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SINGULAR LIMITS OF QUASI-LINEAR HYPERBOLIC SYSTEMS IN A BOUNDED DOMAIN OF $\mathbf{R}^3$ WITH APPLICATIONS TO MAXWELL'S EQUATIONS

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We establish a singular perturbation result for quasi-linear hyperbolic systems in a bounded domain of $\mathbf{R}^3$, depending on a small parameter. We prove and estimate the rate of convergence, as the parameter tends to zero, of uniformly stable solutions of the complete system to a solution of the reduced system. This result is then applied to the study of the convergence of the complete Maxwell equations to the quasi-stationary ones.

1. Introduction. In this paper we are concerned with the singular perturbation problem consisting in the study of the behavior of the solution of an initial-boundary value problem for a quasi-linear hyperbolic system of the type

$$\varepsilon u_{tt} + \sigma u_t + L(u)u = 0 \tag{cs}$$

when the small parameter $\varepsilon > 0$ tends to zero. There are two main questions which are related: (A) whether the solutions of (cs), which we denote by u_ε, tend in some sense to a solution of the parabolic system

$$\sigma u_t + L(u)u = 0, \tag{qs}$$

and (B) to compare in some sense the solutions of (cs) to any solution of (qs) that can be obtained independently.

Singular perturbation problems of this type are extensively considered in Lions' book [**4**], in which however the above type of "hyperbolic $\to$ parabolic" convergence is studied only in the linear case. We present here some results for the quasi-linear case in a rather special situation, considering three-dimensional vectors $u = u(x, t)$ defined on a bounded region $\Omega \times [0, +\infty[$ of $\mathbf{R}^{3+1}$ (although the method of proof suggests that a proper extension would provide results in any number of dimensions). A more stringent limitation is that the coefficients of the elliptic operator $L(u)u$ depend only on the spatial derivatives of u; suitable dependence on u and u_t could be allowed, although it seems possible to obtain appropriate results only if this dependence is on $\varepsilon^{1/2}u_t$ rather than on u_t.

The results we obtain are local in time, and we make essential use of both the strong ellipticity of $L(u)$ and of the presence in (cs) of the "positive" dissipation term σu_t (we mention in passing that such a term is essential also in investigating the long time behavior of the solutions of (cs); see [7]). We are motivated in our study by the "approximation" problem that arises when displacement currents are neglected in the determination of the electromagnetic field in a ferromagnetic material. Such neglect, which is usual for instance in the study of transformer cores, has the effect of reducing the complete system CS of Maxwell's equations, which is of hyperbolic type, to the parabolic type system of the quasi-stationary equations QS. These systems are quasi-linear, because of the nonlinearity of the magnetic characteristic in ferromagnetic materials; and it is precisely in this case that displacement currents are known to be negligible. A first step in studying the related singular perturbation problem was taken in [**6**], where we provided results in all of $\mathbf{R}^3$ showing that indeed the reduced equations QS are the singular limit of the complete equations CS at the vanishing of the dielectric constant ε. However, since the quasilinear nature of the equations is due to the electromagnetic field in a ferromagnetic material, it is physically more relevant to obtain the same results in a bounded region of space. This in general presents more difficulties, both because fewer results are available for the initial-boundary value problem for quasi-linear hyperbolic systems, and because the (rather standard) technique that is used to obtain the necessary stability estimates on the solutions of CS in the whole space would not be effective in a bounded domain, due to the presence of boundary conditions and the prescription of compatibility conditions on the initial data that make direct differentiation of the equations with respect to the space variables of no use. The aim of this work is to show how, using only differentiation with respect to time, it is possible to establish for the bounded domain case essentially the same type of estimate and results that were obtained in [**6**] for the whole space case. Differently than in [**6**], we shall not consider Maxwell's equations directly, which are a first order system, but rather transform them into a second order system of the type (cs), using scalar and vector potentials for the fields, with a procedure already followed in [**8**]. We conclude by mentioning that when the magnetic characteristic is linear, Lions' techniques can be applied directly to the Maxwell equations; an illustration of this can be found in [**9**].

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2. The singular limit problem. Let $\Omega \subseteq \mathbf{R}^3$ be a bounded open domain with smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem for $u_\varepsilon = u_\varepsilon(x, t) \in \mathbf{R}^3$, $x \in \Omega$, $t \geq 0$:

$$(H_\varepsilon)\qquad \begin{cases} (2.1)\quad \varepsilon u_\varepsilon'' + \sigma u_\varepsilon' - \sum_{i,j=1}^{3} a_{ij}(\partial u_\varepsilon)\partial_i\partial_j u_\varepsilon = 0 \\ (2.2)\quad u_\varepsilon(0) = u_{0\varepsilon}, \qquad u_\varepsilon'(0) = u_{1\varepsilon} \\ (2.3)\quad u_\varepsilon|_{\partial\Omega} = 0 \end{cases} \quad \text{in } \Omega,$$

where ε and σ are positive constants, $u' = \partial u/\partial t$, $\partial_j u = {}^t\{\partial_j u^1, \partial_j u^2, \partial_j u^3\}$, $\partial_j u^h = \partial u^h/\partial x_j$ etc.; the a_{ij}'s are 3×3 real valued matrices satisfying ${}^t a_{ij} = a_{ij}$ and ∂u represents the collection of all first order spatial derivatives of u. We assume that for all (sufficiently regular) vector functions $p(x, t) \in \mathbf{R}^3$, the (linear) operator

$$L(p)u = -\sum_{i,j=1}^{3} a_{ij}(p)\partial_i\partial_j u$$

is uniformly strongly elliptic in the sense that the following matrix inequality holds:

$$(2.4)\qquad \exists\lambda_0 \in \mathbf{R}^+;\ \forall\lambda \in \mathbf{R}^3, \forall p, \quad \sum_{i,j=1}^{3} a_{ij}(p)\lambda_i\lambda_j \geq \lambda_0|\lambda|^2;$$

we also assume that the matrices a_{ij} are at least three times continuously differentiable, with uniformly bounded derivatives, so that $L(p)$, considered for each fixed p as a linear operator between two Banach spaces, has uniformly bounded derivatives:

$$(2.5)\qquad \forall p, \quad \|L^{(k)}(p)\| \leq \delta_k, \qquad 0 \leq k \leq 3,$$

the norm being that of the proper spaces of linear functionals (this assumption might be somewhat relaxed, assuming instead for $L(p)$ a polynomial growth of suitable degree). We have explicitly noted in (H_ε) the dependence of the unknown u on the parameter ε, since we want to compare this system with the reduced parabolic system

$$(P)\qquad \begin{cases} (2.6)\quad \sigma u' - \sum_{i,j=1}^{3} a_{ij}(\partial u)\partial_i\partial_j u = 0 \\ (2.7)\quad u(0) = u_0 \\ (2.8)\quad u|_{\partial\Omega} = 0 \end{cases} \quad \text{in } \Omega,$$

formally obtained from (H_ε) by setting $\varepsilon = 0$, and more precisely to investigate the behavior of solutions of (H_ε) as $\varepsilon \to 0$ and the problem of their convergence to the solution of (P). Adapting a definition of Hoppensteadt (see [**2**]), we shall say

DEFINITION. (H_ε) degenerates regularly to (P) on $[0, T]$ as $\varepsilon \to 0$ if $u_{0\varepsilon} \to u_0$ uniformly in $\overline{\Omega}$ and $u_\varepsilon \to u$, $u'_\varepsilon \to u'$ uniformly on $[\tau, T] \times \overline{\Omega}$ $\forall \tau \in]0, T]$, where u_ε and u are solutions respectively of (H_ε) and (P) on $[0, T]$.

