PACIFIC JOURNAL OF MATHEMATICS
Vol 116, No 1, 1985

ON M-STRUCTURES AND STRONGLY REGULARLY STRATIFIABLE SPACES

TAKEMI MIZOKAMI

It is shown that every (strongly) regularly stratifiable space has an M-structure, and it is also shown that a stratifiable space has an M-structure if and only if every closed subset has an open neighborhood base which is interior-preserving on a network.

1. Introduction. We assume the reader is familiar with Ceder's M_i-spaces ($i = 1,2,3$) [**2**] and Borges's stratifiable spaces [**1**]. Ceder proposed in [**2**] the problem whether the reverse implications of $M_1 \to M_2 \to M_3$ ($=$ stratifiable) are true or not. That M_2-spaces coincide with stratifiable spaces was proved independently by Gruenhage [**3**] and by Junnila [**7**]. Thus the problem (P1) remains open.

(P1) *Is every stratifiable space an M_1-space*?

Some of the equivalent and related problems to (P1) are given.

(P2) *Is every closed subspace of an M_1-space M_1*?

(P3) *Is every image of an M_1-space under a closed mapping M_1*?

(P4) *Does every closed subset of an M_1-space have a closure-preserving open neighborhood base in X*?

(P4) is true if (P1) is so, [**8**].

(P5) *Is every M_1-space the image of an M_0-space* ($=$ *a space which has a σ-closure-preserving base consisting of closed and open sets*) *under a perfect mapping*? (Heath and Junnila)

(P6) *Is every stratifiable space the image of a stratifiable space with* $\dim = 0$ *under a perfect mapping*? (Nagami [**10**])

As for the dimension of an M_1-space, Nagata proposed the following problem:

(P7) *Let X be an M_1-space. Is it true that* $\dim X \leq n$ ($\operatorname{Ind} X \leq n$) *if and only if there exists a σ-closure-preserving base $\mathcal{W}$ for X such that* $\dim B(W) \leq n-1$ (*resp.* $\operatorname{Ind} B(W) \leq n-1$) *for every $W \in \mathcal{W}$, where $B(W)$ denotes the boundary of W in X.*

In a quite recent paper [**9**], the author defined a class $\mathcal{M}$ of stratifiable spaces with M-structures and studied the properties of $\mathcal{M}$, some of which are as follows:

(M1) $\mathcal{M} \subset \{M_1\text{-spaces}\}$.

(M2) A space X is an M_0-space if and only if $X \in \mathscr{M}$ and $\dim X = 0$.

(M3) $\mathscr{M}$ is hereditary with respect to subspaces.

(M4) $\mathscr{M}$ has the countably productive property.

(M5) If $X \in \mathscr{M}$, then there exists a perfect mapping f of an M_0-space onto X. Moreover, if $\dim X \leq n$, then f can be chosen so that $\operatorname{ord} f \leq n + 1$.

(M6) Every stratifiable μ-space belongs to $\mathscr{M}$.

From these results it follows that all statements of (P1) to (P7) hold for the class $\mathscr{M}$. This implies that all the problems are solved positively if it can be shown that (P8) is true.

(P8) *Does every stratifiable space have an M-structure?*

By [**5**] and (M3), (P8) is equivalent to (P9).

(P9) *Does every M_1-space have an M-structure?*

On the other hand, recently Tamano defined in [**12**] strongly regularly stratifiable spaces and showed that the statements of (M1), (M3), (M5), (M6) and the if part of (M2) are true for these spaces.

Now, the purpose of this paper is to show two theorems. In Theorem 1 we show that every strongly regularly stratifiable space belongs to $\mathscr{M}$. But we do not know the converse. In Theorem 2 we give a characterization of $\mathscr{M}$ in terms of "an interior-preserving family" and "a network". From this result we recognize that the notion of "an interior-preserving family" plays some role in order to establish the class $\mathscr{M}$.

