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NON LINEAR MULTIPLIERS AND APPLICATIONS

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Non linear multipliers, associated non linear representations and non autonomous systems are studied. A notion of inducing of non linear representation is defined. Various applications are given.

Introduction. A theory of non linear representations of Lie groups was initiated in [5]. Non linear representations are related to non linear physical equations, as shown in [7] where a general program of linearization of a priori non linear field equations was sketched. This program is based on a powerful criteria of linearizability given in [6] and used in several papers to prove linearizability of some physically interesting non linear representations ([6, 7, 8, 15, 17, 18, 19]). On the other hand, it is well known, since Poincaré's work on singular vector fields, that there exist non linearizable non linear representations, and, moreover, it seems true that nature may use representations of this kind when formalizing some of its physical properties. Therefore, it seems interesting to develop techniques for the construction of non linear representations. This was initiated in [12] and [14], and this paper is continuation of this constructive program. When one is concerned with linear representations, a powerful constructive theory is indeed the theory of induced representations. To the author's knowledge, such a theory does not exist in the non linear case, and the first goal of this paper is to define non linear inducing. As we shall see, this definition is based on the notion of non linear multipliers, and when developing our theory, it appeared that this notion has some interest in itself, since it can be used to describe several non linear problems. The second goal is to describe these associated problems and how they can be solved.

We now give a brief description of the paper:

In §0 we recall needed definitions and results about non linear formal representations of Lie groups and Lie algebras in Fréchet spaces.

Let us note that in this short survey, the presentation is slightly different from [5] (which, moreover, was devoted to the case of Banach spaces): we try to keep as close as possible to the usual formalism of differential geometry, using very simple notions of formal differential calculus in Fréchet spaces. For instance, the "Fock-space linear representation" associated to a non linear representation, was one of the main tools of [5], but was not given a geometrical interpretation. We give here a natural realization of this representation on the space of polynomials on the leading space, using the canonical duality between this space and the space of formal power series, on which a linear action is obviously defined from a non linear representation. For more details and developments, see, e.g., [5], [11], or [10].

Now let G be a connected Lie group, Γ a closed subgroup, $\mathfrak{X}$ a manifold with a C^∞ action of G, and E a finite-dimensional space.

In §1, having defined the notion of multiplier on $G \times \mathfrak{X}$ with values in E (1.1), we associate to such a multiplier A a formal representation V^A of G on $C^\infty(\mathfrak{X}, E)$. Equivalence of multipliers is introduced in such a way that equivalent multipliers lead to equivalent formal representations (1.4). We describe the formal representation dV^A of the Lie algebra $\mathfrak{g}$ of G (1.2), and then characterize which representations of $\mathfrak{g}$ on $C^\infty(\mathfrak{X}, E)$ are of the form dV^A (1.3). This last result can be considered as an existence theorem for solutions of some systems of non linear evolution equations.

In §2 we assume $\mathfrak{X} = G/\Gamma$. Then any multiplier extends, in some sense, a formal representation of Γ (2.1). The crucial point is that given multipliers are equivalent if and only if the corresponding formal representations of Γ are so (2, 3). This gives a very simple criteria for instance for linearizability of a multipler A and consequently of the formal representation V^A.

In §3 we introduce systems of non linear non autonomous differential equations on G, with Γ-periodic coefficients which are naturally related to multipliers over $G \times G/\Gamma$. These systems are generalizations to the non abelian case of ordinary total differential equations in $\mathbf{R}^n$. Solvability of such systems is studied in (3.3) and related to a Frobenius compatibility condition. From the relation with multipliers, a natural equivalence of such systems can be introduced (3.6), which turns out to correspond to some kind of Backlund transformation of solutions.

In §4 we study the case when $E = \mathbf{C}$, i.e. the so-called scalar multipliers. Our principal goal in this section is to study the following problem: given a formal representation of Γ in $\mathbf{C}$, is it possible to construct a multiplier over $G \times G/\Gamma$ extending this representation? To our knowledge, except in the trivial case when there is a C^∞ section of the projection $G \to G/\Gamma$ (2.3), there is no general answer to this question, even in the linear case. It is shown in (2.4) that if the answer is yes for the linear part I^1 of the given formal representation I of Γ, then it is yes for the representation I itself (2.4). Therefore, the problem is reduced to the linear case. We describe in (4.1) linear scalar multipliers over $G \times G/\Gamma$ as exponentials of 1-cocycles of the regular representation of the covering $\tilde{G}$ of G on G/Γ. These 1-cocycles are easily obtained by Shapiro's lemma (4.4). Let us note that since $\tilde{G}$ appears, it will not be always possible to define continuous powers of a multiplier, and this explains why discrete phenomena can appear when constructing multiplier representations (4.3). As a corollary of the preceding results, we get (4.6), which can be considered and used as a no-go theorem: several examples are given from (4.6), where it is not possible to construct a linear scalar multiplier extending a character of Γ (4.7). Alternatively, these are examples where it is not possible to realize a C^∞ linearly induced representation as a multiplier representation on the homogeneous space (4.7). Let us mention that the given example of $m^2 = 0$ discrete helicity representations of the Poincaré group is related to some problems which arise in the case of massless particles (see [**4**]).

In §5 we give several criteria of linearization of multipliers. The first kind is (5.4): if $\mathfrak{X}$ is a product of homogeneous spaces of G, and if at least one of the corresponding subgroups is compact, then any multiplier over $G \times \mathfrak{X}$ is linearizable. The second kind is (5.8), which is cohomological, and can apply to cases which are very far from homogeneous spaces: examples are developed in (5.9).

In §6 we develop the announced notion of non linear inducing: we show how to associate to any class of formal representations I of Γ a class of formal representations S of G with leading space the space H^L of the usual linear representations U^L of G C^∞-induced by the linear part L of I, and with linear part U^L. By definition, equivalence is preserved by inducing. Finally, we discuss possible realizations as multiplier representations on the homogeneous space (6.8).

In §7 we specialize to $G = \mathbf{R}^n$, $\Gamma = Z^n$ and $\mathfrak{X} = G/\Gamma$. We first show that induced (non linear) representations can always be realized as multiplier representations over $\mathfrak{X}$ (7.1).

We then give a series of reduction results concerning multipliers over $G \times \mathfrak{X}$, and corresponding non autonomous associated systems of §3. Using equivalence (3.6) we are able to reduce these systems to normal forms as close as possible to autonomous systems. (7.2), (7.4) and (7.5) are extensions to general n of results known for $n = 1$ (see [**1**]). (7.8) and (7.7), which is a particular case) give minimal normal forms. Let us mention that Lemma 7.6 (which shows how to extend formal representation of Γ to formal representations of G under certain conditions) has some interest by itself (especially since its proof is very simple!).

REMARK. §7 shows how the computation of non linear representation induced from Z^n to $\mathbf{R}^n$ gives the reduction to normal forms of systems of non linear non autonomous formal equations with periodic coefficients, satisfying the Frobenius compatibility condition, of type $\partial f/\partial t_i = F_i(t, f)$, $i = 1,\dots,n$. Since it extends the classical non linear Floquet-Liapunov theory **[1]**, which is the case $n = 1$, one may ask some question concerning this particular case: where is, in this picture, the monodromy (or Poincaré) mapping **[1]**? Obviously, it is the inducing representation. We hope that this remark will show that the framework of non linear inducing is quite natural in this context.

In §8 we compute up to equivalence non linear induced formal representations of the hyperbolic group SU(1,1) with linear part a representation of the continuous series. Explicit formulae for multipliers are given (8.2). It can be seen that these formulae can be obtained from general formulae given in (4.8).

Finally, let us mention that we hope to come back later to the following two problems: first, introduce convergence in this theory, which is, at this stage, formal; second, study non autonomous systems with discrete periodicity subgroup Γ in non abelian cases.

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0. Notations. We briefly recall some notation and results about (non linear) formal representations of Lie groups in Fréchet spaces. For more details see, e.g., **[11]**.

(0.1) Given two locally convex topological vector spaces (tlcvs) E and F, we denote by $L(E, F)$ ($L(E)$ when $E = F$) the space of continuous linear mappings from E into F. When $L(E, F)$ is endowed with the topology of convergence of finite sets (resp. on compact sets), we use the notation $L_\sigma(E, F)$ (resp. $L_c(E, F)$). We denote by $L_n(E, F)$ the space of n-linear continuous mappings from E into F. In the following, we identify $L_n(E, F)$ and $L(\hat{\otimes}_n E, F)$. We define the symmetrization $\sigma_n \in L(\oplus_n E)$ by

$$\sigma_n(e_1 \otimes \cdots \otimes e_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}, \qquad e_i \in E,$$

and the symmetric tensors $\hat{\otimes}_{n,s} E$ as the range of the projection σ_n. We identify the space $L_n^s(E, F)$ of continuous n-linear symmetric mappings from E into F with $L(\hat{\otimes}_{n,s} E, F)$; we denote by $F(E, F)$ ($F(E)$ when $E = F$) the product space $F \times \prod_{n \geq 1} L(\hat{\otimes}_{n,s} E, F)$; the elements of $F(E, F)$ are called formal series from E into F, and we use the standard notation $T = \sum_{n \geq 0} T^n$, $T^0 \in F$, $T^n \in L(\hat{\otimes}_{n,s} E, F)$, $n \geq 1$. $F_\sigma(E, F)$ (resp: $F_c(E, F)$) is the tlcvs obtained when endowing $F(E, F)$ with the product topology. Formal series T such that $T^0 = 0$ will be called formal vector fields, and the corresponding subspace of $F(E, F)$ will be denoted by $C(E, F)$.

(0.2) We now assume that E and F are Fréchet spaces and G is a tlcvs. We define the composition product by

$$U \circ T = U^0 + \sum_{n \geq 1} \left(\sum_{p=1}^{n} U^p \sum_{i_1 + \cdots + i_p = n} T^{i_1} \otimes \cdots \otimes T^{i_p} \circ \sigma_n \right),$$

$$U \in F(F, G),\ T \in C(E, F),\ U \circ T \in F(E, G).$$

This product is linear with respect to U and sequentially continuous from $F_\sigma(F, G) \times C_\sigma(E, F)$ into $F_\sigma(E, G)$. Moreover, if $V \in F(G, H)$, $U \in C(F, G)$, $T \in C(E, F)$, one has

$$V \circ (U \circ T) = (V \circ U) \circ T.$$

Invertible elements D of $C(E)$ are characterized by: D^1 invertible in $L(E)$. We denote by $D(E)$ the group of such elements, which we call formal diffeomorphisms.

(0.3) Given a Fréchet space E and a tlcvs F, we define the Fréchet derivative $D(T) \in F(E, L(E, F))$ of a formal series $T \in F(E, F)$ in the following way: first suppose $T = T^0$; then $D(T) = 0$; now suppose $T = T^n$; then $D(T)$ is the unique continuous symmetric $n - 1$ linear mapping such that

$$\left[D(T)(\underbrace{x, \ldots, x}_{n-1}) \right](e) = nT^n(x, \ldots, x, e), \qquad x, e \in E;$$

when $T = \sum_{n \geq 0} T^n$, we define $D(T) = \sum_{n \geq 0} D(T^n)$. Actually, D is a continuous linear mapping from $F_\sigma(E)$ into $F_\sigma(E, L_\sigma(E, F))$.

(0.4) Given a Fréchet space E, and a tlcvs F we define a sequentially continuous bilinear mapping x from $F_\sigma(E, L_\sigma(E, F)) \times F_\sigma(E)$ into $F_\sigma(E, F)$ in the following way: let

$$T = \sum_{n \geq 0} T^n \in F(E, L_\sigma(E, F)), \quad S = \sum_{n \geq 0} S^n \in F_\sigma(E);$$

we define

$$W^n(e_1,\dots,e_n) = \sum_{i=0}^{n} \left[T^i(e_1,\dots,e_i)\right]\left(S^{n-i}(e_{i+1},\dots,e_n)\right),$$

$$n \geq 1,\ e_j \in E,\ W^0 = T^0(S^0);$$

$$V^n = W^n \circ \sigma^n \quad \text{and} \quad T \times S = \sum_{n\geq 0} V^n.$$

Given now $U \in C(E)$, we have $(T \times S)\circ U = (T\circ U) \times (S \circ U)$.