This notion is necessary in that convergence, if it occurs, is in general singular in time, due to the loss of one initial condition: we are actually in the presence of a boundary layer problem in time.

Local existence in time and uniqueness results for (H_ε) can be established for fixed ε using Kato's theory in [**3**]; such results however would not provide uniform estimates with respect to ε, and the (small) time interval of existence might indeed shrink to 0 as $\varepsilon \to 0$. We need therefore at first to ensure the existence of a family of solutions of (H_ε) that are stable with respect to the parameter ε, in the sense that such solutions are all defined on a fixed (small) time interval; and then we must provide uniform bounds on appropriate norms of the solutions, which will imply the convergence of (sequences of) such solutions to a limit that is a solution of (P). These bounds will also provide information relative to the "approximation" problem, permitting us to estimate the error that is made in considering (P) instead of (H_ε), with a suitable choice of u_0, essentially in terms of ε and the difference $u_{1\varepsilon} - u'(0)$ that accounts for the loss in initial data. As a final remark we mention that although we have considered homogeneous systems, the same results would apply in the more general case, at least if the inhomogeneous terms are small enough; this would in particular permit us to consider the problem in non-simply connected domains (see for instance [**10**] for the parabolic case of the quasistationary Maxwell equations).

3. Basic assumptions and main results. We recall that in the theory of quasi-linear hyperbolic systems (see for instance Kato, [**3**], and its references), solutions for a second order system such as (H_ε) are usually sought in the space $C(0, T; H^{s+1}) \cap C^1(0, T: H^s)$ with $s > 1 + n/2$; here $n = 3$ and we choose $s = 3$. We consider therefore the spaces

$$H_0 = \mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^3,$$

$$H_m = \mathbf{H}^m(\Omega) \cap \mathbf{H}^1_0(\Omega), \qquad \mathbf{H}^m(\Omega) = \left(H^m(\Omega)\right)^3 \quad \text{for } 1 \le m \le 4,$$

and note $\|\cdot\|_m$ the H^m norm in H_m. It is well known that in order for (H_ε) to be solvable, a necessary condition is that the initial data (2.2) satisfy certain restrictions, called the compatibility conditions. To describe these, we start with defining the operator $\tilde{M} = \tilde{M}(\varepsilon, \sigma)$ by

$$\tilde{M}w_n = \varepsilon w_{n+1} + \sigma w_n;$$

where w_n is a given sequence. Then, given a pair of smooth vector functions $\{w_0, w_1\}$, we generate a finite sequence $w_j = w_j(w_0, w_1)$ by setting

$$w_0 = w_0; \qquad w_1 = w_1;$$

$$w_2 = -\varepsilon^{-1}\{L(\partial w_0)w_0 + \sigma w_1\};$$

$$w_3 = -\varepsilon^{-1}\{L'(\partial w_0)(\partial w_1)w_0 + L(\partial w_0)w_1 + \sigma w_2\};$$

$$w_4 = -\varepsilon^{-1}\{L''(\partial w_0)(\partial w_1, \partial w_1)w_0 + L'(\partial w_0)(\partial w_2)w_0 + 2L'(\partial w_0)(\partial w_1)w_1 + L(\partial w_0)w_2 + \sigma w_3\}.$$

Defining then the set

$$\mathfrak{D} = \left\{w_0, w_1 \mid M^j w_k \in H_{4-(j+k)} \text{ for } j \leq k, 0 \leq j + k \leq 4\right\},$$

we require that the initial data (2.2) satisfy the regularity, compatibility and boundedness conditions

$$u_{0\varepsilon}, u_{1\varepsilon} \in \mathfrak{D}, \tag{3.1}$$

$$\|u_{0\varepsilon}\|_4 + \varepsilon^{1/2}\|u_{1\varepsilon}\|_3 = O(1) \quad \text{as } \varepsilon \to 0. \tag{3.2}$$

REMARK 1. In [6] the stronger assumption $\|u_{1\varepsilon}\|_3 = O(1)$ was made, but this turns out to be unnecessary. If such is assumed, however, it can be shown that stronger results would follow.

We are now ready to state our main results. Under the assumptions (2.5), (2.6), (3.1) and (3.2) we claim:

THEOREM 1. *There exist positive numbers T_0 and Δ_0 independent of ε such that all problems (H_ε) have solutions $u_\varepsilon \in \bigcap_{j=0}^4 C^j(0, T_0; H_{4-j})$ satisfying the uniform bounds*

$$\forall t \in [0, T_0], \quad \|u_\varepsilon(t)\|_4^2 + \varepsilon\|u_\varepsilon'(t)\|_3^2 + \int_0^t \|u_\varepsilon'(s)\|_3^2\, ds \leq \Delta_0^2, \tag{3.3}$$

$$\forall t \in [0, T_0], \quad \|u_\varepsilon'(t)\|_2^2 \leq \Delta_0^2(1 + \Delta_0)^4(1 + \varepsilon^{-1}e^{-at/\varepsilon}), \tag{3.4}$$

where $0 < a < 2\sigma$.

THEOREM 2. *There exist a subsequence* $\{u_\varepsilon\}$ *and a vector function* u *such that* $u_\varepsilon \to u$ *in* $L^\infty(0, T_0;\ H_4)$ *weak* and* $C(0, T_0;\ H_3)$ *uniformly*; $u'_\varepsilon \to u'$ *in* $L^2(0, T_0;\ H_3)$ *weak and* u *is the unique solution of* (P) *with* $u_0 = \text{w-lim}_{\varepsilon\to 0} u_{0\varepsilon}$ *in* H_4.

(Here and in the sequel, w-lim and s-lim mean limits taken respectively in the weak and in the strong topologies). We note from Theorem 2 that we do not have uniform convergence of u'_ε to u'; this singularity is to be expected, due to the loss of the initial condition on u'. Indeed, from (2.6) we deduce that the solution u of (P) determined by Theorem 2 satisfies $u'' \in L^2(0, T_0;\ H_1)$, so that $u' \in C(0, T_0;\ H_2)$ (and therefore $u \in C(0, T_0;\ H_4)$, see Remark 2); however, there is not, in general, any relation between $u'(0)$ and the $u_{1\varepsilon}$, and even if $\|u_{1\varepsilon}\|_3 = O(1)$, so that there exists $u_1 \in H_3$ such that $u_1 = \text{w-lim}_{\varepsilon\to 0} u_{1\varepsilon}$ in H_3 and $u_1 = \text{s-lim}_{\varepsilon\to 0} u_{1\varepsilon}$ in H_2 (by compactness), it need not to be true that $u_1 = u'(0)$, unless $u_{1\varepsilon}$ and $u_{0\varepsilon}$ satisfy additional restrictions. These are the so called initialization conditions; in the present case a sufficient one is the requirement that