In this paper, all spaces are assumed to be regular Hausdorff and N always denotes the set of all positive integers. If $\mathscr{U}$ is a family of subsets of a space X, then $\mathscr{U}$ is called simply *a family of* X. If each $U \in \mathscr{U}$ is open (closed) in X, then $\mathscr{U}$ is called *an open* (resp. *closed*) family of X. $\mathscr{U}^{\#}$ denotes the union of all members of $\mathscr{U}$. If $\mathscr{U}$ consists of neighborhoods of a subset M in X such that if V is an open set of X with $M \subset V$, then $M \subset U \subset V$ for some $U \subset \mathscr{U}$, then we call $\mathscr{U}$ *a neighborhood base of* M *in* X. Moreover, if every $U \in \mathscr{U}$ is open in X, then $\mathscr{U}$ is called *an open neighborhood base of* M *in* X. Let $\mathscr{K}_i$, $i = 1,\dots,k$, be families of X. Then we denote by $\mathscr{C}(\mathscr{K}_1,\dots,\mathscr{K}_k)$ the family of all members of $\bigcup_{i=1}^{k} \mathscr{K}_i$ and all finite intersections of members of $\bigcup_{i=1}^{k} \mathscr{K}_i$. It is easy to see that $\mathscr{C}(\mathscr{K}_1,\dots,\mathscr{K}_k)$ is a σ-discrete closed family of X if each $\mathscr{K}_i$ is so. For a subset A of X, $\overline{A}$ denotes the closure of A in X. Finally, we give the definition of *pair collections* (see [**12**, Definition 2.1]). Let $\mathscr{P}$ be a collection of pairs (F, U) such that F is a closed set and U is an open set of X with $F \subset U$. In the discussion below, $\mathscr{P}$ denotes a pair collection. Let M be a subset of X and define

$$\mathscr{A}(M, \mathscr{P}) = \{U : (F, U) \in \mathscr{P} \text{ for some closed set } F \text{ of } X \text{ such that } F \cap (X - M) \neq \varnothing\}$$

and

$$A(M, \mathscr{P}) = \mathscr{A}(M, \mathscr{P})^{\#}.$$

2. Theorems. To begin with, we state the definitions of the class $\mathscr{M}$ and strongly regularly stratifiable spaces.

DEFINITION 1 [**9**, Definition 1.1]. Let $\mathscr{U}$, $\mathscr{H}$ be families of a space X. We say that $\mathscr{U}$ is *$\mathscr{H}$-preserving in both sides at a point p* if the following two conditions are satisfied: Let $\mathscr{U}_0$ be an arbitrary subfamily of $\mathscr{U}$.

(1) If $p \in \bigcap\mathscr{U}_0$, then $p \in H \subset \bigcap\mathscr{U}_0$ for some $H \in \mathscr{H}$.

(2) If $p \in X - \mathscr{U}_0^{\#}$, then $p \in H \subset X - \mathscr{U}_0^{\#}$ for some $H \in \mathscr{H}$.

DEFINITION 2 [**9**, Definition 1.2]. Let $\mathscr{P} = \langle\{\mathscr{U}(F) : F \in \mathscr{F}\}, \mathscr{H}\rangle$ be a collection of families of a space X. Then $\mathscr{P}$ is called *an M-structure* of X if the following three conditions are satisfied:

(1) $\mathscr{F}$, $\mathscr{H}$ are σ-discrete closed families of X.

(2) Each $\mathscr{U}(F)$ is a family of open neighborhoods of F in X and is $\mathscr{H}$-preserving in both sides at each point of $X - F$.

(3) If $p \in G$ for a point p and an open set G of X, then there exist $F \in \mathscr{F}$ and $U \in \mathscr{U}(F)$ such that $p \in F \subset U \subset G$.

Let $\mathscr{M}$ be the class of all stratifiable spaces with M-structures.

DEFINITION 3 (Tamano [**12**]). A space X is called *a strongly regularly stratifiable space* if there exists a base $\bigcup_{n=1}^{\infty} \mathscr{B}_n$ for X such that each $\mathscr{B}_n$ has a σ-locally finite, point-finite and finitely approaching stratifier. (As for the terminology used here, see [**12**, Definition 2.1, 3.1, 4.1 and 4.2]. But the meaning is given in the proof below.)