(0.5) Suppose E is a Fréchet space. We define a Lie algebra structure on $C(E)$ by introducing the Jacobi bracket of vector fields:

$$T, S \in C(E), \qquad [T, S] = D(T) \times S - D(S) \times T.$$

To each vector field T is associated a linear continuous mapping π_T of $F(E, F)$ defined by

$$\pi_T(Z) = -D(Z) \times T, \qquad Z \in F(E, F).$$

Let us define a product on $F(E, \mathbf{C})$ by: let

$$Z = \sum_{n\geq 0} Z^n, \quad Y = \sum_{n\geq 0} Y^n F(E, \mathbf{C});$$

set

$$W^n(e_1,\dots,e_n) = \sum_{i=0}^{n} Z^i(e_1,\dots,e_i) Y^{n-i}(e_{i+1},\dots,e_n),$$

$$e_j \in E, n \geq 1, W^0 = Z^0 Y^0, V^n = W^n \circ \sigma^n \text{ and } Z \cdot Y = \sum_{n\geq 0} V^n.$$

It turns out that $F(E, \mathbf{C})$ is an abelian algebra and π_T, $T \in F(E)$, is a derivation of it.

(0.6) Suppose E is a Fréchet space. Given $T = \sum_{n\geq 1} T^n \in C(E)$, we assume there exists a one parameter group $U \in C^\infty(\mathbf{R}, L_\sigma(E))$ such that $d(U_t)/dt = T^1 \circ U(t)$. Then, the equation $(d[V(t)]/dt) = T \circ V(t)$, with initial condition $V(0) = A \in C(E)$, has a unique solution $V \in C^\infty(\mathbf{R}, C_\sigma(E))$. Let S be the solution with initial condition Id_E; one has $S^1 = U$, $S_{t+t'} = S_t \circ S_{t'}$, $\forall t,\ t' \in \mathbf{R}$, and $V_t = S_t \circ A$. The one parameter group S of formal diffeomorphisms is called the flow of the formal vector field T.

(0.7) Suppose E is a Fréchet space, G a Lie group, $\mathfrak{g}_0$ the Lie algebra of G. A formal representation of G in E (resp. $\mathfrak{g}_0$) is a morphism S: $G \to D(E)$ (resp. s: $\mathfrak{g}_0 \to C(E)$) such that $\forall e_1,\ldots,e_n \in E$, $n \geq 1$, $g \to S_g^n(e_1,\ldots,e_n)$ is continuous from G into E. S^1 (resp. s^1) is a continuous linear representation of G (resp. $\mathfrak{g}_0$) in E, called the linear part of S (resp. s). If we assume S^1 is a C^∞ representation, then any of the mappings $(g, e) \to S_g^n(e)$, $g \in G$, $e \in \hat{\otimes}_{n,s} E$, is also C^∞, and therefore we can define the differential dS of S by

$$dS_X = d/dt\left[S_{\exp tX}\right]_{t=0}.$$

dS is a formal representation of $\mathfrak{g}_0$, and the one parameter group $S_t = S_{\exp tX}$ is the unique solution of $(d/dt)[S_t] = dS_X \circ S_t$ with initial condition $S(0) = \mathrm{Id}_E$.

Given formal representations S_1 and S_2 of G in E (resp. s_1 and s_2 of $\mathfrak{g}_0$), we say that S_1 and S_2 (resp. s_1 and s_2) are equivalent if there exists $A \in D(E)$ such that $S_2 = A \circ S_1 \circ A^{-1}$ (resp. $s_2 = (D(A) \times s_1) \circ A^{-1}$). A formal representation S of G (resp. s of $\mathfrak{g}_0$) is linearizable if it is equivalent to its linear part S^1 (resp. s^1).

Let us assume that the linear parts of our formal representations are C^∞ linear representations. Then, as soon as S_1 and S_2 are equivalent, dS_1 and dS_2 are also equivalent. On the other hand, if we assume that G is connected, then S_1 and S_2 are equivalent if and only if dS_1 and dS_2 are equivalent.

(0.8) Keep the assumptions of (0.7). We give a natural introduction to the "Fock-space linear representation" associated to a formal representation, which is probably one of the most useful tools for this type of problem (see e.g. [5]). We denote by A_α the abelian algebra $F_\alpha(E, \mathbf{C})$, $\alpha = \sigma$ or c. Let H be the strict inductive limit $H = \bigcup_{n \geq 0} H_n$, where $H_n = \oplus_{i=0}^n H^i$, $H^0 = \mathbf{C}$, $H^1 = E$, $H^i = \hat{\otimes}_{i,s} E$, $\forall i \geq 2$. Introducing the duality $h \in H, f \in A$, $h = \Sigma_{\text{finite}} h^n, f = \Sigma f^n$, $\langle f \,|h\rangle = \Sigma f^n(h^n)$, one has $A_\alpha = H_\alpha^*$.

Each $T \in D(E)$ (resp. $X \in C(E)$) defines a continuous automorphism (resp. derivation) $\mathscr{A}_T$ (resp. π_X) of A_α by $\mathscr{A}_T(f) = f \circ T^{-1}$ (resp. $\pi_X(f) = -Df \times X$), $\forall f \in A$. $\mathscr{A}$ (resp. π) is an injective morphism from the group $D(E)$ of formal diffeomorphisms (resp. the Lie algebra $C(E)$ of formal vector fields) into the group of automorphisms (resp. the Lie algebra of derivations) of A.

Since $A_\sigma = H_\sigma^*$, by transposition, we define for any $T \in D(E)$ (resp. $X \in C(E)$) a continuous linear mapping L_T (resp. τ_X) from H into H by

$$L_T = {}^t\mathscr{A}_{T^{-1}} \quad \left(\text{resp. } \tau_X = -{}^t\pi_X\right).$$

One has $L_T(H_n) \subset H_n$ (resp. $\tau_X(H_n) \subset H_n$) and

$$\forall h \in H^n, \quad L_T(h) = \sum_{k=1}^{n} \sum_{i_1 + \cdots + i_k = n} \sigma_p\left[T^{i_1} \otimes \cdots \otimes T^{i_k}(h)\right]$$

(resp.

$$\tau_X(h) = \sum_{k=1}^{n} \sum_{p=0}^{k-1} \sigma_k\left[I_p \otimes X^{n-k+1} \otimes I_{k-p-1}(h)\right],$$

where $I_p = Id \,\hat{\otimes}_p\, E$).

Given now a formal representation S of G (resp. s of $\mathfrak{g}$) in E, we define two associated continuous linear representations by

$$\begin{cases} \tilde{S}_g = \mathscr{A}_{S_g}, \\ \check{S}_g = L_{S_g}, \end{cases} \quad g \in G \qquad \left(\text{resp.} \begin{cases} \tilde{s}_X = \pi_{s_X}, \\ \check{s}_X = \tau_{s_X}, \end{cases} \quad X \in \mathfrak{g}\right).$$

The first one acts by automorphisms (resp. derivations) on A_c, and the second acts on H. If we assume S^1 is C^∞, then $\tilde{S}$ and $\check{S}$ are also C^∞, and one has

$$d(\tilde{S}) = \widetilde{dS} \quad \text{and} \quad d(\check{S}) = \widecheck{dS}.$$

$\check{S}$ is the announced natural realization of the "Fock-space linear representation associated to S" intensively used in [5]. Let us now point out why $\check{S}$ (and not $\tilde{S}$) is, in our opinion, the good linear object associated to S. The useful property of $\check{S}$ is the following: denoting $\check{S}_n$ the restriction of $\check{S}$ to H_n, one has an exact sequence

$$0 \to (H_n, \check{S}_n) \to (H_{n+1}, \check{S}_{n+1}) \to \Big(H^n, \bigotimes_{ns} S^1\Big) \to 0,$$

which means $(H_{n+1}, \check{S}_{n+1})$ is an extension of $(H_n, \check{S}_n)$ by $(H^n, \otimes_{ns} S^1)$ (see [9] for details about extensions of representations). Since extensions of representations can be described in terms of 1-cohomology, it turns out that, using $\check{S}$, many problems concerning S will be easily translated into 1-cohomological problems, for which much information is known. For instance, splitting of the above extensions will correspond to linearization of S, and this leads to a simple criterion of linearizability that we shall give in (0.9). As a second application, one has the following integrability theorem which generalizes (0.6).

Given a formal representation s of $\mathfrak{g}$ in E, assume there exists a linear representation S^1 of $\tilde{G}$ (the universal covering of G) such that $dS^1 = s^1$. Then there exists a unique formal representation S of $\tilde{G}$ such that $dS = s$. Assume, moreover, that S^1 is a representation of G and $H^1(G) = \{0\}$ (de Rham's cohomology); then S is a formal representation of G.

(0.9) Keep the notation of (0.8). We now describe a useful criterion for linearization. Let us mention that the old Poincaré condition of formal linearization of a singular vector field in $\mathbf{C}^n$ is a particular case of this criterion. We need some notation: given a linear continuous representation L of G in E, we define a continuous linear representation $L^{(n)}$ in $L_c(\hat{\otimes}_n E, E)$ by

$$L^{(n)}(T)(e_1,\ldots,e_n) = L_g\left[T\left(L_{g-1}e_1,\ldots,L_{n-1}e_n\right)\right],$$
$$T \in L\left(\hat{\otimes}_n E, E\right),\ e_j \in E,\ g \in G.$$

We denote by $H^1(G, L_c(\hat{\otimes}_n E, E))$ the first group of continuous cohomology of G with coefficients in the representation $L^{(n)}$ (see e.g. **[9]** for details about cohomology). Now, here is the condition of linearization: if we assume $H^1(G, L_c(\hat{\otimes}_n E, E)) = \{0\}$, $\forall n \geq 2$, then any formal representation S, such that $S^1 = L$, is linearizable. Similar results hold (using the corresponding cohomology group of $\mathfrak{g}_0$) in the case of formal representations of Lie algebras. Obviously, our condition of linearizability is satisfied when G is compact, or when G is semi-simple connected and E finite dimensional. It is also satisfied in many cases of physical interest (e.g. **[7]**, **[17]**, **[18]**, **[19]**).

(0.10) Assume that $\mathfrak{g}$ is a nilpotent Lie algebra, and s a formal representation of $\mathfrak{g}$ in a finite-dimensional space E. Given $\lambda \in \mathfrak{g}^*$, we define $E_\lambda = \{e \in E \mid (s^1_X - \lambda(X))^n(e) = 0,\ \forall X \in \mathfrak{g}$, for sufficiently large $n\}$. It is well known that E_λ is invariant under s^1, and we can find $\lambda_1,\ldots,\lambda_p \in \mathfrak{g}^*$ such that $E_{\lambda_i} \neq \{0\}$, $\forall i = 1,\ldots,p$, and $E = \oplus_{i=1}^p E_{\lambda_i}$ **[2]**. Given $N = (n_1,\ldots,n_p) \in \mathbf{N}^p$, we let

$$|N| = n_1 + \cdots + n_p, \quad \text{and} \quad \langle N, \lambda\rangle = \sum_{i=1}^p n_i\lambda_i.$$

PROPOSITION. *If $\langle N, \lambda\rangle \neq \lambda_j$, $\forall N \in \mathbf{N}^p$ such that $|N| \geq 2$, $\forall j = 1,\ldots,p$, then s is linearizable.*

Proof. The condition implies that the linear representation deduced from s^1 on $L(\otimes_n E, E)$, $n \geq 2$, does not contain the trivial representation. Since $\mathfrak{g}$ is nilpotent, we conclude that $H^1(\mathfrak{g}, L(\otimes_n E, E)) = \{0\}$, $\forall n \geq 2$, and then apply (0.9). □

REMARK. When $\mathfrak{g} = \mathbf{R}$, (0.10) is the well known Poincaré theorem of linearization of a formal singular vector field.

In the rest of the paper, G is a connected Lie group, $\mathfrak{g}_0$ its Lie algebra, $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$, and E a finite-dimensional (complex) vector space.

1. Multipliers and associated formal representations. Let $\mathfrak{X}$ be a differentiable manifold denumerable at infinity. We suppose that G is a Lie transformation group of $\mathfrak{X}$. We denote by $(g, x) \rightsquigarrow g \cdot x$, $g \in G$, $x \in \mathfrak{X}$, the action of G on $\mathfrak{X}$, and by U the regular representation of G on $C^\infty(\mathfrak{X}, E)$.