$$\|\sigma u_{1\varepsilon} + L(\partial u_{0\varepsilon}) u_{0\varepsilon}\|_0 \to 0 \quad \text{as } \varepsilon \to 0; \tag{3.5}$$

in fact if it so happens, we have that, since $u_0 = \text{s-lim}_{\varepsilon\to 0} u_{0\varepsilon}$ in H_3, $L(\partial u_0) u_0 = \text{s-lim}_{\varepsilon\to 0} L(\partial u_{0\varepsilon}) u_{0\varepsilon}$ in H_1, and from

$$\begin{aligned}\|\sigma u_1 - \sigma u'(0)\|_0 \le & \|\sigma u_1 - \sigma u_{1\varepsilon}\|_0 + \|\sigma u_{1\varepsilon} + L(\partial u_{0\varepsilon}) u_{0\varepsilon}\|_0 \\ & + \|L(\partial u_0) u_0 - L(\partial u_{0\varepsilon}) u_{0\varepsilon}\|_0\end{aligned}$$

the fact that $u_1 = u'(0)$. We do not require here any such initialization condition (we remark that (3.5), together with (3.2), implies that $\|u_{1\varepsilon}\|_0 = O(1)$, which is in itself one type of initialization condition); indeed, we have a boundary layer in time. Estimates on the differences $u_\varepsilon - u$ and $u'_\varepsilon - u'$ are provided by

THEOREM 3. *Suppose* u *is a solution of* (P) *with* u_0 *given in* H_4 *such that* $u \in C(0, T;\ H_4)$ *and* $u' \in L^2(0, T;\ H_3)$ *for* $T \le T_0$. *Then* $\forall \eta > 0$

$$\begin{aligned}&\sup_{t\in[0,T]} \|u_\varepsilon(t) - u(t)\|_{4-\eta} + \left(\int_0^T \|u'_\varepsilon - u'\|_{3-\eta}^2\right)^{1/2} \\ &\quad \le k_{T,\eta}\left[\varepsilon^{1/2}\left(1 + \|u_{1\varepsilon} - u'(0)\|_0\right) + \|u_{0\varepsilon} - u_0\|_1\right]^{\eta/3}.\end{aligned} \tag{3.6}$$

This Theorem allows us to estimate the rate of convergence of u_ε and u'_ε if $\|u_{0\varepsilon} - u\| \to 0$ as $\varepsilon \to 0$; moreover, as a consequence for the boundary layer problem, we have

COROLLARY. *Suppose in addition that* $\|u_{0\varepsilon} - u_0\|_1 = O(\varepsilon^{1/2})$ *and* $\|u_{1\varepsilon}\|_0 = O(1)$ *as* $\varepsilon \to 0$. *Then* (H_ε) *degenerates regularly to* (P) *on* $[0, T]$ *as* $\varepsilon \to 0$.

REMARK 2. Let u be the solution of (P) determined by Theorem 2: then, as we have already mentioned, $u' \in C(0, T_0;\ H_2)$ and therefore $L(\partial u)u \in C(0, T_0;\ H_2)$. Since u is known to be in $C(0, T_0;\ H_3)$, we can regard $\forall t \in [0, T_0]$ $L(\partial u(t))$ as a linear strongly elliptic operator with coefficients in $\mathbf{H}^2(\Omega)$. It has been shown in [5] that this is sufficient to conclude that $\forall t$ $u(t) \in H_4$ (in fact, the Sobolev index 2 is greater than $n/2 = 3/2$ here), so that $u \in C(0, T_0;\ H_4)$. We point out that this argument does not depend on the particular values of s and n considered here, and the condition $s > 1 + n/2$ is sufficient to conclude in the same way that $u \in C(0, T_0;\ H_{s+1})$ if $u' \in L^2(0, T_0;\ H_s)$.

REMARK 3. An estimate analogous to (3.6) could be obtained for $\eta = 0$ (see Remark 5 in §6).

4. Proof of Theorem 1. The proof of Theorem 1 is established with a standard fixed point technique, carrying out suitable a priori estimates on the solutions of the linear systems obtained by linearization of (H_ε). More precisely, consider, for $T > 0$ the space

$$S_T = \bigcap_{j=0}^{4} C^j(0, T;\ H_{4-j})$$

and for $\varphi \in S_T$ the weighted norm

$$\begin{aligned}[\varphi]_T^2 &= \sup_{t\in[0,T]} [\varphi(t)]_1^2 + \int_0^T [\varphi]_2^2 \\ &\equiv \sup_{t\in[0,T]} \left\{\|\varphi(t)\|_4^2 + \varepsilon\|\varphi'(t)\|_3^2 + \|M\varphi'(t)\|_2^2 + \varepsilon\|M\varphi''(t)\|_1^2\right\} \\ &\quad + \int_0^T \left\{\|\varphi'\|_3^2 + \|M\varphi''\|_1^2\right\}\end{aligned}$$

where we have defined the linear operator $M = M(\varepsilon, \sigma, \partial_t)$ by

$$Mf = \varepsilon f' + \sigma f.$$

Theorem 1 is proved by a contraction argument on the subspace

$$\begin{aligned}X = X_{T_0,\Delta_0} = \Big\{ f \in S_{T_0} \big| f(0) = u_{0\varepsilon}, f'(0) = u_{1\varepsilon};\ [f]_{T_0} \le \Delta_0, \\ \|\varphi'(t)\|_2 \le \Delta_0(1 + \Delta_0)^2(1 + \varepsilon^{-1/2}e^{-at/2\varepsilon})\Big\}\end{aligned}$$

where $T_0 > 0$ and $\Delta_0 > 0$ are to be determined: more precisely, for $\varphi \in X$ let $\psi = \partial\varphi$ and define $u = \mathscr{A}(\varphi)$ to be the solution of the linear problem

$$(L_{0\varepsilon}) \qquad \begin{cases} \varepsilon u'' + \sigma u' + L(\psi)u = 0 \\ u(0) = u_{0\varepsilon}, \qquad u'(0) = u_{1\varepsilon} \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

We claim that

PROPOSITION 1. *There exist $T_0 > 0$ and $\Delta_0 > 0$ independent of* ε *such that*

(4.1) $\mathscr{A}$ *maps* X_{T_0,Δ_0} *into itself*;

(4.2) $\mathscr{A}$ *is a contraction in* X_{T_0,Δ_0} *for the norm*

$$\|f\|_w^2 = \sup_{t\in[0,T_0]} \left\{\|f(t)\|_1^2 + \varepsilon\|f'(t)\|_0^2\right\}.$$

The conclusion of Theorem 1 would then follow easily from Proposition 1.