LEMMA 1. *Let M be a closed subset of a stratifiable space and $\mathscr{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a family of open neighborhoods of M in X, which has a σ-locally finite, point-finite and finitely approaching pair cover $\mathscr{P}$ such that*

$$\overline{A(U_\lambda, \mathscr{P})} \cap M = \varnothing$$

for each $\lambda \in \Lambda$. *Then there exists a collection*

$$\langle \mathscr{V}(M) = \{V_\lambda : \lambda \in \Lambda\}, \mathscr{K}(M)\rangle$$

of families of X *satisfying the following two conditions*:

(1) $\mathscr{K}(M)$ *is a* σ*-discrete closed family of* X.

(2) *Each* V_λ *is an open neighborhood of* M *in* X *such that* $M \subset V_\lambda \subset U_\lambda$ *and* $\mathscr{V}(M)$ *is* $\mathscr{K}(M)$*-preserving in both sides at each point of* $X - M$.

Before the proof, we recall the method of Siwiec and Nagata. They showed in **[11]** that a space X is a σ-space if and only if X has a σ-closure-preserving closed network. If we read their proof, then we easily understand the fact that if $\mathscr{B}$ is a closure-preserving closed family of a σ-space X, then there exists a σ-discrete closed network $\mathscr{F}$ of X such that $\mathscr{B}$ is $\mathscr{F}$-preserving in both sides at each point of X. In the proof below, we say that *by the method of Siwiec and Nagata* we obtain $\mathscr{F}$ with this property.

Proof. For each pair $(F, U) \in \mathscr{P}$, choose an open set $V(F, U)$ of X such that $F \subset V(F, U) \subset \overline{V(F, U)} \subset U$. Note that $\{\overline{V(F, U)}: (F, U) \in \mathscr{P}\}$ is σ-locally finite and point-finite in X because $\mathscr{P}$ is so. Since X is a σ-space, by the method of Siwiec and Nagata there exists a σ-discrete closed family $\mathscr{K}_1$ such that each $\overline{V(F, U)}$ is the union of members of $\mathscr{K}_1$. For each $\lambda \in \Lambda$, let

$$\mathscr{V}(\lambda) = \{\overline{V(F, U)} : F \cap (X - U_\lambda) \neq \varnothing\} \quad \text{and}$$

$$V(\lambda) = X - \mathscr{V}(\lambda)^{\#}.$$

Then each V_λ is an open neighborhood of M in X such that $M \subset V_\lambda \subset U_\lambda$ because

$$\overline{A(U_\lambda, \mathscr{P})} \cap M = \varnothing$$

and $\{\overline{V(F, U)} : (F, U) \in \mathscr{P}\}$ is locally finite at each point of U_λ. Since $\mathscr{P}$ is point-finite,

$$\mathscr{W} = \{X - U : (F, U) \in \mathscr{P} \text{ for some closed set } F \text{ of } X\}$$

is a closure-preserving closed family of X. By the method fo Siwiec and Nagata, we can get a σ-discrete closed network $\mathscr{K}_2$ of X such that $\mathscr{W}$ is $\mathscr{K}_2$-preserving in both sides at each point of $X - M$. Let

$$\mathscr{V}(M) = \{V_\lambda : \lambda \in \Lambda\} \quad \text{and} \quad \mathscr{K}(M) = \mathscr{C}(\mathscr{K}_1, \mathscr{K}_2).$$

We shall show that $\mathscr{V}(M)$ is $\mathscr{K}(M)$-preserving in both sides at each point of $X - M$. Let $\mathscr{V}_0 = \{V_\lambda : \lambda \in \Lambda_0\}$ be an arbitrary subfamily of $\mathscr{V}(M)$.

Assume $p \in \bigcap \mathscr{V}_0$. Since $\mathscr{W}$ is $\mathscr{K}_2$-preserving in both sides at p, there exists $K_1 \in \mathscr{K}_2$ such that $p \in K_1$ and

$$K_1 \cap \bigcup\{U : (F, U) \in \mathscr{P} \text{ for some closed set } F \text{ of } X \text{ and } p \notin U\} = \varnothing.$$

Since $\mathscr{P}$ is point-finite, there exists $K_2 \in \mathscr{K}_2$ such that

$$p \in K_2 \subset \bigcap\{U - \overline{V(F,U)} : (F,U) \in \mathscr{P} \text{ and } p \in U - \overline{V(F,U)}\}.$$