We define an associative product on $C^\infty(\mathfrak{X}, C(E))$ by

$$A, B \in C^\infty(\mathfrak{X}, C(E)), \qquad (A \circ B)(x) = A(x) \circ B(x),$$

and a Lie algebra structure by

$$A, B \in C^\infty(\mathfrak{X}, C(E)), \qquad [A, B](x) = [A(x), B(x)].$$

We define a linear mapping $\hat{}$ from $C^\infty(\mathfrak{X}, C(E))$ into $C(C^\infty(\mathfrak{X}, E))$ by

$$A \in C^\infty(\mathfrak{X}, C(E)), \qquad \hat{A} = \sum_{n \geq 1} \hat{A}^n,$$

where

$$[\hat{A}^n(f_1, \ldots, f_n)](x) = A^n(x)[f_1(x), \ldots, f_n(x)], \quad f_j \in C^\infty(\mathfrak{X}, E), \ x \in \mathfrak{X}.$$

It is easily seen that $\widehat{A \circ B} = \hat{A} \circ \hat{B}$, and

$$\widehat{[A, B]} = [\hat{A}, \hat{B}].$$

1.1. Definition. An element A of $C^\infty(G \times \mathfrak{X}, C(E))$ is called a multiplier if

$$A(gg', x) = A(g, x) \circ A(g', g^{-1}x), \qquad \forall g, g' \in G, \ x \in \mathfrak{X},$$
$$A(1, x) = \mathrm{Id}_E, \qquad \forall x \in \mathfrak{X}.$$

Set $A_g = A(g, \cdot)$ and $V_g^A = \hat{A}_g \circ U_g$, $g \in G$. From (1.1) V^A is a formal representation of G in $C^\infty(\mathfrak{X}, E)$, which we call the (formal) representation associated to A. Obviously A^1 is a (linear) multiplier and the linear part of V^A is V^{A^1}. Moreover, V^{A^1} is a C^∞ linear representation of G in $C^\infty(\mathfrak{X}, E)$, and therefore (by (0,7)), we can introduce the formal representation dV^A of $\mathfrak{g}$.

1.2. Proposition. *Let U denote the regular representation of G on $C^\infty(\mathfrak{X}, E)$, and set $dA_X = (d/dt)[A_{\exp tX}]_{t=0}$, $X \in \mathfrak{g}_0$. Then $dV^A = dU + \widehat{dA}$.*

Proof. Let $B(g, x) = A(g, gx), g \in G, x \in \mathfrak{X}$, and $B_g = B(g, \cdot)$. Since $V_g^A = U_g \circ \hat{B}_g$ and U_g is linear, when differentiating we obtain $\forall X \in \mathfrak{g}_0$,

$$dV_X^A = dU_X + d(\hat{B})_X, \quad \text{where} \quad d(\hat{B})_X = d/dt\left[\hat{B}_{\exp tX}\right]_{t=0}.$$

Setting $dB_X = (d/dt)[B_{\exp tX}]_{t=0}$, we have $d(\hat{B})_X = \widehat{dB_X}$. Denoting by $\mathcal{U}$ the regular representation of G on $C^\infty(\mathfrak{X}, F(E))$, we obtain $B_g = \mathcal{U}_{g^{-1}}(A_g)$ and, therefore,

$$dB_X = -d\mathcal{U}_X(A_1) + dA_X = dA_X,$$

since $A_1(x) = \mathrm{Id}_E, \forall x \in \mathfrak{X}$. □

1.3. Proposition. *Denote by $\mathcal{U}$ the regular representation of G on $C^\infty(\mathfrak{X}, C(E))$. Given $F \in L(\mathfrak{g}, C^\infty(\mathfrak{X}, C(E)))$, we set $s = dU + \hat{F}$.*

(1) *s is a formal representation of $\mathfrak{g}$ in $C^\infty(\mathfrak{X}, E)$ if and only if*

$$F_{[X,Y]} = d\mathcal{U}_X(F_Y) - d\mathcal{U}_Y(F_X) + [F_X, F_Y], \qquad \forall X, Y \in \mathfrak{g}.$$

(2) *When (1) is satisfied, there exits a unique multiplier A on $\tilde{G} \times X$ ($\tilde{G}$ the universal covering of G) such that $s = dV^A$.*

Proof.

(1) $$[s_X, s_Y] = dU_{[X,Y]} + (dU_X \circ \hat{F}_Y - D(\hat{F}_Y) \times dU_x) + (D(\hat{F}_X) \times dU_Y - dU_Y \circ \hat{F}_X) + \widehat{[F_X, F_Y]}.$$

Since $\widehat{\mathcal{U}_g(F)} = U_g \circ \hat{F} \circ U_{g^{-1}}$, we have

$$dU_X \circ \hat{F} = D(\hat{F}) \times dU_X + d\widehat{\mathcal{U}_X(F)}, \qquad \forall F \in C^\infty(\mathfrak{X}, C(E)).$$

Therefore

$$[s_X, s_Y] = dU_{[X,Y]} + \left(\widehat{d\mathcal{U}_X(F_Y) - d\mathcal{U}_Y(F_X) + [F_X, F_Y]}\right)$$

and (1) follows.

(2) We must show that there exists a multiplier A such tht $dA = F$. We first prove there exists a linear multiplier A^1 such that $dA^1 = F^1$: For fixed $X \in \mathfrak{g}_0$ and $x \in \mathfrak{X}$, denote by $C_X(x)$ the vector field on E defined by $C_X(x)(f)_{(e)} = -df_{(e)}(F_X^1(x, e))$, where $f \in C^\infty(E)$, $e \in E$. we can consider the vector fields dU_X on $\mathfrak{X}$ and $C_X(x)$ on E as vector fields on $\mathfrak{X} \times E$ by setting

$$\begin{cases} dU_X(f)(x, e) = [dU_X(f_e)](x), \\ C_X(f)(x, e) = [C_X(x)(f_x)]e, \end{cases}$$

$$\text{where} \quad f \in C^\infty(\mathfrak{X} \times E),\ f_x = f(x, \cdot),\ f_e = f(\cdot, e).$$

Introducing $\tau_X = dU_X + F^1_X$ and using (1), we obtain

$$\tau_{[X,Y]} = [\tau_X, \tau_Y], \qquad \forall X, Y \in \mathfrak{g}_0.$$

For fixed X in $\mathfrak{g}_0$, let $B^1(t,x)$ be the solution of the linear differential equation

$$\begin{cases} \dfrac{d}{dt}[B^1(t,x)] = F^1_X(\exp tX \cdot x)[B^1(t,x)], & \forall t \in \mathbf{R},\ x \in \mathfrak{X}, \\ B^1(0,x) = \mathrm{Id}_E. \end{cases}$$

The flow of τ_X is the mapping $(x, e) \rightsquigarrow (\exp tX \cdot x, B^1(t,x)(e))$, and therefore τ_X is a complete vector field. Using Palais's theorem [3], we conclude that there exists an action T of $\tilde{G}$ on $\mathfrak{X} \times E$ such that τ is the differential of T. Let $T_g(x,e) = (g \cdot x, B^1(g,x,e))$; when developing $T_{gg'} = T_g \circ T_{g'}$, we obtain $B^1(gg', x) = B^1(g, g'x) \circ B^1(g', x)$, and therefore $A^1(g,x) = B(g, g^{-1}x)$ is a linear multiplier on $\tilde{G} \times \mathfrak{X}$. Obviously, $dA^1 = F^1$.

We now construct A. We begin by the construction of $A(\exp tX, x)$ for fixed X in $\mathfrak{g}_0$. Given $H \in C^\infty(\mathfrak{X}, C(E))$, we solve the equation

$$\text{(i)} \qquad \frac{d}{dt}[C(t,x)] = F_X(\exp tX \cdot x) \circ C(t,x), \quad C(0,x) = H(x).$$

Set $F(t,x) = F_X(\exp tX, x)$, $F^1(t,x) = F^1_X(\exp tX, x)$, $A^1(t,x) = A^1(\exp tX, x)$, and $D(t,x) = A^1(-t,x) \circ C(t,x)$; we see that C is a solution of (i) if and only if

$$\text{(ii)} \quad \begin{cases} \dfrac{d}{dt}[D(t,x)] = -A^1(-t,x) \circ F^1(t,x) \circ A^1(t,tx) \circ D(t,x) \\ \qquad\qquad\qquad\qquad + A^1(-t,x) \circ F(t,x) \circ A^1(t,tx) \circ D(t,x), \\ D(0,x) = H(x). \end{cases}$$

Setting $D(t,x) = \Sigma_{n \geq 1} D^n(t,x)$, we obtain $D^1(t,x) = H^1(x)$, and then we compute $D^n(t,x)$ by induction, since (at the nth order) the right member of (ii) only contains terms of order strictly less than n.

Let $B(t,x)$ be the solution of (i) such that $B(0,x) = \mathrm{Id}_E$. Since $B(t,x) \circ H$ is still a solution of (i), with initial condition $H(x)$, we obtain $C(t,x) = B(t,x) \circ H(x)$. Now, for fixed $t' \in \mathbf{R}$,

$$E(t,x) = B(t + t', \exp - t'X \cdot x)$$

is the solution of

$$\frac{d}{dt}[E(t,x)] = F_X(\exp X) \circ E(t,x), \quad E(0,x) = B(t', \exp - t'X \cdot x).$$

Therefore

$$E(t, x) = B(t, x) \circ B(t', \exp - t'X \cdot x),$$

and we obtain

$$B(t + t', x) = B(t, \exp t'X \cdot x) \circ B(t, x).$$

Setting $A(t, x) = B(t, \exp - tX \cdot x)$, we get a multiplier on $\mathbf{R} \times \mathfrak{X}$ such that $dA = F_X$. Let $V = V^A$; we get a formal representation of $\mathbf{R}$ on $C^\infty(\mathfrak{X}, E)$. By (0.7), V is the unique solution of

$$\text{(iii)} \qquad \frac{d}{dt}[V_t] = s_X \circ V_t, \qquad V_0 = \mathrm{Id}_{C^\infty(X,E)}.$$

On the other hand, s^1 is a linear representation of $\mathfrak{g}_0$, integrable to the representation V^{A^1} of $\tilde{G}$. Owing to (0.8) or [5], there exists a formal representation W of $\tilde{G}$ on $C^\infty(\mathfrak{X}, E)$ such that $dW = s$. $W_{\exp tX}$ and V_t are both solutions of (iii), which proves that $V_t = W_{\exp tX}$.

We introduce some notation: given $e \in E$, set $\xi_e(x) = e$, $\forall x \in \mathfrak{X}$, $F^n(g, x) = W_g^n[\xi_e](x)$, and $F(t, x) = \Sigma_{n \geq 1} F^n(g, x)1$. Since $V_t = W_{\exp tX}$, we have

$$A^n(t, x)(e) = V_t^n(\xi_e)(x) = F^n(\exp tX, x)(e).$$

We conclude that $A(t, x) = F(\exp tX, x)$. Since X is arbitrary in $\mathfrak{g}_0$, we have proved there exists an element $A \in C^\infty(\tilde{G} \times \mathfrak{X}, C(E))$ such that $A_X(t, x) = A(\exp tX, x)$ is an $\mathbf{R} \times \mathfrak{X}$ multiplier for each X in $\mathfrak{g}_0$. Moreover, $W_{\exp tX} = V_t^{A_X} = V_{\exp tX}^A$.

Using the mappings ξ_e, $e \in E$, and the fact that W is a representation of $\tilde{G}$, it follows that A satisfies the conditions of a multiplier in a neighbourhood of 1 in $\tilde{G}$. Therefore, V^A satisfies the conditions of a representation in a neighbourhood of 1 in $\tilde{G}$. Since $\tilde{G}$ is simply connected, we conclude that V^A is a representation of $\tilde{G}$, and A is a multiplier over $\tilde{G} \times \mathfrak{X}$. Since $W = V^A$ and, by definition, $dW = s$, we obtain $dV^A = s$. □

1.4. Definition. Multipliers A and A' on $G \times \mathfrak{X}$ are equivalent if there exists $F \in C^\infty(\mathfrak{X}, C(E))$ such that

$$F(g) \in D(E), \quad \forall g \in G,$$

and

$$A'(g, x) = F(x)^{-1} \circ A(g, x) \circ F(g^{-1}x), \quad \forall g \in G, \ x \in \mathfrak{X}.$$

We say that A is linearizable (resp. trivial) if it is equivalent to a linear multiplier (resp. to the trivial multiplier), i.e. a multiplier with values in $L(E)$ (resp. $A(g, x) = \mathrm{Id}_E$, $\forall g \in G, x \in \mathfrak{X}$).