REMARK 4. Because of (3.1), the initial data in $(L_{0\varepsilon})$ satisfy the regularity and compatibility conditions necessary in order that the system have a solution $u \in S_{T_0}$. This can be shown by a direct application of Kato's result in §2.6 of [**3**]; the reservation made there on the regularity result that is needed for the operator $L(\psi)$ has been eliminated in the already mentioned paper [**5**]. Before we proceed to the proof of Proposition 1, it is useful to recall the integration by parts formulas we shall use in the sequel. Noting by (,) the ordinary inner product in $\mathbf{L}^2(\Omega)$, we have:

LEMMA 1. $\forall p \in C^1(\overline{\Omega})$, $\forall u \in \mathbf{H}^2(\Omega)$, $\forall v \in H_1$,

$$(4.3) \qquad (L(p)u, v) = \sum (a_{ij}(p)\partial_i u, \partial_j v) - \frac{1}{2}\xi_0(p, \partial p; \partial u, v),$$

where $\{f, g\} \mapsto \xi_0(p, \partial; f, g)$ is for all fixed p a bilinear continuous form on $\mathbf{L}^2(\Omega)$ such that, because of (2.5), *for suitable $M(p)$ and $M > 0$*

(4.4)

$$\forall f\,\forall g \in \mathbf{L}^2(\Omega), \quad |\xi_0(p, \partial p; f, g)| \le M(p)\|f\|_0\|g\|_0 \le M|p|_1\|f\|_0\|g\|_0$$

(*here and in the sequel we write* $|\cdot|_m$ *for the norm in* $C^m(\overline{\Omega})$).

LEMMA 2. $\forall p \in C^0(0, T; C^1(\bar{\Omega})) \cap C^1(0, T; C^0(\bar{\Omega}))$, $\forall u \in L^2(0, T; \mathbf{H}^2(\Omega)) \cap H^1(0, T; H_1)$,

$$(4.5)\quad (L(p)u, u') = \frac{1}{2}\frac{d}{dt}\sum\left(a_{ij}(p)\partial_i u, \partial_j u\right) - \frac{1}{2}\xi_1(p, p'; \partial u, \partial u) - \frac{1}{2}\xi_0(p, \partial p; \partial u, u'),$$

where $\{f, g\} \to \xi_1(p, p'; f, g)$ *has analogous properties as* ξ_0; (4.5) *follows immediately from* (4.3), *recalling that* $a_{ij} = {}^t a_{ji}$.

We define then for fixed p the quadratic forms on $\mathbf{H}^1(\Omega)$

$$Q_1(u) = \left(\Sigma a_{ij}(p)\partial_i u, \partial_j u\right),$$
$$Q_2(u) = \xi_1(p, p'; \partial u, \partial u)$$

and the bilinear form on $\mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$

$$Q_3(f, g) = \xi_0(p, \partial p; \partial f, g)$$

and notice that because of (2.4), (2.5) and Poincare's inequality, Q_1 is an equivalent norm on H_1 and Q_2 and Q_3 satisfy

$$(4.6)\quad |Q_2(u)| \le c|p'|_0\|u\|_1^2,$$

$$(4.7)\quad |Q_3(f, g)| \le c|p|_1\|f\|_1\|g\|_0.$$

5. Proof of Proposition 1. For $\varphi \in X$, set $\psi = \partial\varphi$ and differentiate $(L_{0\varepsilon})$ with respect to time (this procedure is rather formal, since u is in general not regular enough to make sense out of all the differentiations; because of the linearity of the system however, full justification of the procedure could be given considering instead a regularization of u in time, for instance by means of the Friedrichs mollifier). We obtain for $0 \le k \le 3$ the sequences of systems

$$(L_{h\varepsilon})\qquad \begin{cases} \varepsilon u^{2+h} + \sigma u^{1+h} + L(\psi)u^h = G_h \\ u^h(0) = u_{h\varepsilon}, \qquad u^{1+h}(0) = u_{1+h,\varepsilon} \quad \text{in } \Omega, \\ u^h|_{\partial\Omega} = 0 \end{cases}$$

where $u^h = \partial_t^h u$ and $G_h = L(\psi)u^h - \partial_t^h L(\psi)u$. By a linear combination of these and the linearity of M we also have

$$(L_{h\varepsilon}^*)\qquad \begin{cases} \varepsilon M u^{2+h} + \sigma M u^{1+h} + L(\psi)Mu^h = \tilde{M}G_h \\ Mu^h(0) = \tilde{M}u_{h\varepsilon}, \qquad Mu^{1+h}(0) = \tilde{M}u_{1+h,\varepsilon} \quad \text{in } \Omega, \\ Mu^h|_{\partial\Omega} = 0 \end{cases}$$

which we shall consider only for $h = 1, 2$. Because of the uniform ellipticity of $L(\psi)$ and Poincare's inequality we have

$$
(5.1)\qquad \begin{cases} \|u\|_{j+2} \le \nu\{\|Mu'\|_j + \|u\|_1\}, & j = 0,1,2, \\ \|Mu'\|_2 \le \nu\{\|M^2u''\|_0 + \|\tilde{M}G_1\|_0 + \|Mu'\|_1\}, & \\ \|u'\|_{j+1} \le \nu\{\|Mu''\|_j + \|G_1\|_j + \|u'\|_0\}, & j = 0,1, \end{cases}
$$

so that we are led to consider the "energy" norms

$$
\mathscr{E}_1^2(u) = \{\|u\|_1^2 + \varepsilon\|u'\|_1^2 + \|Mu'\|_1^2 + \|M^2u''\|_0^2 + \varepsilon\|Mu''\|_1^2\},
$$

$$
\mathscr{E}_2^2(u) = \{\|u'\|_1^2 + \|Mu''\|_1^2\}.
$$

We proceed now to carry out suitable a priori estimates for such norms from $(L_{h\varepsilon})$ and $(L^*_{h\varepsilon})$. We have from $(L_{0\varepsilon})$, times u' and recalling (4.5):

$$
(5.2)\qquad \frac{d}{dt}\{\varepsilon\|u'\|_0^2 + Q_1(u)\} + 2\sigma\|u'\|_0^2 = Q_2(u) + Q_3(u, u');
$$

from $(L^*_{1\varepsilon})$, times Mu'':

$$
(5.3)\qquad \begin{aligned} &\frac{d}{dt}\{\varepsilon\|Mu''\|_0^2 + Q_1(Mu')\} + 2\sigma\|Mu''\|_0^2 \\ &\qquad = Q_2(Mu') + Q_3(Mu', Mu'') + 2(\tilde{M}G_1, Mu'') \end{aligned}
$$

and finally from $(L^*_{2\varepsilon})$, times M^2u'':

$$
(5.4)\qquad \begin{aligned} &\frac{d}{dt}\{\|M^2u''\|_0^2 + \varepsilon Q_1(Mu'')\} + 2\sigma Q_1(Mu'') \\ &\qquad = \varepsilon Q_2(Mu'') + Q_3(Mu'', M^2u'') + 2(\tilde{M}G_2, M^2u''). \end{aligned}
$$