Then easily we have $p \in K_1 \cap K_2 = K \in \mathscr{K}(M)$ and $K \subset \bigcap \mathscr{V}_0$. Assume $p \in X - \mathscr{V}_0^{\#}$. Since $\{\overline{V(F,U)} : (F,U) \in \mathscr{P}\}$ is point-finite at p, we easily have an intersection K of finitely many members of $\mathscr{K}_1$ (and hence $K \in \mathscr{K}$) such that $p \in K$ and

$$K \subset \bigcap\{\overline{V(F,U)} : p \in \overline{V(F,U)}, (F,U) \in \mathscr{P}\}.$$

This implies that

$$p \in K \subset X - \mathscr{V}_0^{\#}.$$

For brevity, we say that M has *a frame of s.r. stratifier* $\langle \mathscr{U}, \mathscr{P} \rangle$ when $\mathscr{U}$ and $\mathscr{P}$ satisfy the condition stated in the assumption of Lemma 1.

LEMMA 2. *Let $\mathscr{F}$ be a σ-discrete closed family of a stratifiable space X such that each $F \in \mathscr{F}$ has a frame of s.r. stratifier*

$$\langle \mathscr{U}(F) = \{U_\lambda : \lambda \in \Lambda(F)\}, \mathscr{P}(F) \rangle.$$

Then there exists a collection $\langle \{\mathscr{V}(F) : F \in \mathscr{F}\}, \mathscr{K} \rangle$ of families of X satisfying the following three conditions:

(1) *$\mathscr{K}$ is a σ-discrete closed family of X.*

(2) *For each $F \in \mathscr{F}$, $\mathscr{V}(F) = \{V_\lambda : \lambda \in \Lambda(F)\}$ is $\mathscr{K}$-preserving in both sides at each point of $X - F$.*

(3) *For each $F \in \mathscr{F}$, $\mathscr{V}(F)$ is a family of open neighborhoods of F in X such that $F \subset V_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda(F)$.*

Proof. Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$, where each $\mathscr{F}_n$ is a discrete closed family of X. Let $\{V_F : F \in \mathscr{F}_n\}$ be an open family of X such that $F \subset V_F$ for each $F \in \mathscr{F}_n$ and $\{\overline{V}_F : F \in \mathscr{F}_n\}$ is discrete in X. By Lemma 1, for each $F \in \mathscr{F}$ there exists a collection $\langle \mathscr{V}'(F), \mathscr{K}(F) \rangle$ satisfying the conditions (1) and (2) of Lemma 1 with $F = M$ and $\Lambda = \Lambda(F)$. Let

$$\mathscr{V}(F) = \mathscr{V}'(F)/V_F$$

for each $F \in \mathscr{F}$. Let $\mathscr{K}_1$ be a σ-discrete closed network of X. Set

$$\mathscr{K}_2 = \bigcup\{\mathscr{K}(F)/\overline{V}_F : F \in \mathscr{F}\},$$

$$\mathscr{K}_3 = \{\overline{V}_F - V_F : F \in \mathscr{F}\} \quad \text{and} \quad \mathscr{K} = \mathscr{C}(\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3).$$

Then it is easy to see that $\langle\{\mathscr{V}(F) : F \in \mathscr{F}\}, \mathscr{K}\rangle$ is the required one.

THEOREM 1. *If a space X is strongly regularly stratifiable, then $X \in \mathscr{M}$.*

Proof. X is stratifiable by [**12**, Theorem 3.3]. It remains to prove that X has an M-structure. By the assumption, there exists a base $\bigcup_{n=1}^{\infty} \mathscr{B}_n$ for X such that each $\mathscr{B}_n$ has a σ-locally finite, point-finite and finitely approaching stratifier $\langle \mathscr{P}_{nk}\rangle_{k\in N}$ and X has a σ-discrete closed network $\mathscr{F}$. For each n, $k \in N$ let $\mathscr{F}_{nk} = \mathscr{F}$. For each n, $k \in N$ and each $F \in \mathscr{F}_{nk}$, let

$$\mathscr{U}_{nk}(F) = \{U \in \mathscr{B}_n : F \subset U \text{ and } \overline{A(U, \mathscr{P}_{nk})} \cap F = \varnothing\}$$

and write

$$\mathscr{U}_{nk}(F) = \{U_\lambda : \lambda \in \Lambda_{nk}(F)\}.$$

These imply that F has a frame of s.r. stratifier $\langle \mathscr{U}_{nk}(F), \mathscr{P}_{nk}\rangle$. Therefore by Lemma 2, there exists a collection

$$\langle\{\mathscr{V}_{nk}(F) = \{V_\lambda : \lambda \in \Lambda_{nk}(F)\} : F \in \mathscr{F}_{nk}\}, \mathscr{K}_{nk}\rangle$$

of families of X satisfying the following three conditions:

(1) $\mathscr{K}_{nk}$ is a σ-discrete closed family of X.

