phi_y * \chi_E(t), \qquad (t, y) \in (\mathbf{R}^2_+)^2.$$

Now, $V_E(t, y)$ is essentially the density of E in $R_{t,y}$. In particular, if this density is greater than $1 - \varepsilon$, ε small, then $V_E(t, y) > 10^{-6}$.

Merryfield's lemma is [**8**]:

LEMMA. *Let $u \in H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, $p < 2$, and let $u^*_{100} \leq \lambda$ on $E \subset \mathbf{R}^2$. Then*

$$\int_{(\mathbf{R}^2_+)^2} y_1 y_2 |\nabla_1 \nabla_2 u|^2 V_E^2(t, y)\, dt\, dy \leq C\lambda^2 |E|.$$

(Note: Merryfield states this for E open, but openness, as his proof shows, is not required.)

Let us set $G^k = F^k \setminus E^{k+1}$. Merryfield's lemma says that

$$\int_{\mathbf{R}^2_+} y_1 y_2 |\nabla_1 \nabla_2 u|^2 V_{G^k}^2(t, y)\, dt\, dy \leq C 2^{2k} |G^k| \leq C 2^{2k} |E^k|.$$

Therefore, we will have (2) (and be done) if we can show

$$V_{G^k} > 10^{-6} \quad \text{on } T^k.$$

Take $(t, y) \in T^k$. Then $R_{t,y} \subset F^k$ but $R_{t,y} \not\subset F^{k+1}$. So there is an $x \in R_{t,y} \cap (F^k \setminus F^{k+1})$. Since $x \notin F^{k+1}$, $M\chi_{E_1^{k+1}}(x) \leq \varepsilon$. From the definition of M, this implies

$$|R_{t,y} \cap E^{k+1}| / |R_{t,y}| \leq \varepsilon.$$

Since $R_{t,y} \subset F^k$,

$$|R_{t,y} \cap (F^k \setminus E^{k+1})| / |R_{t,y}| \geq 1 - \varepsilon.$$

But $F^k \setminus E^{k+1} = G^k$, and this implies that $V_{G^k}(t, y) > 10^{-6}$, for ε small. □

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