From (5.2), (5.3) and (5.4) we get

$$
(5.5)\qquad \begin{aligned} &\frac{d}{dt}\mathscr{N}_1^2(u(t)) + 2\sigma\mathscr{N}_2^2(u(t)) \\ &\quad = \{Q_2(u) + Q_2(Mu') + \varepsilon Q_2(Mu'')\} \\ &\qquad + \{Q_3(u, u') + Q_3(Mu', Mu'') + Q_3(Mu'', M^2u'')\} \\ &\qquad + 2(\tilde{M}G_1, Mu'') + 2(\tilde{M}G_2, M^2u'') \\ &\quad = A_1 + A_2 + A_3 \end{aligned}
$$

with the obvious definition of $\mathscr{N}_j$ and A_j; we remark that since Q_1 is an equivalent norm on H_1, the $\mathscr{E}_j$ and the $\mathscr{N}_j$ are equivalent. We claim now that the right side of (5.5) satisfies the estimate

$$
(5.6)\qquad A_1 + A_2 + A_3 \le \theta\mathscr{N}_2^2(u(t)) + c_\theta([\varphi]_2 + \|\varphi'\|_2^2)\lambda(\Delta)\mathscr{N}_1^2(u(t))
$$

where $\theta > 0$ is arbitrary, $c_\theta > 0$ is independent of ε and $\lambda(\Delta) = (1 + \Delta)^8$. At first we have, recalling (4.6) and (4.7):

$$|A_1| \le c_1\|\psi'\|_2\left\{\|u\|_1^2 + \|Mu'\|_1^2 + \varepsilon\|Mu''\|_1^2\right\}, \tag{5.7}$$

$$\begin{aligned} |A_2| &\le c_2\|\psi\|_3(\|u\|_1 + \|Mu'\|_1 + \|M^2u''\|_0) \\ &\quad \times(\|u'\|_0 + \|Mu''\|_0 + \|Mu''\|_1) \\ &\le c_\theta\|\psi\|_3^2(\|u\|_1^2 + \|Mu'\|_1^2 + \|M^2u''\|_0^2) \\ &\quad + \theta(\|u'\|_0^2 + \|Mu''\|_0^2 + \|Mu''\|_1^2). \end{aligned} \tag{5.8}$$

To estimate A_3 we need to estimate the G_j and their combinations MG_j. This is done recalling (2.5), resorting again to Sobolev's imbedding theorems and the elliptic estimates (5.1); we need only to remark that the linearity of the derivatives of $L(\psi)$ permits us to express suitably the linear combinations MG_j in terms of the combinations $M\psi'$, $M\psi''$ and Mu'. We compute

$$\begin{aligned} -G_1 &= L'(\psi)(\psi')u, \\ -G_2 &= L''(\psi)(\psi',\psi')u + L'(\psi)(\psi'')u + 2L'(\psi)(\psi')u', \\ -G_3 &= L'''(\psi)(\psi',\psi',\psi')u + 3L''(\psi)(\psi',\psi'')u + 3L''(\psi)(\psi',\psi')u' \\ &\quad + L'(\psi)(\psi''')u + 3L'(\psi)(\psi'')u' + 3L'(\psi)(\psi')u'', \end{aligned}$$

so that by opportune linear combinations we obtain

$$\begin{aligned} -\tilde{M}G_1 &= \varepsilon L''(\psi)(\psi',\psi')u + L'(\psi)(M\psi')u + 2\varepsilon L'(\psi)(\psi')u', \\ -\tilde{M}G_2 &= \varepsilon L'''(\psi)(\psi',\psi',\psi')u + 3L''(\psi)(\psi',M\psi')u \\ &\quad + 3\varepsilon L''(\psi)(\psi',\psi')u' + L'(\psi)(M\psi'')u + 3L'(\psi)(M\psi')u' \\ &\quad + 3L'(\psi)(\psi')Mu' - 4\sigma L'(\psi)(\psi')u' - 2\sigma L''(\psi)(\psi',\psi')u. \end{aligned}$$

Because of (2.5) and the Sobolev' imbedding theorems we have then

$$\begin{aligned} c_3^{-1}\|G_1\|_1 &\le \|\psi\|_3\|\psi'\|_1\|u\|_3 + \|\psi'\|_1\|u\|_4, \\ c_4^{-1}\|\tilde{M}G_1\|_0 &\le \left(\varepsilon\|\psi'\|_2^2 + \|M\psi'\|_1\right)\|u\|_3 + \varepsilon\|\psi'\|_2\|u'\|_2, \\ c_5^{-1}\|\tilde{M}G_2\|_0 &\le \left(\varepsilon\|\psi'\|_2^2\|\psi'\|_1 + \|\psi'\|_1\|M\psi'\|_1 + \|\psi'\|_1^2\right)\|u\|_3 \\ &\quad + \|M\psi''\|_0\|u\|_4 + \left(\varepsilon\|\psi'\|_1^2 + \|M\psi'\|_1 + \|\psi'\|_1\right)\|u'\|_3 \\ &\quad + \|\psi'\|_2\|Mu'\|_2. \end{aligned}$$

Combining these with the elliptic estimates (5.1) we arrive after a rather lengthy but straightforward computation at

$$c_6^{-1}\|\tilde{M}G_1\|_0 \le \left(\varepsilon\|\psi'\|_2^2 + \|M\psi'\|_1\right)\left(\|Mu'\|_1 + \|u\|_1\right) + \varepsilon\|\psi'\|_2\left(\|Mu''\|_0 + \|u'\|_0\right);$$

$$\begin{aligned} c_7^{-1}\|\tilde{M}G_2\|_0 \le & \Big\{\|\psi'\|_2\big(1 + \|M\psi'\|_1 + \varepsilon\|\psi'\|_2^2 + \|\psi\|_3\|M\psi'\|_1 \\ & + \|M\psi'\|_1^2 + \varepsilon\|\psi\|_3\|\psi'\|_1^2 + \varepsilon\|M\psi'\|_1\|\psi'\|_2^2 + \varepsilon^2\|\psi'\|_2^4\big) \\ & + \|\psi'\|_1^2\big(1 + \|\psi\|_3 + \|M\psi'\|_2 + \varepsilon\|\psi'\|_2^2\big) \\ & + \|M\psi''\|_0\big(1 + \|M\psi'\|_0 + \varepsilon\|\psi'\|_2^2\big)\Big\} \\ & \cdot\left(\|Mu'\|_1 + \|u\|_1\right) \\ & + \Big\{\|M\psi'\|_1 + \varepsilon\|\psi'\|_2^2 + \varepsilon\|\psi'\|_2\|M\psi''\|_0 + \varepsilon\|M\psi'\|_1\|\psi'\|_2^2 \\ & + \varepsilon^2\|\psi'\|_2^4 + \|\psi'\|_1\big(1 + \varepsilon\|\psi'\|_2^2\big)\Big\} \\ & \cdot\left(\|u'\|_0 + \|Mu''\|_1\right) \\ & + \Big\{\|\psi'\|_2\big(1 + \|M\psi'\|_1 + \varepsilon\|\psi'\|_2^2\big) + \|M\psi''\|_0 + \|\psi'\|_1^2\Big\}\|M^2u''\|_0. \end{aligned}$$