Note that V^A and $V^{A'}$ are equivalent as soon as A and A' are equivalent (the intertwining mapping is $\hat{F}$).

2. Multipliers on homogeneous spaces and applications. In this section we fix a closed subgroup Γ of G and assume that $\mathfrak{X} = G/\Gamma$ with the canonical action. The following lemma is easily obtained from (1.1).

2.1. LEMMA. *Given a multiplier A on $G \times \mathfrak{X}$, we set $b(g) = A(g^{-1}, \dot{1})$; $I_\gamma = A(\gamma, \dot{1})$, $B(g, x) = A(g, gx)$, $c(g) = B(g, \dot{1})$ for $g \in G, \gamma \in \Gamma, x \in \mathfrak{X}$.*

I is a formal representation of Γ in E, called the formal representation associated to A. Moreover, one has:

$$\begin{gathered} b(g\gamma) = I_{\gamma^{-1}} b(g), \quad b(1) = \mathrm{Id}_E, \\ A(g, \dot{h}) = b(h)^{-1} \circ b(g^{-1}h), \qquad \forall g, h \in G, \ \gamma \in \Gamma, \\ B(gg', x) = B(g, g'x) \circ B(g, x), \\ c(g) = b(g)^{-1}, \quad c(g\gamma) = c(g) \circ I_\gamma, \quad c(1) = \mathrm{Id}_E, \\ B(g, \dot{h}) = c(gh) \circ c(h)^{-1}, \qquad \forall g, g', h \in G, \ x \in \mathfrak{X}, \ \gamma \in \Gamma. \end{gathered}$$

2.2. PROPOSITION. *Multipliers A and A' are equivalent if and only if the associated representations I and I' are equivalent.*

COROLLARY. *A is linearizable if and only if I is linearizable.*

Proof. Assume A and A' are equivalent, then obviously I and I' are equivalent.

Conversely, assume there exists $U \in D(E)$ such that $I' = U \circ I \circ U^{-1}$. Using the mappings b and b' of (2.1) we set $f(g) = b(g)^{-1} \circ U^{-1} \circ b'(g)$. We see that $f(1) = U^{-1}$, and since $f(g\gamma) = f(g)$, $\forall g \in G, \gamma \in \Gamma$, we can find $F \in C^\infty(\mathfrak{X}, C(E))$ such that $F(\dot{g}) = f(g)$, $\forall g \in G$. One has $b'(g) = F(\dot{1})^{-1} \circ b(g) \circ F(\dot{g})$, and, by (2.1),

$$A'(g, x) = F(x)^{-1} \circ A(g, x) \circ F(g^{-1}x). \qquad \square$$

2.3. DEFINITION. Given a formal representation I of Γ in E, we say that I extends to the multiplier A on $G \times \mathfrak{X}$ if $I_\gamma = A(\gamma, \dot{1})$, $\forall \gamma \in \Gamma$.

Note that I might extend to several multipliers on $G \times \mathfrak{X}$, but, owing to (2.3), they are all equivalent.

2.4. PROPOSITION. *Assume there exists a C^∞ section s of the projection π: $G \to G/\Gamma = \mathfrak{X}$; then any formal representation of Γ extends to a multiplier on $G \times \mathfrak{X}$.*

Proof. Set $\gamma(g) = s(\pi(g))^{-1}g$, and define $A(g, x) = I^{-1}_{\gamma(g^{-1}s(x))}$, $g \in G$, $x \in \mathfrak{X}$. A is a multiplier on $G \times \mathfrak{X}$ which extends I. □

2.5. PROPOSITION. *Given a formal representation I of Γ in E, assume that I^1 extends to a (linear) multiplier on $G \times \mathfrak{X}$; then I extends to a multiplier on $G \times \mathfrak{X}$.*

Proof. Choose $f \in C^\infty(G)$ such that $\gamma \rightsquigarrow f(g\gamma)$ is compactly supported in Γ, and $\int_\Gamma f(g\gamma)\, d\gamma = 1$ for any $g \in G$ **[21]**.

Let $F(g) = \int_\Gamma I^1_\gamma f(g\gamma) I^{-1}_\gamma\, d\gamma$. We first note that $F^1(g) = \mathrm{Id}_E$, $\forall g \in G$, and, using the invariance of $d\gamma$, that $F(g\gamma) = I^1_{\gamma-1}F(g)I_\gamma$, $\forall g \in G$, $\gamma \in \Gamma$. Let us now introduce $d(g) = (b^1(g))^{-1} \circ F(g)$ (see (2.1) for notation). By (2.1) and the preceding results, $d(g\gamma) = d(g) \circ I_\gamma$, $\forall g \in \Gamma$, $\gamma \in \Gamma$, and $d^1(g) = (b^1(g))^{-1}$. Therefore, $d(g) \in D(E)$, $\forall g \in G$, and we can define $b(g) = (d(g))^{-1} \circ d(1)$. From the properties of d, we now deduce that $b(g\gamma) = I_{\gamma-1} \circ b(g)$, $\forall g \in G$, $\gamma \in \Gamma$, $b(1) = \mathrm{Id}_E$, and $b(\gamma) = I_{\gamma-1}$, $\forall \gamma \in \Gamma$.

Let us now define a multiplier $\tilde{A}$ over $G \times G$ by the formula

$$\tilde{A}(g, h) = (b(h))^{-1} \circ b(g^{-1}h), \qquad \forall g, h \in G.$$

Owing to the properties of b, $\tilde{A}$ satisfies

$$\tilde{A}(\gamma, 1) = I_\gamma, \quad \tilde{A}(g, h\gamma) = \tilde{A}(g, h), \qquad \forall g, h \in G, \gamma \in \Gamma.$$

It results from the last equality that there exists a multiplier A on $G \times \mathfrak{X}$ such that $A(g, \dot{h}) = \tilde{A}(g, h)$, $\forall g, h \in G$. Since $A(\gamma, \dot{1}) = I_\gamma$, $\forall \gamma \in \Gamma$, the proof is complete.

3. **Systems associated to multipliers.** In this section we let Γ be a closed subgroup of G, and $\mathfrak{X}$ the homogeneous space G/Γ.

Given a tlcvs F, to each element X of $\mathfrak{g}_0$ we associate the vector field X acting on $C^\infty(G, F)$ by

$$X(f)_g = \frac{d}{dt}[f(\exp(-tX) \cdot g)]_{t=0}, \qquad \forall f \in C^\infty(G, F).$$

We identify $C^\infty(\mathfrak{X}, F)$ and the subspace of Γ-periodic functions in $C^\infty(G, F)$, i.e. $\{f \in C^\infty(G, F)/f(g\gamma) = f(g),\ \forall g \in G,\ \gamma \in \Gamma\}$. Using this notation, a multiplier over $G \times \mathfrak{X}$ is actually a multiplier over $G \times G$ which is Γ periodic with respect to the second variable.

We will now study the solvability of some systems related to multipliers, as we shall see. Given $F \in L(\mathfrak{g}, C^\infty(E))$, we consider:

$$\text{(i)} \qquad \begin{cases} X(f)_g = -F_X(g) \circ f(g), & \forall g \in G,\ X \in \mathfrak{g}, \\ f(1) = T, \end{cases}$$

$$\text{(ii)} \qquad \begin{cases} X(h)_g = -\pi_{F_X(g)}(h(g)), & \forall g \in G,\ X \in \mathfrak{g}\ (\text{notation } (0.5)), \\ h(1) = T, \end{cases}$$

where T is a given element of $C(E)$ (the initial condition), and the unknown f (resp. h) is an element of $C^\infty(G, C(E))$.

3.1. Definition. (i) (resp. (ii)) is integrable if there exists a solution c satisfying $c(1) = \mathrm{Id}_E$ (resp. $b(1) = \mathrm{Id}_E$); c is called the fundamental solution. We say that the Cauchy problem for (i) (resp. (ii)) can be solved if there exists a unique solution satisfying the given initial condition for any initial condition in $C(E)$.

We often make use of the following technical lemma.

3.2. Lemma. *Given $f, h \in C^\infty(G, C(E))$ and $X \in \mathfrak{g}_0$, one has*

$$X(f \circ h)_g = X(f)_g \circ h(g) + (D(f(g)) \circ h(g)) \times X(h)_g, \quad \forall g \in G.$$

3.3. Proposition. *The following statements are equivalent*:

(1) (i) *is integrable.*

(2) (ii) *is integrable.*

(3) *The Cauchy problem for* (i) *can be solved.*

(4) *The Cauchy problem for* (ii) *can be solved.*

(5) *There exists a unique multiplier A over $G \times \mathfrak{X}$ such that $F = dA$.*

Whenever these conditions are satisfied one has

(6) *F satisfies the Frobenius compatibility condition*

$$F_{[X,Y]} = X(F_Y) - Y(F_X) + [F_X, F_Y], \quad \forall X, Y \in \mathfrak{g}.$$

Moreover, the fundamental solution c of (i) *satisfies $c(g) \in D(E)$, $\forall g \in G$ and $c(g\gamma) = c(g) \circ c(\gamma)$, $\forall g \in G$, $\gamma \in \Gamma$. c is related to the fundamental solution b of* (i) *by $b(g) = c(g)^{-1}$, $\forall g \in G$. The general solution f of* (i) *(resp. h of* (i)*) is $f(g) = c(g) \circ T$ (resp. $h(g) = T \circ b(g)$). The multiplier of* (5) *is*

$$A(g, x) = b(x)^{-1} \circ b(g^{-1}x), \quad \forall g, x \in G.$$

Finally, if G is simply connected, (6) *is equivalent to* (1)–(5).

In order to prove (2.7), we first prove the following lemma.

3.4. LEMMA. *Suppose there exists a solution f of* (i) *(resp. h of* (ii)) *such that $f(1) \in D(E)$ (resp. $h(1) \in D(E)$). Then* (i) *(resp.* (ii)) *is integrable, and there exists a fundamental solution c (resp. b) such that $c(g) \in D(E)$ (resp. $b(g) \in D(E)$), $\forall g \in G$.*

Proof. f^1 is a solution of the linear system $X(f^1)_g = -F_X^1(g) \circ f^1(g)$, $\forall g \in G$, $X \in \mathfrak{g}$, which can be written

$$\frac{d}{dt}[f^1(\exp tX \cdot g)] = F_X^1(\exp tX \cdot g) \circ f^1(\exp X \cdot g),$$

from which we deduce that

$$\det[f^1(\exp tX \cdot g)] = \left[\exp\left[\int_0^t \operatorname{Tr}\left(F_X^1(\exp sX \cdot g)\right) ds\right]\right] \det\left(f^1(g)\right).$$

Recalling that $f(1) \in D(E)$ if and only if $\det(f^1(1)) \neq 0$, we obtain

$$\det\left(f^1(\exp X)\right) \neq 0, \quad \forall X \in \mathfrak{g}$$

and, by induction, $\det(f^1(\exp X_1 \cdots \exp X_p)) \neq 0$, $\forall X_1, \ldots, X_p \in \mathfrak{g}$. Therefore $f(g) \in D(E)$, $\forall g \in G$ (G is connected !). We now introduce $c(g) = f(g) \circ (f(1))^{-1}$ and see that c is a solution of (i) satisfying $c(1) = \mathrm{Id}_E$ (i.e. a fundamental solution) and $c(g) \in D(E)$, $\forall g \in G$. Similar arguments hold in case (ii). □

Proof of (2.7). Assume (5). Using (1.2) and (1.3)(1), we obtain (6). With the notation of (2.1), we have

$$\frac{d}{dt}[c(\exp tX \cdot g)]_{t=0} = \frac{d}{dt}[B(\exp tX, g) \circ c(g)]_{t=0} = dB_X(g) \circ c(g),$$

$$\begin{aligned} \frac{d}{dt}[b(\exp - tX \cdot g)]_{t=0} &= \frac{d}{dt}[b(g) \circ A(\exp tX, g)]_{t=0} \\ &= D(b(g)) \times dA_X(g) = -\pi_{dA_X(g)}(b(g)). \end{aligned}$$

It is proved in the proof of (1.2) that $dA = dB$. Moreover, $c(1) = b(1) = \mathrm{Id}_E$ ((2.1)). Therefore we obtain (1) and (2).

Assume (6) and G simply connected. Using (1.3)(2), we obtain (5).

Assume (1). Given a fundamental solution c, we apply (3.4) and see that $c(g) \in D(E)$, $\forall g \in G$.