(2) For each $F \in \mathscr{F}_{nk}$, $\mathscr{V}_{nk}(F)$ is $\mathscr{K}_{nk}$-preserving in both sides at each point of $X - F$.

(3) For each $F \in \mathscr{F}_{nk}$, $\mathscr{V}_{nk}(F)$ is a family of open neighborhoods of F in X such that $F \subset V_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda_{nk}(F)$.

Let

$$\mathscr{K} = \bigcup\{\mathscr{K}_{nk} : n, k \in N\}$$

and for each $F \in \mathscr{F}_{nk}$, let $\mathscr{W}(F) = \mathscr{V}_{nk}(F)$. Then it is easily seen that $\mathscr{K}$ is a σ-discrete closed family and $\mathscr{W}(F)$ is $\mathscr{K}$-preserving in both sides at each point of $X - F$. Let $p \in G$ for a point p and an open set G of X. Take $U \in \mathscr{B}_n$ and $k \in N$ such that $p \in U \subset G$ and $p \notin \overline{A(U, \mathscr{P}_{nk})}$. Since $\mathscr{F}_{nk}$ is a network of X, there exists $F \in \mathscr{F}_{nk}$ such that $p \in F \subset X - \overline{A(U, \mathscr{P}_{nk})}$. Then $U = U_\lambda \in \mathscr{U}_{nk}(F)$ for some $\lambda \in \Lambda_{nk}(F)$. Hence we have

$$p \in F \subset V_\lambda \subset U_\lambda \subset U \subset G.$$

This proves that $\langle\{\mathscr{V}(F) : F \in \bigcup\{\mathscr{F}_{nk} : n, k \in N\}\}, \mathscr{K}\rangle$ is an M-structure of X.

A family $\mathscr{U}$ of a space X is said to be *interior-preserving at a point p in X* when $p \in \bigcap\{\operatorname{int} U : U \in \mathscr{U}_0\}$ if and only if $p \in \operatorname{Int}\bigcap\mathscr{U}_0$ for each subfamily $\mathscr{U}_0$ of $\mathscr{U}$ and is said to be *interior-preserving in X* if $\mathscr{U}$ is so at each point of X in X.

DEFINITION 4. Let $\mathcal{U}$ be a family of a space X. We say that $\mathcal{U}$ is *interior-preserving on a network* if there exists a σ-discrete closed network $\mathcal{F}$ of X such that for each $F \in \mathcal{F}$, $\mathcal{U}/F$ is interior-preserving in the subspace F.

LEMMA 3 [**9**, Lemma 3.5]. *For a stratifiable space X the following statements are equivalent*:

(1) $X \in \mathcal{M}$.

(2) *Every closed subset M of X has a collection $\langle \mathcal{U}(M), \mathcal{F} \rangle$ of families of X such that $\mathcal{U}(M)$ is an open neighborhood base of M in X and is $\mathcal{F}$-preserving in both sides at each point of $X - M$.*

LEMMA 4. *Let $\mathcal{U}$ be an open family of a stratifiable space X such that $\mathcal{U}$ is interior-preserving on a network. Then there exists a σ-discrete closed family $\mathcal{K}$ of X such that $\mathcal{U}$ is $\mathcal{K}$-preserving in both sides at each point of X.*

Proof. Suppose that we are given such a family $\mathcal{U}$ stated in the lemma. Let $\mathcal{F}$ be a σ-discrete closed network of X such that $\mathcal{U}/F$ is interior-preserving in the subspace F for each $F \in \mathcal{F}$. Since $\{F - U : U \in \mathcal{U}\}$ is a closure-preserving closed family of F, by the method of Siwiec and Nagata there exists a σ-discrete closed family $\mathcal{K}(F)$ of F such that $\{F - U : U \in \mathcal{U}\}$ is $\mathcal{K}(F)$-preserving in both sides at each point of F. Set

$$\mathcal{K} = \bigcup \{ \mathcal{K}(F) : F \in \mathcal{F} \}.$$

Then $\mathcal{K}$ is a σ-discrete closed family of X. It is easy to see that $\mathcal{U}$ is $\mathcal{K}$-preserving in both sides at each point of X.