Recalling the bounds implied on $\psi = \partial\varphi$ by the choice $\varphi \in X$ we have therefore, writing T and Δ instead of T_0 and Δ_0, and setting $h(\Delta) = (1 + \Delta)^2$:

$$2(\tilde{M}G_1, Mu'') \le c_\theta\Delta^2\left(\|Mu'\|_1^2 + \|u\|_1^2\right) + \theta\|Mu''\|_0^2 + c_8\|\psi'\|_2\left(\varepsilon\|Mu'''\|_0^2 + \varepsilon\|u'\|_0^2\right);$$

$$\begin{aligned} 2c_9^{-1}\left(\tilde{M}G_2, M^2u''\right) \le\ & h(\Delta)\left\{\|\psi'\|_2 h(\Delta) + \|\psi'\|_1^2 + \|M\psi''\|_0\right\} \\ & \times\left(\|Mu'\|_1^2 + \|u\|_1^2 + \|M^2u''\|_0^2\right) \\ & + c_\theta h^2(\Delta)\left\{h^2(\Delta) + \|\psi'\|_1^2\right\}\|M^2u''\|_0^2 \\ & + \theta\|u'\|_0^2 + \theta\|Mu''\|_1^2, \end{aligned}$$

with arbitrary $\theta > 0$ and c_θ determined accordingly. These inequalities, together with (5.7) and (5.8) permit us to deduce (5.6) easily. Taking θ so small that $a = 2\sigma - \theta > 0$, we have then from (5.5) and (5.6) that

$$\text{(5.9)}\quad \mathscr{N}_1^2(u(t)) + a\int_0^t \mathscr{N}_2^2(u(t)) \leq \mathscr{N}_1^2(u(0)) + c_\theta\lambda(\Delta)\int_0^t\left([\varphi]_2 + \|\varphi'\|_2^2\right)\mathscr{N}_1^2(u(t))$$

from which we get at first by Gronwall's inequality

$$\text{(5.10)}\qquad \mathscr{N}_1^2(u(t)) \leq \mathscr{N}_1^2(u(0)) \exp c_\theta\lambda(\Delta)\int_0^t\left([\varphi]_2 + \|\varphi'\|_2^2\right).$$

Since $\varphi \in X$ we have

$$\int_0^t\left([\varphi]_2 + \|\varphi'\|_2^2\right) \leq \Delta t^{1/2} + \Delta^2 t + a^{-1}(1+\Delta)^4\Delta^2(1 - e^{-at/\varepsilon}) \equiv \alpha_0(t, \Delta);$$

moreover the uniform bounds (3.2) on the initial data are sufficient to ensure that $\mathscr{N}_1(u(0)) \leq k$, independently of ε. Therefore, setting

$$\alpha(t, \Delta) = c_\theta\lambda(\Delta)\alpha_0(t, \Delta),$$

we have from (5.10)

$$\text{(5.11)}\qquad \mathscr{N}_1^2(u(t)) \leq k^2 \exp \alpha(T, \Delta).$$

Let now m be a sufficiently large integer, and choose at first $\Delta = \Delta_0$ such that $\Delta_0 > k/m$, and subsequently $T = T_0$ so that

$$\alpha(T_0, \Delta_0) \leq \ln(\Delta_0/km)^2$$

(we observe that $\alpha(T, \Delta_0)$ is an increasing function of T such that $\alpha(0, \Delta_0) = 0$); from (5.11) we have then that

$$\mathscr{N}_1(u(t)) \leq \Delta_0/m \quad \forall t \in [0, T_0],$$

and since from (5.1) we have for all $t \in [0, T_0]$

$$[u(t)]_1 \leq c\Delta_0^2\mathscr{E}_1(u(t)) \leq c\Delta_0^2\mathscr{N}_1(u(t)),$$

we can deduce that

$$\sup_{t\in[0, T_0]} [u(t)]_1 \leq \Delta_0/2$$

if m is chosen large enough. From this inequality and (5.9) it is then immediate to recover the analogous estimate

$$\int_0^{T_0}[u]_2^2 \leq \left(\frac{\Delta_0}{2}\right)^2;$$

and finally, to recover the analogous of estimate (3.4), we observe that since

$$\|L(\psi)u\|_2 \leq ch(\Delta)\|u\|_4 \leq c\Delta h(\Delta),$$

we have from $(L_{0\varepsilon})$ that for all $\theta > 0$

$$\varepsilon\frac{d}{dt}\|u'\|_2^2 + 2\sigma\|u'\|_2^2 \leq c_\theta(\Delta h(\Delta))^2 + \theta\|u'\|_2^2;$$

setting again $a = 2\sigma - \theta > 0$ we have therefore

$$\varepsilon e^{-at/\varepsilon}\frac{d}{dt}e^{at/\varepsilon}\|u'\|_2^2 \leq c_\theta(\Delta h(\Delta))^2$$

whence

$$\|u'\|_2^2 \leq \|u_{1\varepsilon}\|_2^2 e^{-at/\varepsilon} + a^{-1}c_\theta(\Delta h(\Delta))^2$$

from which (choosing possibly a larger value for Δ_0)

$$\|u'\|_2 \leq \Delta_0 h(\Delta_0)(1 + \varepsilon^{-1/2}e^{-at/2\varepsilon}).$$

The proof of (4.1) of Proposition 1 can therefore be concluded easily. The proof of (4.2) is then standard, and actually identical to the one given for the analogous claim in [**6**], so that we omit it; we need only to remark that since the weaker norm $\|\cdot\|_w$ is considered, the boundary condition $u|_{\partial\Omega} = 0$ is sufficient to perform the integration by parts in the space variables that is used to obtain (4.2). The proof of Proposition 1, and consequently of Theorem 1, can therefore be completed.

6. Proof of Theorems 2 and 3. In this section we follow closely the procedure of §8 of [**6**].

6.1. Because of (3.3) and (3.4) we have that, as $\varepsilon \to 0$:

$$u_\varepsilon \quad \text{is bounded in } C(0, T; H_4),$$
$$u'_\varepsilon \quad \text{is bounded in } L^2(0, T; H_3)$$

(we have set $T_0 = T$ for simplicity). There exist therefore a subsequence, still denoted u_ε, and a vector function w such that

$$u_\varepsilon \to w \quad \text{in } L^\infty(0, T; H_4) \text{ weak*},$$
$$u'_\varepsilon \to w' \quad \text{in } L^2(0, T; H_3) \text{ weak},$$

so that by compactness

$$u_\varepsilon \to w \quad \text{in } C(0, T; H_3) \text{ uniformly}$$

and because of the regularity of L

$$L(u_\varepsilon)u_\varepsilon \to L(w)w \quad \text{in } C(0, T; H_1).$$

Let now ψ be a smooth function such that $\psi(x, T) = 0$: from (2.1) we get, upon integrating by parts

$$\int_0^T\{-\varepsilon(u'_\varepsilon, \psi') + \sigma(u'_\varepsilon, \psi') + (L(u_\varepsilon)u_\varepsilon, \psi)\} = \varepsilon(\psi(0), u_{1\varepsilon});$$

letting $\varepsilon \to 0$ and recalling (3.2) we have then

$$\int_0^T \{\sigma(w', \psi) + (L(w)w, \psi)\} = 0,$$

and because of the arbitrariness of ψ

$$\sigma w' + L(w)w = 0.$$

w is therefore a solution of (P) on $[0, T]$, with $u_0 = \text{w-lim}_{\varepsilon\to 0} u_{0\varepsilon}$ in H_4 (which exists because of (3.2)). By standard monotonicity methods, such a solution is easily seen to be unique.