We introduce $B(g, x) = c(gx) \circ c(x)^{-1}$, $A(g, x) = B(g, g^{-1}x)$, and easily verify that A is a multiplier over $G \times G$. Using the definition of B, we have

$$\frac{d}{dt}[c(\exp tX\cdot g)]_{t=0}=\frac{d}{dt}[B(\exp tX,g)\circ c(g)]=dB_X(g)\circ c(g).$$

It is proved in the proof of (1.2) that $dA = dB$, and we obtain

$$X(c)_g=-dA_X(g)\circ c(g).$$

Comparing with (i), we deduce that $F = dA$.

Now set $b(g)=c(g)^{-1}$. Since $A(g,x)=b(x)^{-1}\circ b(g^{-1}x)$, we have

$$\frac{d}{dt}[b(\exp -tX\cdot g)]_{t=0}=\frac{d}{dt}[b(g)\circ A(\exp tX,g)]_{t=0}$$
$$=D(b(g))\times dA_X(g)=-\pi_{dA_X(g)}(b(g))=-\pi_{F_X(g)}(b(g)).$$

This proves (2).

Now set $f(g)=c(g)\circ T$, it is easily seen that f is a solution of (i) and $f(1)=T$.

Conversely, suppose f is a solution of (i) and $f(1)=T$. Introducing $f'(g)=c(g)^{-1}\circ f(g)$ and using (2.6), we obtain

$$\begin{aligned}X(f')_g&=X(b)_g\circ f(g)+(D(b(g))\circ f(g))\times X(f)_g\\&=X(b)_g\circ f(g)-(D(b(g))\circ f(g))\times(F_X(g)\circ f(g))\\&=(X(b)_g-D(b(g))\times F_X(g))\circ f(g)\\&=\left(X(b)_g+\pi_{F_X(g)}(b(g))\right)\circ f(g)=0.\end{aligned}$$

Therefore $f'(g)=T$ and $f(g)=c(g)\circ T$. This proves (3). Similarly, set $h(g)=T\circ b(g)$; it is easily seen that h is a solution of (ii) and $h(1)=T$. Suppose now that h is a solution of (ii) and $h(1)=T$. We introduce $h'(g)=h(g)\circ b(g)^{-1}$ and use (3.2) to obtain

$$\begin{aligned}X(h')_g&=X(h)_g\circ c(g)+(D(h(g))\circ c(g))\times X(c)_g\\&=X(h)_g\circ c(g)-(D(h(g))\circ c(g))\times(F_X(g)\circ c(g))\\&=(X(h)_g-D(h(g))\times F_X(g))\circ c(g)\\&=\left(X(h)_g+\pi_{F_X(g)}(h(g))\right)\circ c(g)=0.\end{aligned}$$

Therefore $h'(g)=T$ and $h(g)=T\circ b(g)$. This proves (4).

Let us now introduce, for fixed γ in Γ, $c^\gamma(g)=c(g\gamma)$, $g\in G$.

$$X(c^\gamma)_g=-F_X(g\gamma)\circ c(g\gamma)=-F_X\circ c^\gamma(g).$$

Since $c^{\gamma}(1) = c(\gamma)$, using the preceding results, we get $c^{\gamma}(g) = c(g) \circ c(\gamma)$, i.e. $c(g\gamma) = c(g) \circ c(\gamma)$, $\forall g \in G$, $\gamma \in \Gamma$. Going back to the definition of A and using the last relation, we see that A is a multiplier over $G \times \mathfrak{X}$. This proves (5).

Summarizing, starting from (1), we have proven (2)–(5). Moreover, we have proven the last assumption of (3.3) during the proof.

Finally, let us assume (2). Since the proof is quite similar, we omit details:

Given a fundamental solution b of (ii), we introduce the multiplier $A(g, x) = b(x)^{-1} \circ b(g^{-1}x)$ over $G \times G$, $c(g) = b(g)^{-1}$, $B(g, x)$ $(= A(g, gx)) = c(gx) \circ c(x)^{-1}$. Differentiating the relation of the definition of A, we obtain $F = dA$; differentiating the relation of the definition of B we obtain that c is a fundamental solution of (i). This proves (1). □

3.5. Proposition. *Suppose that a given* $F \in L(\mathfrak{g}, C^{\infty}(\mathfrak{X}, C(E)))$ *satisfies the Frobenius condition* ((3.3)(6)) *and define* F' *by*

$$F'_X(g) = -X(l)_g \circ l(g)^{-1} + \big(D(l(g)) \times F_X(g)\big) \circ l(g)^{-1},$$

$$\forall g \in G, X \in \mathfrak{g},$$

for fixed l *in* $C^{\infty}(\mathfrak{X}, C(E))$ *such that* $l(g) \in D(E)$, $\forall g \in G$. *Then* F' *also satisfies the Frobenius condition.*

Proof. Using (3.3) there exists a multiplier A over $\tilde{G} \times \mathfrak{X}$ such that $dA = F$. We define a multiplier A' over $\tilde{G} \times \mathfrak{X}$ by $A'(g, x) = l(x) \circ A(g, x) \circ l(g^{-1}x)^{-1}$. Differentiating the equality $A'(g, x) \circ l(g^{-1}x) = l(x) \circ A(g, x)$ at the point 1, we obtain $F' = dA'$. □

3.6. Proposition. *Given the integrable systems* (i) *and* (i)′ *with corresponding functions* F *and* F', *corresponding multipliers* A *and* A', *fundamental solutions* c *and* c', $I = c|\Gamma$ *and* $I' = c'|\Gamma$, *the following statements are equivalent*:

(1) *A and A' are equivalent multipliers.*

(2) *I and I' are equivalent formal representations of* Γ.

(3) *There exists* $l \in C^{\infty}(\mathfrak{X}, C(E))$ *such that*: $l(g) \in D(E)$, $\forall g \in G$, *and*

$$F'_X(g) = -X(l)_g \circ l(g)^{-1} + \big(D(l(g)) \times F_X(g)\big) \circ l(g)^{-1},$$

$$\forall g \in G,\ X \in \mathfrak{g}.$$

(4) *There exists $l \in C^\infty(\mathfrak{X}, C(E))$ such that $l(g) \in D(E)$, $\forall g \in G$, and f' is a solution of* (i)$'$ *if and only if there exists a solution f of* (i) *such that $f'(g) = l(g) \circ f(g)$, $\forall g \in G$.*

Whenever these conditions hold, we say that (i) *and* (i)$'$ *are equivalent.*

Proof. (1) and (2) are equivalent by (2.2).

Assuming (1), there exists $l \in C^\infty(\mathfrak{X}, D(E))$ such that $A'(g, x) = l(x) \circ A(g, x) \circ l(g^{-1}x)^{-1}$. Differentiating the equality $A'(g, x) \circ l(g^{-1}x) = l(x) \circ A(g, x)$ at the point; we obtain (3).

Moreover, one has $c'(g) = l(g) \circ c(g) \circ l(1)^{-1}$; f' is a solution of (i)$'$ if and only if $f'(g) = c'(g) \circ T$, $\forall g \in G$, for a given $T \in C(E)$ (see (3.3)). Setting $f(g) = c(g) \circ l(1)^{-1} \circ T$, we obtain that f' is a solution of (i)$'$ and only if $f'(g) = l(g) \circ f(g)$, $\forall g \in G$, i.e. (4).

Assuming (3), we see that the multipliers A' and $A''(g, x) = l(x) \circ A(g, x) \circ l(g^{-1}x)^{-1}$ satisfy $dA' = dA''$ and, therefore, coincide. So we obtain (1).

Assuming (4), we get $c'(g) = l(g) \circ c(g) \circ T$, $\forall g \in G$. Taking $g = 1$, we get $T = l(1)^{-1}$. From $c'(g) = l(g) \circ c(g) \circ l(1)^{-1}$, we deduce that

$$A'(g, x) = l(x) \circ A(g, x) \circ l(g^{-1}x)^{-1}, \quad \forall g, x \in G$$

(see (2.1)). So we obtain (1). □

4. Scalar multipliers. In this section, $\mathfrak{X}$ is a differentiable manifold, denumerable at infinity, and G acts as a Lie transformation group of $\mathfrak{X}$. We assume $E = \mathbf{C}$. Multipliers over $G \times \mathfrak{X}$ with values in $C(\mathbf{C})$ are called scalar multipliers.

We denote by U the regular representation of G on $C^\infty(X)$.

4.1. PROPOSITION. *Let ξ be an element of $Z^1(G, U)$ and T a formal representation of* $\mathbf{R}$ *in* $\mathbf{C}$; *then*

$$A(g, x) = T_{\xi_g(x)}, \quad \forall g \in G,\ x \in \mathfrak{X},$$

defines a scalar multiplier over $G \times \mathfrak{X}$. Whenever T and T' are equivalent representations, or whenever ξ and ξ' are equivalent cocycles, then the corresponding multipliers are equivalent.

We omit the proof, which is a trivial calculation. We next give an answer to the inverse problem of (4.1) in the case of a linear scalar multiplier.

4.2. PROPOSITION. *Given a linear scalar multiplier A over $G \times \mathfrak{X}$ there exists $\xi \in Z^1(\tilde{G}, U)$ ($\tilde{G}$ the universal covering of G) such that $A(g, x) = \exp \xi_g(x)$, $\forall g \in \tilde{G}$, $x \in \mathfrak{X}$. If $H^1(G) = \{0\}$ (de Rham's cohomology), then $\xi \in Z^1(G, U)$.*

Proof. Since A is linear scalar, one has

$$dA_{[X,Y]} = dU_X(dA_Y) - dU_Y(dA_X), \quad \forall X, Y \in \mathfrak{g},$$

and therefore $dA \in Z^1(\mathfrak{g}, dU)$. Using **[16]**, there exists $\xi \in Z^1(\tilde{G}, U)$ (resp. $Z^1(G, U)$ if $H^1(G) = \{0\}$) such that $dA = d\xi$. For fixed X in $\mathfrak{g}_0$, let us note that $\theta_t(x) = A(\exp tX, x)$, $t \in \mathbf{R}$, $x \in X$. Using the multiplier's relation, we obtain

$$d\theta_t/dt = \theta_t \cdot U_t[dA_X] = \theta_t \cdot U_t(d\xi_X).$$

Therefore

$$\theta_t = \exp \int_0^t U_s(d\xi_X)\, ds = \exp \xi_{\exp tX}$$

(see **[16]**).

This proves that A and the multiplier $\exp \xi$ coincide on a neighborhood of 1, and therefore everywhere by connectedness. □

(4.3) Given a linear scalar multiplier A, and using (4.2), we can define a continuous series A^α, $\forall \alpha \in \mathbf{C}$, of linear scalar multipliers if the associated cocycle lies in $Z^1(G, U)$. If it lies strictly in $Z^1(G, U)$ we can only define a discrete series A^n, $\forall n \in Z$. Let us show by an example how to translate this remark in terms of the theory of linear representations of G:

(1) Set $G = \mathrm{SU}(1,1)$, $\mathfrak{X} = [z \in \mathbf{C} \mid |z| < 1]$, with action

$$g \cdot z = \frac{\bar{\alpha} z + \bar{\beta}}{\beta z + \alpha}, \quad \text{if } g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \ |\alpha|^2 - |\beta|^2 = 1.$$

We introduce the multiplier $A(g, z) = (\bar{\alpha} - \beta z)$. It is easily seen that $A = \exp \xi$, where ξ is a cocycle in $Z^1(\tilde{G}, U)$ which does not belong to $Z^1(G, U)$. It is well known that the representations V^{A^n} are realizations (when restricted to adapted subspaces) of the discrete series of representations of G.

(2) Set $G = \mathrm{SU}(1,1)$, $\mathfrak{X} = \{ z \in \mathbf{C} \mid |z| = 1\}$, with action

$$g \cdot z = \frac{\bar{\alpha} z + \bar{\beta}}{\beta z + \alpha}, \quad \text{if } g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \ |\alpha|^2 - |\beta|^2 = 1.$$

We introduce the multiplier $A(g, z) = |\bar{\alpha} - \beta z|^2$. It is easily seen that $A = \exp \xi$, where ξ is a cocycle in $Z^1(G, U)$. The representations V^{A^α},

$\alpha \in \mathbf{C}$, are realizations of one of the continuous series of representations of G.