LEMMA 5 [**4**, Lemma 4.4]. *Let $\mathcal{U}$ be an open cover of a stratifiable space X, and let $\mathcal{V}$ be a locally finite family such that every union of members of $\mathcal{V}$ is regular open. Then there exists a locally finite open refinement $\mathcal{W}$ of $\mathcal{U}$ such that every union of members of $\mathcal{V} \cup \mathcal{W}$ is regular open.*

LEMMA 6. *Let $\mathcal{F}_n$, $n \in N$, be a closed family and $\{U_F : F \in \mathcal{F}_n\}$, $n \in N$, a discrete open family of a stratifiable space X such that $F \subset U_F$ for each $F \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then there exists families $\mathcal{V}_n = \{V_F : F \in \mathcal{F}_n\}$, $n \in N$, of X such that $F \subset V_F \subset U_F$ for each $F \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$ and for each $n \in N$ every union of members of $\bigcup_{j=1}^{n} \mathcal{V}_j$ is regular open.*

Proof. We easily obtain $\mathscr{V}_1$. Assume that $\mathscr{V}_1, \ldots, \mathscr{V}_n$ have been constructed. Let $\mathscr{U}$ be an open cover of X such that $\mathscr{U}$ is a star-refinement of

$$\{U_F\colon F \in \mathscr{F}_{n+1}\} \cup \{X - \mathscr{F}_{n+1}^{\#}\}$$

and let $\mathscr{V} = \bigcup_{j=1}^{n} \mathscr{V}_j$. By the preceding lemma, there exists a locally finite open refinement $\mathscr{W}$ of $\mathscr{U}$ such that every union of members of $\mathscr{V} \cup \mathscr{W}$ is regular open. For each $F \in \mathscr{F}_{n+1}$, let $V_F = S(F, \mathscr{W})$. Then it is easy to see that $\mathscr{V}_{n+1} = \{V_F\colon F \in \mathscr{F}_{n+1}\}$ satisfies the required conditions. This completes the proof.

THEOREM 2. *For a stratifiable space X the following statements are equivalent*:

(1) $X \in \mathscr{M}$.

(2) *Every closed subset of X has a closure-preserving open neighborhood base which is interior-preserving on a network.*

(3) *Every closed subset of X has an open neighborhood base which is interior-preserving on a network.*

Proof. (1) $\to$ (2): Let M be a closed subset of $X \in \mathscr{M}$. By Lemma 3, M has a collection $\langle \mathscr{U}(M) = \{U_\alpha\colon \alpha \in A\}, \mathscr{F}\rangle$ of families stated in (2) of the lemma, where we assume that every finite intersection of members of $\mathscr{F}$ belongs to $\mathscr{F}$. We recall that X has a monotonically normal operator $D(M, N)$, [**6**]. Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$, where each $\mathscr{F}_n$ is a discrete closed family of X. For each $n \in N$, let $\{U'_F\colon F \in \mathscr{F}_n\}$ be a discrete open family of X such that $F \subset U'_F$ for each $F \in \mathscr{F}_n$. We assume that $F \cap W = \varnothing$ for every $F \in \mathscr{F}$. For each $n \in N$ and each $F \in \mathscr{F}_n$, we choose an open set U_F of X such that

$$(1)\quad F \subset U_F \subset \overline{U_F} \subset U'_F \cap D\left(F, M \cup \left(\bigcup\left\{F' \in \bigcup_{t=1}^{n} \mathscr{F}_t, F' \cap F = \varnothing\right\}\right)\right).$$

Note that each $U_\alpha = X - \mathscr{F}_\alpha^{\#}$, where

$$\mathscr{F}_\alpha = \{F \in \mathscr{F}\colon F \cap U_\alpha = \varnothing\}.$$

We apply Lemma 6 to $\mathscr{F}_n$, $\{U_F\colon F \in \mathscr{F}_n\}$, $n \in N$, to get $\mathscr{V}_n = \{V_F\colon F \in \mathscr{F}_n\}$, $n \in N$, satisfying the following conditions:

$$(2)\qquad F \subset V_F \subset U_F \quad \text{for every } F \in \bigcup_{n=1}^{\infty} \mathscr{F}_n.$$

(3) For every $n \in N$, every union of members of $\bigcup_{t=1}^{n} \mathscr{V}_t$ is regular open in X.