6.2. Suppose now that (P) has a solution u satisfying the assumptions of Theorem 3: then it follows from (2.6) that

$$u' \in C(0, T; H_2), \qquad u'' \in C(0, T; H_0) \cap L^2(0, T; H_1).$$

From (2.1) and (2.6) we have

$$\begin{cases} \varepsilon u_\varepsilon'' + \sigma u_\varepsilon' + L(\partial u_\varepsilon)u_\varepsilon = 0, \\ \varepsilon u'' + \sigma u' + L(\partial u)u = \varepsilon u'', \end{cases}$$

from which, setting $w_\varepsilon = u_\varepsilon - u$:

$$\text{(6.1)} \quad \varepsilon w_\varepsilon'' + \sigma w_\varepsilon' + L(\partial u_\varepsilon)w_\varepsilon = [L(\partial u) - L(\partial u_\varepsilon)]u - \varepsilon u'' = W_\varepsilon.$$

We have, recalling (4.5) (with $p = \partial u_\varepsilon$):

$$\text{(6.2)} \qquad \frac{d}{dt}\left\{\varepsilon\|w_\varepsilon'\|_0^2 + Q_1(w_\varepsilon)\right\} + 2\sigma\|w_\varepsilon'\|_0^2 = Q_2(w_\varepsilon) + Q_3(w_\varepsilon, w_\varepsilon') + 2(W_\varepsilon, w_\varepsilon')$$

and since by (3.3) and the regularity of L

$$|Q_2(w_\varepsilon)| \le c\|u_\varepsilon'\|_3\|w_\varepsilon\|_1^2,$$

$$|Q_3(w_\varepsilon, w_\varepsilon')| \le c\|u_\varepsilon\|_4\|w_\varepsilon\|_1\|w_\varepsilon'\|_0 \le c_\theta\|w_\varepsilon\|_1^2 + \theta\|w_\varepsilon'\|_0^2,$$

$$\|W_\varepsilon\|_0 \le c\|w_\varepsilon\|_1 + \varepsilon\|u''\|_0,$$

we obtain from (6.2) that for suitable a and $b > 0$

$$\frac{d}{dt}\left\{\varepsilon\|w_\varepsilon'\|_0^2 + Q_1(w_\varepsilon)\right\} + a\|w_\varepsilon'\|_0^2 \le b\left(1 + \|u_\varepsilon'\|_3\right)\|w_\varepsilon\|_1^2 + \varepsilon\|u''\|_0^2$$

whence (after possibly renaming the constants)

$$\varepsilon\|w_\varepsilon'\|_0^2 + \|w_\varepsilon\|_1^2 + a\int_0^t\|w_\varepsilon'\|_0^2$$
$$\le b\left(\varepsilon\|u_{1\varepsilon} - u'(0)\|_0^2 + \|u_{0\varepsilon} - u_0\|_1^2 + \int_0^t\left(1 + \|u_\varepsilon'\|_3\right)\|w_\varepsilon\|_1^2 + \varepsilon\int_0^t\|u''\|_0^2\right),$$

from which it is immediate, using Gronwall's inequality and recalling that $\int_0^t \|u_\varepsilon'\|_3^2 \le$ const., to deduce that

$$(6.3)\quad \varepsilon\|w_\varepsilon'\|_0^2 + \|w_\varepsilon\|_1^2 + \int_0^T \|w_\varepsilon'\|_0^2 \le k_T\left[\varepsilon\|u_{1\varepsilon} - u'(0)\|_0^2 + \|u_{0\varepsilon} - u_0\|_1^2 + \varepsilon\right].$$

We now have for all $\eta > 0$, using well known interpolation inequalities:

$$(6.4)\quad \begin{aligned} \|w_\varepsilon\|_{4-\eta} &\le c_\eta \|w_\varepsilon\|_4^{1-\eta/3} \|w_\varepsilon\|_1^{\eta/3}, \\ \|w_\varepsilon'\|_{3-\eta} &\le c_\eta \|w_\varepsilon'\|_3^{1-\eta/3} \|w_\varepsilon'\|_0^{\eta/3}, \\ \int_0^T \|w_\varepsilon'\|_{3-\eta}^2 &\le c_\eta \left(\int_0^T \|w_\varepsilon'\|_3^2\right)^{1-\eta/3} \left(\int_0^T \|w_\varepsilon'\|_0^2\right)^{\eta/3}, \end{aligned}$$

so that because of (3.3) we get from (6.3)

$$\varepsilon^{1/2}\|w_\varepsilon'\|_{3-\eta} + \|w_\varepsilon\|_{4-\eta} \le k_{T,\eta}\left[\varepsilon^{1/2}\|u_{1\varepsilon} - u'(0)\|_0 + \|u_{0\varepsilon} - u_0\|_1 + \varepsilon^{1/2}\right]^{\eta/3},$$

$$\int_0^T \|w_\varepsilon'\|_{3-\eta}^2 \le k_{T,\eta}\left[\varepsilon\|u_{1\varepsilon} - u'(0)\|_0^2 + \|u_{0\varepsilon} - u_0\|_1^2 + \varepsilon\right]^{\eta/3},$$

that is (3.6).

REMARK 5. Sharper estimates could be obtained if stronger norms of w were considered instead of the one in (6.2), and even the case $\eta = 0$ could be treated. Such estimates would be established using the same "elliptic procedure" used in §5.

We conclude by proving the Corollary to Theorem 3 (we remark that its additional assumptions are not enough to control the difference w_ε', unless some initialization conditions such as (3.5) are imposed). However we have from (6.1):

$$(6.5)\quad \begin{aligned} &\varepsilon(w_\varepsilon'', w_\varepsilon') + \sigma(w_\varepsilon', w_\varepsilon') \\ &\qquad = (L(\partial u)u - L(\partial u_\varepsilon)u_\varepsilon - \varepsilon u'', w_\varepsilon') \equiv (\Lambda_\varepsilon, w_\varepsilon'); \end{aligned}$$

acting as before, and using interpolation, we have

$$\|L(\partial u)u - L(\partial u_\varepsilon)u_\varepsilon\|_0 \le c\|w_\varepsilon\|_2 \le c\|w_\varepsilon\|_4^{1/3}\|w_\varepsilon\|_1^{2/3}$$

so that from (3.3) and (6.3) it follows that

$$\|\Lambda_\varepsilon\|_0 = O(\varepsilon^{1/3}) \quad \text{as } \varepsilon \to 0,$$

and we can deduce from (6.5) that, for suitable a and $k > 0$

$$\varepsilon\frac{d}{dt}\|w_\varepsilon'\|_0^2 + a\|w_\varepsilon'\|_0^2 \le k\varepsilon^{2/3},$$

whence

$$(6.6) \qquad \|w_\varepsilon'\|_0^2 \le e^{-at/\varepsilon}\|u_{1\varepsilon} - u'(0)\|_0^2 + k\varepsilon^{2/3} \le c(e^{-at/\varepsilon} + \varepsilon^{2/3}).$$

Then, since for $\eta > 1$

$$\|w_\varepsilon'\|_{3-\eta} \le c_\eta \|w_\varepsilon'\|_2^{(3-\eta)/2} \|w_\varepsilon'\|_0^{(\eta-1)/2}$$

we have, recalling (3.4) and (6.6):

$$\|w_\varepsilon'\|_{3-\eta} \le c(1 + \varepsilon^{-1/2}e^{-at/2\varepsilon})^{(3-\eta)/2}(e^{-at/2\varepsilon} + \varepsilon^{1/2})^{(\eta-1)/2}$$

so that $\|w_\varepsilon'\|_{3-\eta} \to 0$ uniformly on $[\tau, T]$ $\forall \tau \in]0, T]$. If in particular $\eta < 3/2$, $\mathbf{H}^{3-\eta}(\Omega) \hookrightarrow (C^0(\overline{\Omega}))^3$, so that (H_ε) degenerates regularly to (P) as $\varepsilon \to 0$.