4.4. LEMMA [13] or [11]. *Assume $\mathfrak{X} = G/\Gamma$ and define a linear mapping R from $Z^1(G,U)$ into $Z^1(\Gamma, \mathrm{Id}_{\mathbf{C}})$ by $R(\xi)_\gamma = \xi_\gamma(\dot{1})$, $\forall \gamma \in \Gamma, \xi \in Z^1(G,U)$. R is onto and induces an isomorphism of $H^1(G,U)$ and $H^1(\Gamma, \mathrm{Id}_{\mathbf{C}})$.*

Proof. Denote by $\tilde{U}$ the left regular representation of G on $C^\infty(G)$. Given $\xi \in Z^1(G,U)$, we define $\tau \in C^\infty(G)$ by $\tau(g) = \xi_{g^{-1}}(\dot{1})$ and easily obtain

$$\xi_{g'}(\dot{g}) = \tilde{U}_{g'}(\tau)(g) - \tau(g), \qquad \forall g, g' \in G.$$

Assuming that $R(\xi) = 0$, one has $\tau_{g\gamma} = \tau_g$, $\forall g \in G$, $\gamma \in \Gamma$, and by the later formula, $\xi \in B^1(G,U)$. Obviously, $R(B^1(G,U)) = \{0\}$.

We next prove that R is onto: given $\theta \in Z^1(\Gamma, \mathrm{Id}_{\mathbf{C}})$, we choose an element ψ of $C^\infty(G)$ such that $\gamma \rightsquigarrow \psi(g\gamma)$ is compactly supported in Γ and $\int_\Gamma \psi(g\gamma)\,d\gamma = 1$, $\forall g \in G$ [21], and we set $\tau(g) = \int_\Gamma \theta_\gamma \psi(g\gamma)\,d\gamma - \int_\Gamma \theta_\gamma \psi(\gamma)\,d\gamma$. Using the invariance of $d\gamma$, we obtain $\tau(g\gamma) = \tau(g) + \theta_{\gamma-1}$, $\forall g \in G$, $\gamma \in \Gamma$. Since $\tau(1) = 0$, we obtain $\tau(\gamma) = \theta_{\gamma-1}$, $\forall \gamma \in \Gamma$. Setting $\tilde{\xi}_g = \tilde{U}_g(\tau) - \tau$, we obtain an element of $Z^1(G, \tilde{U})$ satisfying $\tilde{\xi}_\gamma(1) = \theta\gamma$, $\forall \gamma \in \Gamma$. Since $\tilde{\xi}_{g'}(g\gamma) = \tilde{\xi}_{g'}(g)$, $\forall g, g' \in G$, $\gamma \in \Gamma$, there exists $\xi \in Z^1(G,U)$ such that $\xi_{g'}(\dot{g}) = \tilde{\xi}_{g'}(g)$, $\forall g, g' \in G$. Obviously, $R(\xi) = \theta$. □

4.5. COROLLARY. *Assume $\mathfrak{X} = G/\Gamma$. Assume, moreover, that Γ is connected and $H^1(\Gamma) = \{0\}$ (de Rham's cohomology). Then any (linear) character of Γ can be extended to a multiplier over $G \times \mathfrak{X}$.*

Proof. dI is a cocycle of the trivial representation of γ, and since $H^1(\Gamma) = \{0\}$, there exists a cocycle θ on Γ such that $dI = d\theta$ [16]. Define $I'_\gamma = \exp \theta_\gamma$, $\forall \gamma \in \Gamma$; I' is a character of Γ. Since $dI' = dI$, the connectedness of Γ implies $I' = I$. From (4.4) we can find $\xi \in Z^1(G,U)$ such that $R(\xi) = \theta$.

Now the multiplier $\exp \xi$ extends I. □

On the other hand:

4.6. PROPOSITION. *Assume $H^1(G) = \{0\}$ (de Rham's cohomology) and $\mathfrak{X} = G/\Gamma$; let K be a compact subgroup of Γ and I a linear character of Γ; if there exists a linear scalar multiplier A over $G \times \mathfrak{X}$ extending I, then one has $I|_K = \mathrm{Id}_{\mathbf{C}}$.*

Proof. Let $\xi \in Z^1(G, U)$ such that $A = \exp \xi$ ((4.2)). We have $I_\gamma = A(\gamma, \dot{1}) = \exp R(\xi)_\gamma$, $\forall \gamma \in \Gamma$. $R(\xi)_{|K}$ is a cocycle of the trivial representation of K and, since K is compact, $R(\xi)_{|K} = 0$ and $I_{|K} = \mathrm{Id}_{\mathbf{C}}$. □

(4.7) In physics, linear induced representations are often realized as multiplier representations on homogeneous spaces. This can be done because one is dealing with unitary representations, and therefore the involved functions (and multipliers) are only assumed to be Borel functions. Since there always exists a Borel section of the projection $G \to G/\Gamma$, realization on the homogeneous space follows. Nevertheless, as soon as one is concerned with differentiability properties, singularities do appear, and this is the case, for instance, for massless particles [**4**]. Let us show how this can be seen using (4.6).

Given a (linear) character I of Γ, we denote by H^I the space

$$H^I = \left\{ f \in C^\infty(G) | f(g\gamma) = I_{\gamma^{-1}} f(g), \forall g \in G, \gamma \in \Gamma \right\}.$$

We define the C^∞ (linearly) induced representation U^I as the restriction of the left regular representation of G to H^I.

PROPOSITION. *Assume that I extends to a multiplier A over $G \times \mathfrak{X}$. Then V^A and U^I are equivalent (linear) representations of G.*

Proof. We introduce the function c of (2.1). Recall that $c(g\gamma) = c(g) I_\gamma$, $\forall g \in G$, $\gamma \in \Gamma$, and set $T(f)(g) = c(g) f(g)$, $f \in H^I$. Obviously T is an isomorphism of H^I onto $C^\infty(\mathfrak{X})$, and $V^A = T \circ U^I \circ T^{-1}$. □

We know cases where (4.7) can be applied (see (2.2) and (4.5)). Let us now give some no-go examples, where U^I cannot be realized as a multiplier representation on $C^\infty(G/\Gamma)$:

(1) Set $G = \mathrm{SO}(3)$, $\Gamma = \mathrm{SO}(2)$, and apply (4.6) to $K = \Gamma$: the only (linear) character of Γ that can be extended to a multiplier over $G \times G/\Gamma$ is the trivial one.

(2) Set $G = \mathrm{SL}(2, \mathbf{C})$, acting on $\mathbf{R}^4$ via the standard surjection $\mathrm{SL}(2, \mathbf{C}) \to \mathrm{SO}_0(3,1)$. Let x be a given point of the vertexless forward light cone, and Γ the stabilizer of x. Actually, Γ is a semi-direct product $K \cdot \mathbf{R}^2$, where K is a one-dimensional torus. Using (4.6) we see that characters of Γ that can be extended to multipliers over G/Γ are necessarily trivial on K. Using physics terminology this result shows that singularities do appear in the case of massless particles with non zero helicity (see [**4**] for development).

(4.8) Let us return to the construction of (non linear) scalar multipliers.

Given a linear scalar multiplier m over $G \times \mathfrak{X}$, we define a linear representation V^n on $C^\infty(\mathfrak{X})$ by

$$V_g^n(F)(x) = \left(m_g(x)\right)^{-(n-1)} F(g^{-1}x).$$

PROPOSITION. *Given $\xi \in Z^1(G, V^n, C^\infty(\mathfrak{X}))$, $n \geq 2$, the formal development of*

$$A_g(x)(z) = \frac{m(g,x)z}{\left(1 - \xi_g(x) m(g,x)^{n-1} z^{(n-1)1/(n-1)}\right)},$$

$$g \in G, x \in \mathfrak{X}, \ z \in \mathbf{C},$$

defines a scalar multiplier on $G \times \mathfrak{X}$. A is linearizable if and only if ξ is a coboundary.

Proof. The first assertion is a simple calculation. Assume next that

$$\xi_g(x) = (m(g,x))^{-(n-1)} f(g^{-1}x) - f(x), \qquad f \in C^\infty(\mathfrak{X}).$$

The formal development of

$$T_x(Z) = \frac{z}{\left[1 - f(x)z^{n-1}\right]1(n-1)}, \qquad x \in \mathfrak{X}, \ z \in \mathbf{C},$$

defines an element T of $C^\infty(\mathfrak{X}, D(\mathbf{C}))$, and it is easily seen that

$$m_g(x) \circ T(g^{-1}x) = T(x) \circ A(g,x), \quad \forall g \in G, \ x \in X. \qquad \square$$

5. Linearization results.

5.1. PROPOSITION. *Given a multiplier A over $G \times \mathfrak{X}$, for any compact subgroup K of G there exists an equivalent multiplier A' such that A' is linear on K.*

Proof. Define

$$B(x) = \int_K A^1(k,x) \circ A(k^{-1}, k^{-1}x)\, dk, \quad \forall x \in \mathfrak{X}.$$

Since $B^1(x) = \mathrm{Id}_E$, we get that $B(x) \in D(E)$, $\forall x \in \mathfrak{X}$. Moreover, using the invariance of dk, one obtains

$$A^1(k,x) \circ B(k^{-1}x) = B(x) \circ A(k,x), \quad \forall k \in K, \ x \in \mathfrak{X}.$$

It results from this last equality that the equivalent multiplier

$$A'(g, x) = B(x) \circ A(g, x) \circ B(g^{-1}x)^{-1}, \qquad g \in G,\ x \in \mathfrak{X},$$

is linear on K. $\square$

5.2. COROLLARY. *Assume G is compact; then any multiplier A over $G \times \mathfrak{X}$ is linearizable.*

5.3. PROPOSITION. *Let M be a manifold on which G acts as a Lie transformation group. Suppose $\mathfrak{X} = G/\Gamma \times M$, where Γ is a compact subgroup of G and the action of G in $\mathfrak{X}$ is given by $g \cdot (\dot{h}, m) = (g \cdot \dot{h}, g, m)$, $g, h \in G$, $m \in M$. Then any multiplier over $G \times \mathfrak{X}$ is linearizable.*

Proof. Using (5.1) we can suppose that $A|_\Gamma = A^1|_\Gamma$. We introduce

$$b(g, m) = \left[A^1_{g-1}(\dot{1}, g^{-1}m)\right]^{-1} \circ A_{g-1}(\dot{1}, g^{-1}m), \qquad g \in G,\ m \in M.$$

Using the multiplier's relation, we see that

$$A_g(\dot{g}', m) = (b(g', m))^{-1} \circ A^1_g(\dot{g}', m) \circ b(g^{-1} \cdot (g', m)),$$
$$\forall g, g' \in G,\ m \in M.$$

Moreover, for any $g \in G$, $k \in \Gamma$, $m \in M$, one has

$$\begin{aligned} b(gk, m) &= \left[A^1_{k^{-1}g^{-1}}(\dot{1}, k^{-1}g^{-1}m)\right]^{-1} \circ A_{k^{-1}g^{-1}}(\dot{1}, k^{-1}g^{-1}m) \\ &= \left[A^1_{g^{-1}}(\dot{1}, g^{-1}m)\right]^{-1} \circ \left[A^1_{k^{-1}}(\dot{1}, k^{-1}g^{-1}m)\right]^{-1} \\ &\quad \circ \left[A_{k^{-1}}(\dot{1}, k^{-1}g^{-1}m)\right] \circ \left[A_{g^{-1}}(\dot{1}, g^{-1}m)\right] \\ &= b(g, m) \quad \text{since } A|_\Gamma = A^1|_\Gamma. \end{aligned}$$

Introducing $B \in C^\infty(\mathfrak{X}, C(E))$ such that $B(\dot{g}, m) = b(g, m)$, we obtain

$$A_g(\dot{g}', m) = [B(\dot{g}', m)]^{-1} \circ A^1_g(\dot{g}', m) \circ B(g^{-1}(\dot{g}', m)),$$

i.e. A is linearizable. $\square$

5.4. COROLLARY. *Assume $\mathfrak{X} = G/\Gamma_1 \times \cdots \times G/\Gamma_p$. If one of the subgroups Γ_i is compact, then any multiplier over $G \times \mathfrak{X}$ is linearizable.*

5.5. PROPOSITION. *With the notation of* (5.3), *suppose $\mathfrak{X} = G \times M$. Then any multiplier over $G \times \mathfrak{X}$ is trivial.*

Proof. Introduce $b(g, m) = A_{g^{-1}}(1, g^{-1}m)$, $g \in G$, $m \in M$, and use the beginning of the proof of (5.3). $\square$