For each $\alpha \in A$, let

$$H_\alpha = X - \bigcup\{V_F : F \in \mathscr{F}_\alpha\} \quad \text{and} \quad V_\alpha = \operatorname{Int} H_\alpha.$$

Claim 1. *Every V_α is an open neighborhood of M in X.*
Since by (1) and (2)

$$\overline{D(X - U_\alpha, M)} \cap M = \varnothing \quad \text{and} \quad V_F \subset D(X - U_\alpha, M)$$

for each $F \in \mathscr{F}_\alpha$, we have $M \subset V_\alpha$. This proves Claim 1.

Claim 2. $\overline{V_\alpha} = H_\alpha$ *for each* $\alpha \in A$.
To see this, it suffices to show that $\bigcup\{V_F : F \in \mathscr{F}_\alpha\}$ is regular open. Assume that

$$p \in \operatorname{Int}\overline{\left(\bigcup\{V_F : F \in \mathscr{F}_\alpha\}\right)} - \bigcup\{V_F : F \in \mathscr{F}_\alpha\}.$$

Since $p \in U_\alpha - M$ and $\mathscr{U}(M)$ is $\mathscr{F}$-preserving in both sides at p, there exists $F(p) \in \mathscr{F}_n$, $n \in N$, such that

$$p \in F(p) \subset U_\alpha - M.$$

By (1) and (2) we observe that

$$F(p) \cap \overline{D(X - U_\alpha, M \cup F(p))} = \varnothing$$

and

$$V_F \subset D(X - U_\alpha, M \cup F(p))$$

for each $F \in \mathscr{F}_\alpha \cap (\bigcup_{t=n}^{\infty} \mathscr{F}_t)$. Therefore we have

$$p \notin \overline{\bigcup\left\{V_F : F \in \mathscr{F}_\alpha \cap \left(\bigcup_{t=n}^{\infty} \mathscr{F}_t\right)\right\}}.$$

Thus p belongs to the interior of the closure of

$$\bigcup\left\{V_F : F \in \mathscr{F}_\alpha \cap \left(\bigcup_{t=1}^{n-1} \mathscr{F}_t\right)\right\}.$$

This contradicts (3).

Claim 3. $\mathscr{V} = \{V_\alpha : \alpha \in A\}$ *is closure-preserving in* X.
By virtue of Claim 2, it suffices to show that $\{H_\alpha : \alpha \in A\}$ is closure-preserving in X. Let A_0 be an arbitrary subset of A and suppose that

$$p \in X - \bigcup\{H_\alpha : \alpha \in A_0\}.$$

Let

$$A_1 = \{\alpha \in A_0 : p \in X - U_\alpha\} \quad \text{and} \quad A_2 = A_0 - A_1.$$

Since $\mathscr{U}(M)$ is $\mathscr{F}$-preserving in both sides at p, there exist $F_1 \in \mathscr{F}$ and $F_2 \in \mathscr{F}_n$, $n \in N$, such that

$$p \in F_1 \subset X - \bigcup\{U_\alpha : \alpha \in A_1\}$$

and

$$p \in F_2 \subset \bigcap\{U_\alpha : \alpha \in A_2\}.$$

Observe by (1) that for each $F \in (\bigcup_{\alpha \in A_2} \mathscr{F}_\alpha) \cap (\bigcup_{t=n}^{\infty} \mathscr{F}_t)$ it is true that $V_F \cap F_2 = \varnothing$. Since $\{V_F : F \in \bigcup_{t=1}^{n-1} \mathscr{F}_t\}$ is locally finite at p, there exists an open neighborhood O of p such that

$$O \cap \bigcup\{H_\alpha : \alpha \in A_2\} = \varnothing.$$

It is easy to see that $N = O \cap V_{F_1}$ is an open neighborhood of p such that $N \cap H_\alpha = \varnothing$ for every $\alpha \in A_0$, proving that $p \notin \overline{\bigcup\{H_\alpha : \alpha \in A_0\}}$.