7. Application to Maxwell's equations. We consider the following system of the complete Maxwell's equations:

$$(\mathrm{cs}) \qquad \begin{cases} (7.1) \quad \begin{cases} D' + j - \operatorname{curl} H = 0, \\ B' + \operatorname{curl} E = 0, \end{cases} & (7.2) \quad \begin{cases} \operatorname{div} D = 0, \\ \operatorname{div} B = 0, \end{cases} \\ (7.3) \quad \begin{cases} J = \sigma E, \quad D = \varepsilon E, \\ H = \zeta(B), \end{cases} & (7.4) \quad \begin{cases} D(0) = D_0, \\ B(0) = B_0, \end{cases} \\ (7.5) \quad n \times D = 0 \quad \text{on } \partial\Omega & \end{cases}$$

where n is the outward normal to $\partial\Omega$, ε and σ are positive constants and ζ: $\mathbf{R}^3 \to \mathbf{R}^3$ is a nonlinear function. It is well known that (7.2) are redundent if they are satisfied by the initial data (7.4), and that if $n \cdot B_0 = 0$ on $\partial\Omega$ then the additional boundary condition

$$(7.6) \qquad n \cdot B = 0 \quad \text{on } \partial\Omega$$

can be derived from (7.5) and the second of (7.1). It was shown in **[8]** that because of (7.1), (7.2) and (7.6), scalar and vector potentials φ and u can be determined to satisfy the coupled system of equations

$$(7.7) \qquad \begin{cases} \operatorname{curl} u = B \\ \operatorname{div} u' + \varepsilon\varphi' + \sigma\varphi = 0 \quad \text{in } \Omega, \\ u' + \nabla\varphi = -E \end{cases}$$

$$(7.8) \qquad \begin{cases} n \times u = 0 \\ \operatorname{div} u = 0 \end{cases} \quad \text{on } \partial\Omega,$$

so that (CS) transforms into the second order hyperbolic system

$$(M_\varepsilon) \qquad \begin{cases} \varepsilon u_\varepsilon'' + \sigma u_\varepsilon' + \operatorname{curl} \zeta(\operatorname{curl} u_\varepsilon) - \nabla \operatorname{div} u_\varepsilon = 0 & \\ u_\varepsilon(0) = u_{0\varepsilon}, \qquad u_\varepsilon'(0) = u_{1\varepsilon} & \text{in } \Omega, \\ n \times u_\varepsilon = 0, \qquad \operatorname{div} u_\varepsilon = s & \text{on } \partial\Omega, \end{cases}$$

where curl $u_{0\varepsilon} = B_0$ and $\varepsilon u_{1\varepsilon} = -D_0$. The corresponding quasistationary equations can be transformed into the parabolic system

$$(M_0) \qquad \begin{cases} \sigma u' + \operatorname{curl} \zeta(\operatorname{curl} u) - \nabla \operatorname{div} u = 0 & \\ u(0) = u_0, & \text{in } \Omega, \\ n \times u = 0, \quad \operatorname{div} u = 0 & \text{on } \Omega, \end{cases}$$

which is related to the fields B and E by the coupled relations

$$(7.9) \qquad \begin{cases} B = \operatorname{curl} u, \\ E = -u' + \sigma^{-1}\nabla \operatorname{div} u. \end{cases}$$

These systems are somewhat different from those considered in §2, in that the first order differential operator div appears in the boundary conditions. It will be shown however that, as a peculiarity of Maxwell's equations, the divergence of u enjoys the same regularity as u itself. Indeed, we need to modify the spaces H_j in the following way, defining

$$H_0 = \{u \in \mathbf{L}^2(\Omega) \mid \operatorname{div} u \in L^2(\Omega)\},$$

$$H_m = \{u \in \mathbf{H}^m(\Omega) \mid \operatorname{div} u \in H^m(\Omega);\ n \times u = \operatorname{div} u = 0 \text{ on } \partial\Omega\}$$

for $1 \leq m \leq 4$; we recall from [1] that H_m coincides with the space

$$\{u \in \mathbf{L}^2(\Omega) \mid \operatorname{curl} u \in \mathbf{H}^{m-1}(\Omega),\ \operatorname{div} u \in H^m(\Omega);$$
$$n \times u = \operatorname{div} u = 0 \text{ on } \partial\Omega\}$$

on which the norms

$$\|\operatorname{curl} u\|_{m-1} + \|\operatorname{div} u\|_m, \qquad \|u\|_m + \|\operatorname{div} u\|_m$$

are equivalent. We shall denote $|\cdot|_m$ the first of these norms in H_m. In [8] a local existence result in time for (M_ε) in a bounded domain was established, for fixed ε, adapting Kato's results of [3]; such results however are not stable with respect to ε. We recall from that paper that the non linear operator $u \mapsto \operatorname{curl} \zeta\ (\operatorname{curl} u)$ can be written in explicit fashion as a quasilinear operator

$$u \mapsto T(\operatorname{curl} u)u = \sum_{i,j=1}^{3} \alpha_{ij}(\operatorname{curl} u)\partial_i\partial_j u$$

with suitable 3×3 matrices α_{ij} obtained by direct differentiation of ζ; adding to this the additional term $-\nabla \operatorname{div} u$ we obtain the operator

$$u \mapsto L(\operatorname{curl} u)u = \sum_{i,j=1}^{3} a_{ij}(\operatorname{curl} u)\partial_i\partial_j u.$$

Under the assumptions made on ζ as in [8] and [6], that is essentially the requirements that ζ be a strongly monotone asymptotically linear function, derivative of a convex function F: $\mathbf{R}^3 \to \mathbf{R}$ whose derivatives up to the fifth order at least are uniformly bounded, it was shown in [**8**] that ${}^t a_{ij} = a_{ji}$, that the operator $L(p)u$ is uniformly strongly elliptic, that the boundary conditions (7.8) are complementing and that integration by parts formulas analogous to (4.3) and (4.5) hold. We can therefore apply Theorems 1, 2 and 3 to (M_ε), provided we can take care of the additional regularity required of div u.

This is done observing that, as a straightforward computation shows, $\operatorname{div} T(u)u = 0$ because of the symmetry of $\zeta'(u)$ that follows from the assumption $\zeta = \partial F$. Since $\operatorname{div} L(u)u = -\operatorname{div}\nabla \operatorname{div} u$, we derive from the linearized form of (M_ε) that $v = \operatorname{div} u$ is a solution of the linear problem with constant coefficients

$$\begin{cases} \varepsilon v'' + \sigma v' - \operatorname{div} \nabla v = 0, \\ v(0) = \operatorname{div} u_{0\varepsilon}, \qquad v'(0) = \operatorname{div} u_{1\varepsilon} \quad \text{in } \Omega, \\ v|_{\partial\Omega} = 0, \end{cases}$$

to which classical results apply.

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