5.6. COROLLARY. *Any multiplier over $G \times G^n$ is trivial.*

5.7. EXAMPLE. Set $G = \mathbf{R}$, acting on $\mathfrak{X} = \mathbf{R}^n$ by $t \cdot (x_1,\dots,x_n) = (x_1 + t,\dots,x_n + t)$, t, $x_i \in \mathbf{R}$. Given $F \in C^\infty(\mathbf{R}^n, C(\mathbf{C}^p))$, consider the equation

$$\frac{dV_t}{dt} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \circ V_t + F(x_1 \cdots x_n) \circ V_t \tag{1}$$

with initial condition $V_0 = \mathrm{Id}_{C^\infty(\mathbf{R}^n,\mathbf{C}^p)}$ and unknown function $V \in \mathbf{C}^\infty(\mathbf{R}, C(C^\infty(\mathbf{R}^n, \mathbf{C}^p)))$. Using (1.3) and (0.6), we obtain that (1) has a unique solution, given by $V_t = V_t^A$, where A is a multiplier over $\mathbf{R} \times \mathbf{R}^n$. By (5.6), A is trivial, and therefore (1) is equivalent (in the sense of (0.7)) to

$$\frac{dV_t}{dt} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \circ V_t, \qquad V_0 = \mathrm{Id}_{C^\infty(\mathbf{R}^n,\mathbf{C}^p)}. \tag{2}$$

(5.8) We next give a linearization result for scalar multipliers. We recall the notation of (4.7): given a linear scalar multiplier m over $G \times \mathfrak{X}$, we define a linear representation V^n of G on $C^\infty(\mathfrak{X})$ by

$$V_g^n(f)_{(x)} = \left(m_g(x)\right)^{-(n-1)} f(g^{-1}x), \qquad x \in \mathfrak{X},\ g \in G,\ f \in C^\infty(\mathfrak{X}).$$

Identifying $C^\infty(\mathfrak{X})$ and $C^\infty(\mathfrak{X}, L(\otimes_{n,s} \mathbf{C}, \mathbf{C}))$, one obtains $V_g^n(f) = m_g \circ U_g(f) \circ m_g^{-1}$, where U is the regular representation of G in $C^\infty(\mathfrak{X})$.

5.8. PROPOSITION. *Assume that $H^1(G, V^n, C^\infty(\mathfrak{X})) = \{0\}$, $\forall n = 2, 3, \dots$. Then any scalar multiplier A over $G \times \mathfrak{X}$, such that $A^1 = m$, is linearizable.*

Proof. Set $B_g(x) = A_g(x) \circ (A_g^1(x))^{-1}$, $g \in G$, $x \in \mathfrak{X}$. From the formula

$$B_{gg'}(x) = B_g(x) \circ m_g(x) + B_{g'}(g^{-1}x) \circ \left[m_g(x)\right]^{-1},$$

we deduce

$$B_{gg'}^2 = V_g^2\left(B_{g'}^2\right) + B_g^2, \quad \text{i.e.} \quad B^2 \in Z^1(G, V^2).$$

Therefore, there exists f^2 such that

$$B_g^2(x) = m_g(x) \circ f^2(g^{-1}x) \circ m_g(x)^{-1} - f^2(x).$$

Therefore, we have

$$\left(I + f^2(x)\right) \circ B_g(x) = m_g(x) \circ \left(I + f^2(g^{-1}x)\right) \circ m_g(x)^{-1}$$

at orders 1 and 2 and

$$\left(I + f^2(x)\right) \circ A_g(x) = m_g(x) \circ \left(I + f^2(g^{-1}x)\right)$$

at orders 1 and 2. Introducing the multiplier

$$\overset{(2)}{A}_g(x) = \left(I + f^2(x)\right) \circ A_g(x) \circ \left(I + f^2(g^{-1}x)\right)^{-1},$$

we have

$$\overset{(2)}{A}{}^1_g(x) = m_g(x), \quad \text{and} \quad \overset{(2)}{A}{}^2_g(x) = 0.$$

Repeating the same argument, we can construct step by step a series $\overset{(n)}{A}$ of multipliers such that

$$\begin{cases} \overset{(n)}{A}_g(x) = \left(I + f^n(x)\right) \circ \cdots \circ \left(I + f^2(x)\right) \circ A_g(x) \\ \qquad\qquad \circ \left(I + f^2(g^{-1}x)\right)^{-1} \circ \cdots \circ \left(I + f^n(g^{-1}x)\right)^{-1}, \\ \overset{(n)}{A}{}^1_g = m_g(x), \quad \overset{(n)}{A}{}^2_g(x) = \cdots = \overset{(n)}{A}{}^n_g(x) = 0. \end{cases}$$

The sequence $(I + f^n(x)) \circ \cdots \circ (I + f^2(x))$ converges in the natural topology of formal series (see [**6**]) to an element f of $C^\infty(\mathfrak{X}, D(\mathbf{C}))$, and one has

$$m_g(x) = f(x) \circ A_g(x) \circ f(g^{-1}x)^{-1}, \qquad \forall g \in G,\ x \in \mathfrak{X}. \qquad \square$$

5.9. Example. Whenever $\mathfrak{X} = G/\Gamma$ it can be seen that (5.8) is nothing but linearization condition (0.9) for the associated formal representation I of Γ (see (2.1) and (2.2)). Therefore, we get nothing new. The interesting case is the case when $\mathfrak{X}$ is very far from being of type G/Γ. Let us develop an example:

Set $G = \mathbf{R}^p$, acting on $\mathbf{R}^n$ by the trivial action. Using (4.2) any scalar linear multiplier m over $G \times \mathbf{R}^n$ can be written $m_g(x) = \exp \xi_g(x)$, $g \in G$, $x \in \mathbf{R}^n$, where ξ is a fixed element of $Z^1(G, \mathrm{Id}_{C^\infty(\mathbf{R}^n)})$. Actually, $Z^1(G, \mathrm{Id}_{C^\infty(\mathbf{R}^n)}) = L(\mathfrak{g}, C^\infty(\mathbf{R}^n))$. Introducing a basis X_i of $\mathfrak{g}$, and setting $d_i = d_{X_i}$, $d \in L(\mathfrak{g}, C^\infty(\mathbf{R}^n))$, we can identify $L(\mathfrak{g}, C^\infty(\mathbf{R}^n))$ and $C^\infty(\mathbf{R}^n, \mathbf{C}^p)$. Coming back to the multiplier m, and setting $d = d\xi$, we obtain

$$\xi_g(x) = \sum_{i=1}^{P} g_i d_i(x) = \langle g | d(x) \rangle, \qquad g = (g_i) \in G,\ x \in \mathbf{R}^n,$$

and therefore

$$m_g(x) = \exp\langle g|d(x)\rangle, \qquad g \in G,\ x \in \mathbf{R}^n.$$

In the following, we fix a nonzero element d of $C^\infty(\mathbf{R}^n, \mathbf{C}^p)$. Setting $d_X = \langle X|d\rangle$, $X \in \mathfrak{g}$, we introduce $\mathfrak{Z} = \{x \in \mathbf{R}^n | d_X(x) = 0,\ \forall X \in \mathfrak{g}\}$, and $\mathfrak{X}$ will be the complement of $\mathfrak{Z}$ in $\mathbf{R}^n$. By definition $\mathfrak{X}$ is a non void open subset of $\mathbf{R}^n$.

(5.9) PROPOSITION. *Let A be a scalar multiplier over $G \times \mathfrak{X}$ such that $A^1 = m$; then A is linearizable.*

Proof. Introduce

$$V_g^{n+1}(f)(x) = \big(\exp n\langle g|d(x)\rangle\big)(f(x)), \qquad f \in C^\infty(\mathfrak{X}),\ g \in G,\ x \in \mathfrak{X},$$

and let us show that $H^1(G, V^{n+1}) = \{0\}$, $\forall n \geq 1$. This will be achieved if we prove that $H^1(\mathfrak{g}; dV^{n+1}) = \{0\}$, $\forall n \geq 1$. So let $\xi \in Z^1(\mathfrak{g}, dV^{n+1})$; the cocycle relation gives

(i) $\quad \xi_{[X,Y]}(x) = 0 = n\big(d_X(x)\xi_Y(x) - d_Y(x)\xi_X(x)\big),$

$$\forall X, Y \in \mathfrak{g},\ x \in \mathfrak{X}.$$

We define a function f on $\mathfrak{X}$ in the following way: given $x \in \mathfrak{X}$, we can find $X \in \mathfrak{g}$ such that $d_X(x) \neq 0$; set

$$f(x) = \frac{1}{n}\frac{\xi_X(x)}{d_X(x)}.$$

Using (i) we see that the definition is independent of X. Since d_X remains non zero in a neighbourhood of x, the function

$$y \rightsquigarrow \frac{1}{n}\frac{\xi_X(y)}{d_X(y)}$$

is C^∞ in this neighbourhood, and therefore $f \in C^\infty(\mathfrak{X})$.

By definition we have $\xi_X(x) = nd_X(x)f(x)$ whenever $d_X(x) \neq 0$. When $d_X(x) = 0$, since the linear form $Y \rightsquigarrow d_Y(x)$ is nonzero, we can find $Y \in \mathfrak{g}$ such that $d_Y(x) \neq 0$, and using (i) we see that $\xi_X(x) = 0$, and therefore $\xi_X(x) = nd_X(x)f(x)$ also in this case. This proves that ξ is the coboundary of f. Now applying (5.8) we obtain the desired result. □

For instance, let us now take $G = \mathbf{R}^4$, $\mathfrak{X}$ the forward vertexless light cone, and the following linear multiplier m over $G \times \mathfrak{X}$:

$$g \in G, \quad x = (x_i) \in \mathbf{R}^4, \quad x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0, \quad x_0 > 0,$$

$$m_g(x) = \exp\langle g|d\rangle$$

where

$$d_j(x) = ix_j, \quad j = 1,2,3, \qquad d_0(x) = i\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}.$$

(One can recognize the multiplier corresponding to the translation part of a massless representation of the Poincaré group.) Actually the preceding proposition applies, and therefore any scalar multiplier A over $G \times \mathfrak{X}$, such that $A^1 = m$, is linearizable.

6. Inducing non linear representations. In this section Γ is a closed subgroup of G, $\mathfrak{X} = G/\Gamma$, L is a linear representation of Γ in a (finite-dimensional) space E. We denote by H^L the space $\{f \in C^\infty(G, E) | f(g\gamma) = L_{\gamma^{-1}} f(g), \forall \gamma \in \Gamma, g \in G\}$ and by U^L the restriction of the left regular representation of G to H^L: U^L is called the representation of G induced (in C^∞ sense) by the representation L of Γ (see e.g. [**9**]).

6.1. Definition. Given a multiplier A over $G \times G$, A is called an L-multiplier if

$$A(g, x\gamma) = L_{\gamma^{-1}} \circ A(g, x) \circ L_\gamma, \quad \forall g, x \in G, \gamma \in \Gamma,$$

$$A^1(g, x) = \mathrm{Id}_E, \quad \forall g, x \in G.$$

From the first condition, one easily shows that V^A (see (1.1)) can be "restricted" to H^L; we denote by W^A the formal representation of G in H^L obtained by this "restriction". By the second condition, one sees that the linear part of W^A is U^L.

6.2. Proposition. *Given an L-multiplier A, set $I_\gamma = A(\gamma, 1) \circ L_\gamma, \gamma \in \Gamma$; then I is a formal representation of Γ in E and $I^1|_\Gamma = L$.*

Proof. Trivial calculation.

6.3. Definition. Given a second linear representation L' of Γ in E, a L-multiplier A and a L'-multiplier A', we say that A and A' are s-equivalent if there exists $F \in C^\infty(G, C(E))$ such that

$$F(g) \in D(E), \ \forall g \in G, \quad F(x\gamma) = L'_{\gamma^{-1}} \circ F(x) \circ L_\gamma, \quad \forall x \in G, \ \gamma \in \Gamma,$$

$$A'(g, x) = F(x) \circ A(g, x) \circ F(g^{-1}x)^{-1}, \quad \forall g, x \in G.$$

6.4. Proposition. *Assume that A and A' are s-equivalent; then W^A and $W^{A'}$ are equivalent formal representations of G.*

Proof. Using (6.3) one sees that $\hat{F} \in C(H^L, H^{L'})$ and intertwines A and A'. □

6.5. PROPOSITION. *A and A' are s-equivalent if and only if the corresponding I and I' (see (6.2)) are equivalent.*

Proof. Assume that A and A' are s-equivalent. Using the notation of (6.3) and (6.1), we see that $F^1(x) \circ F^1(g^{-1}x) = \mathrm{Id}_E$, $\forall g, x \in G$. Using (6.3) once more, we obtain $L'_\gamma = F^1(1) \circ L_\gamma \circ F^1(1)^{-1}$, and then $I'_\gamma = F(1) \circ I_\gamma \circ F(1)^{-1}$.