Claim 4. *$\mathscr{V}$ is interior-preserving on a network.*

For each $p \in X - M$, choose $F(p) \in \mathscr{F}$ such that

$$p \in F(p) \subset \bigcap\{U_\alpha : \alpha \in A, p \in U_\alpha\} - \bigcup\{U_\alpha : \alpha \in A, p \notin U_\alpha\}.$$

Set

$$\mathscr{F}' = \{M\} \cup \{F(p) : p \in X - M\}.$$

Obviously $\mathscr{V}/M$ is interior-preserving in the subspace M. Let $F = F(p) \in \mathscr{F}'$ be an arbitrary member. We shall show that $\mathscr{V}/F$ is interior-preserving in the subspace F. Let $\mathscr{V}_0 = \{V_\alpha : \alpha \in A_0\}$ be an arbitrary subfamily of $\mathscr{V}$ and suppose that $x \in (\bigcap \mathscr{V}_0) \cap F$. Let

$$N(x) = F \cap \left(X - \bigcup\left\{ \overline{V}_{F'} : x \notin \overline{V}_{F'} \text{ and } F' \in \bigcup_{t=1}^{n} \mathscr{F}_t \right\}\right),$$

where $F \in \mathscr{F}_n$, $n \in N$. By Claim 2 and the proof of Claim 3, and by the fact that

$$F \cap \left[\bigcup\left\{ \overline{V}_{F'} : F' \in \left(\bigcup_{\alpha \in A_0} \mathscr{F}_\alpha \right) \cap \left(\bigcup_{t=n}^{\infty} \mathscr{F}_t \right)\right\}\right] = \varnothing,$$

we have

$$N(x) \subset \bigcap\{V_\alpha : \alpha \in A_0\}.$$

This shows that $\mathscr{V}/F$ is interior-preserving in the subspace F. From this it is easily seen that there exists a σ-discrete closed network on which $\mathscr{V}$ is interior-preserving. Hence we complete the proof of (1) $\rightarrow$ (2). (2) $\rightarrow$ (3) is trivial. (3) $\rightarrow$ (1) follows from Lemmas 3 and 4.

Acknowledgment. After the completion of the paper, the author found that the argument of Lemma 1 applies easily to the case of regularly stratifiable spaces, and consequently reached to the result that all regularly stratifiable spaces belong to $\mathscr{M}$.

REFERENCES

[1] C. R. Borges, *On stratifiable spaces*, Pacific J. Math., **17** (1966), 1–16.

[2] J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **11** (1961), 105–125.

[3] G. Gruenhage, *Stratifiable spaces are M_2*, Topology Proc., **1** (1976), 221–226.

[4] G. Gruenhage, *On the $M_3 \Rightarrow M_1$ question*, Topology Proc., **5** (1980), 77–104.

[5] R. W. Heath and H. J. Junnila, *Stratifiable spaces are subspaces and continuous images of M_1-spaces*, Proc. Amer. Math. Soc., **83** (1981), 146–148.

[6] R. W. Heath, D. J. Lutzer and P. L. Zenor, *Monotonically normal spaces*, Trans. Amer. Math. Soc., **178** (1973), 481–493.

[7] H. J. Junnila, *Neighbornets*, Pacific J. Math., **76** (1978), 83–108.

[8] T. Mizokami, *On a certain class of M_1-spaces*, Proc. Amer. Math. Soc., **87** (1983), 357–362.

[9] T. Mizokami, *On M-structures*, to appear in Top. Appl.

[10] K. Nagami, *Normality of products*, Actes Congres Inter. Math., **2** (1970), 33–37.

[11] F. Siwiec and J. Nagata, *A note on nets and metrization*, Proc. Japan Acad., **44** (1968), 623–627.

[12] K. Tamano, *Stratifiable spaces defined by pair collections*, to appear in Topology Appl.

Received May 17, 1983 and in revised form August 24, 1983.

JOETSU UNIVERSITY OF EDUCATION
YAMAYASHIKI, JOETSU
NIIGATA 943 JAPAN