Now assume there exists $U \in D(E)$ such that $I' = U^{-1} \circ I \circ U$. We introduce $b(g) = A(g^{-1}, 1)$, $b'(g) = A'(g^{-1}, 1)$, and $f(g) = b(g)^{-1} \circ U \circ b'(g)$, $g \in G$. Since $b(g\gamma) = I_{\gamma^{-1}} \circ b(g) \circ L_\gamma$ and $b'(g\gamma) = I'_{\gamma^{-1}} \circ b'(g) \circ L_{\gamma^{-1}}$, $\forall g \in G, \gamma \in \Gamma$, we have

$$f(g\gamma) = L_{\gamma^{-1}} \circ f(g) \circ L'_\gamma, \quad \forall g \in G, \ \gamma \in \Gamma.$$

Moreover $f^1(g) = U^1$, and therefore $f(g) \in D(E)$, $\forall g \in G$. Since $A(g, x) = b(x)^{-1} \circ b(g^{-1}x)$ and $A'(g, x) = b'(x)^{-1} \circ b'(g^{-1}x)$, we get

$$A'(g, x) = f(x)^{-1} \circ A(g, x) \circ f(g^{-1}x), \quad \forall g, x \in G. \qquad \square$$

(6.6) Let us denote by $\mathfrak{R}_L$ the set of equivalence classes of formal representations of Γ in E having L as linear part, and by $\mathfrak{M}_L$ the set of s-equivalence classes of L-multipliers.

PROPOSITION. *The correspondence $A \rightsquigarrow I$ of (6.2) induces a one-to-one mapping from $\mathfrak{M}_L$ onto $\mathfrak{R}_L$.*

Proof. Using (6.5) we see that this correspondence induces a one-to-one mapping from $\mathfrak{M}_L$ into $\mathfrak{R}_L$. It remains to show that this mapping is onto. Let us assume given a formal representation I of Γ in E such that $I^1 = L$. We choose $f \in C^\infty(G)$ such that $\gamma \rightsquigarrow f(g\gamma)$ is compactly supported in Γ and $\int_\Gamma f(g\gamma)\, d\gamma = 1$, $\forall g \in G$ **[21]**, and we set

$$c(g) = \int_\Gamma L_\gamma\big(f(g\gamma) \circ I_\gamma^{-1}\big)\, d\gamma.$$

Using the invariance of $d\gamma$ and the linearity of L, we obtain

$$c(g\gamma) = L_{\gamma-1} \circ c(g) \circ I_\gamma, \quad \forall g \in G, \ \gamma \in \Gamma.$$

Moreover, $c_g^1 = \mathrm{Id}_E$, $\forall g \in G$. Let us introduce $b(g) = [c(g)]^{-1}$ and $A(g, x) = b(x)^{-1} \circ b(g^{-1}x)$. From the properties of c, it is easily seen that A is a L-multiplier. Since $b(g\gamma) = I_{\gamma-1} \circ b(g) \circ L_\gamma$, $\forall g \in G$, $\gamma \in \Gamma$, we have

$$A(\gamma, 1) \circ L_\gamma = b(1)^{-1} \circ I_\gamma \circ b(1),$$

from which we deduce that $A(\cdot, 1) \circ L$ is equivalent to I. The proof is complete since $A^1(\gamma, 1) \circ L_\gamma = L_\gamma$. □

(6.7) Now we can define non linear inducing. Given an inducing class in $\mathfrak{N}_L$ of formal representations of Γ, we use (6.6) and pick any multiplier A in the corresponding class of L-multipliers in $\mathfrak{M}_L$; then we define the induced class of formal representations of G to be the class of the formal representation W^A (see (6.1)). From (6.4) our definition is coherent, since it is independent of the choice of A. By definition the linear part of W^A is exactly U^L, i.e. the linear part of any representation in the induced class is equivalent to the linear representation C-infinitely induced by the linear part of any representation in the inducing class. Finally, let us note that if L belongs to the inducing class, then U^L belongs to the induced class, i.e. if the inducing is linearizable, so is the induced.

6.8. Proposition. *Assume there exists a linear multiplier M on $G \times \mathfrak{X}$ extending L (see* (2.3)*). Given a formal representation I of Γ in E, with linear part L and a multiplier A corresponding to I from* (6.6)*, then there exists a multiplier B on $G \times \mathfrak{X}$ extending I, and the formal representations V^B and W^A of G are equivalent.*

Proof. The first assertion is proved by (2.4). By (6.6) there exists $U \in D(E)$ such that $A(\gamma, 1) \circ L_\gamma = U \circ I_\gamma \circ U^{-1}$, $\forall \gamma \in \Gamma$. Denote by J the representation $U \circ I \circ U^{-1}$ of Γ, and set $b(g) = A(g, 1)$, $c(g) = B(g, \dot{1})$, $g \in G$. One gets

$$b(g\gamma) = J_{\gamma^{-1}} \circ b(g) \circ L_\gamma, \quad c(g\gamma) = I_{\gamma^{-1}} \circ c(g), \qquad \forall g \in G, \ \gamma \in \Gamma.$$

Therefore, $F(g) = b(g)^{-1} \circ U \circ c(g)$ satisfies $F(g\gamma) = L_{\gamma^{-1}} \circ F(g)$, $\forall g \in G$, $\gamma \in \Gamma$. Moreover, since $A(g, x) = b(x)^{-1} \circ b(g^{-1}x)$ and $B(g, \dot{x}) = c(x)^{-1} \circ c(g^{-1}x)$, $\forall g, x \in G$, we obtain

$$B(g, \dot{x}) = F(x)^{-1} \circ A(g, x) \circ F(g^{-1}x), \qquad \forall g, x \in G.$$

From the relation $F(x\gamma) = L_{\gamma^{-1}} \circ F(x)$, $\forall x \in G, \gamma \in \Gamma$, one sees that

$$\hat{F} \in C(H^{\mathrm{Id}_E}, H^L) = C(C^\infty(\mathfrak{X}, E), H^L),$$

and the relation $B(g, \dot{x}) = F(x)^{-1} \circ A(g, x) \circ F(g^{-1}x)$ shows that

$$V_g^B = \hat{F}^{-1} \circ W_g^A \circ \hat{F}, \qquad \forall g \in G. \qquad \square$$

(6.9) Obviously, if the subgroup Γ is compact, any inducing representation is linearizable, and therefore any induced representation is linearizable. Let us now develop an example of linearization in the non compact case. We assume that N is a closed connected normal nilpotent subgroup of Γ assumed connected. Let $\mathfrak{n}$ be the Lie algebra of N, set $\pi = dL|_{\mathfrak{n}}$, and let $\lambda_1,\ldots,\lambda_p \in \mathfrak{n}^*$ be the weights of π (see (0.10)). Given $N = (n_1,\ldots,n_p) \in \mathbf{N}^p$, we set $|N| = \sum_{i=1}^p n_i$ and $\langle N, \lambda\rangle = \sum_{i=1}^p n_i\lambda_i$.

6.9.1. PROPOSITION. *Assume that* $\langle N, \lambda\rangle \neq \lambda_j$, $\forall j = 1,\ldots,p$, $\forall N$ *such that* $|N| \geq 2$. *Then any formal representation of G induced from a formal representation of* Γ *with linear part L is linearizable.*

Proof. From (0.9) it is sufficient to prove that $H^1(\underline{\gamma}, L(\otimes_n E, E)) = \{0\}$, $\forall n \geq 2$. We use the Hochschild-Serre sequence [4]

$$0 \to H^1\Big(\underline{\gamma}/\mathfrak{n}, L\Big(\bigotimes_n E, E\Big)^{\mathfrak{n}}\Big) \to H^1\Big(\underline{\gamma}, L\Big(\bigotimes_n E, E\Big)\Big)$$
$$\to H^1\Big(\mathfrak{n}, L\Big(\bigotimes_n E, E\Big)\Big)^{\underline{\gamma}} \to H^2\Big(\underline{\gamma}/\mathfrak{n}, L\Big(\bigotimes_n E, E\Big)^{\mathfrak{n}}\Big) \to \cdots.$$

From the condition $\langle N, \lambda\rangle \neq \lambda_j$, we deduce that $L(\otimes_n E, E)^{\mathfrak{n}} = \{0\}$. Therefore,

$$H^1\Big(\underline{\gamma}, L\Big(\bigotimes_n E, E\Big)\Big) = H^1\Big(\mathfrak{n}, L\Big(\bigotimes_n E, E\Big)\Big)^{\underline{\gamma}}.$$

But since $L(\otimes_n E, E)^{\mathfrak{n}} = \{0\}$, and since $\mathfrak{n}$ is a nilpotent Lie algebra, one has $H^1(\mathfrak{n}, L(\otimes_n E, E)) = \{0\}$ and, therefore $H^1(\mathfrak{n}, L(\otimes_n E, E))^{\underline{\gamma}} = \{0\}$. □

6.9.2. Assume that G is the Euclidean group $G = \mathrm{SO}(n)\cdot\mathbf{R}^n$. Unitary irreducible faithful linear representations of G are obtained when inducing linear representations of $\Gamma = \mathrm{SO}(n-1)\cdot\mathbf{R}^n$ of the form

$$L(\theta, r) = \sigma(\theta)\cdot\theta(r), \qquad \theta \in \mathrm{SO}(n-1),\ r \in \mathbf{R}^n,$$

where σ is a unitary irreducible representation of $\mathrm{SO}(n-1)$, and θ a unitary non trivial character of $\mathbf{R}^n$. Applying 6.9.1 with $N = \mathbf{R}^n$, we obtain that any formal induced representation of G with a unitary irreducible faithful linear part is linearizable.

6.9.3. Assume that G is the Poincaré group $\mathrm{SO}_0(3,1)\cdot\mathbf{R}^4$ and U is a linear representation of G C^∞-induced by one of the representations of the (corresponding) little group Γ which are used to obtain the usual unitary $m^2 \geq 0$ representations. Since the restriction to $\mathbf{R}^4$ of the linear representations of Γ inducing U is a non trivial character, (6.9.1) applies. Therefore, any induced representation of G with linear part U is linearizable.

7. Some examples in the abelian case. In this section we assume $G = \mathbf{R}^n$, $\Gamma = Z^n$ and $\mathfrak{X} = G/\Gamma$. As before, E is a finite-dimensional complex space. Given a Fréchet space F, we identify $C^\infty(\mathfrak{X}, F)$ with the space of Γ-periodic F-valued functions over G. Our first proposition shows that induced representations from Γ to G can always be realized as multiplier representations on $\mathfrak{X}$.

7.1. PROPOSITION. *Given a formal representation I of Γ in E, there exists a multiplier A over $G \times \mathfrak{X}$ which extends I. The formal representation V^A of G belongs to the class induced by I.*

Proof. By (2.4) we have only to prove (7.1) when I is a linear representation of Γ. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbf{R}^n$, and set $I_j = I_{e_j}$, $j = 1, \ldots, n$. We can find K_j, $j = 1, \ldots, n$, in $L(E)$ such that $[K_l, K_m] = 0$, $\forall l, m$, and $I_j = \exp K_j$, $\forall j$. Setting

$$S_{t_1 e_1 + \cdots + t_n e_n} = \exp(t_1 K_1 + \cdots + t_n K_n), \qquad t_i \in \mathbf{R},$$

we get a linear representation S of $\mathbf{R}^n$ extending I. Since S is a multiplier over $G \times \mathfrak{X}$, we obtain the first part of (7.1). For the second part, we apply (6.9). □

We now give a description of multipliers over $G \times \mathfrak{X}$, and its interpretation in terms of systems of non autonomous differential equations of §3. Precisely, we shall consider systems of type

$$\text{(i)} \qquad \frac{\partial f}{\partial t_i} = F_i(t) \circ f(t), \qquad i = 1, \ldots, n, \; t \in G,$$

where F is a given element of $L(\mathfrak{g}, C^\infty(\mathfrak{X}, C(E)))$, $F_i = F_{-\partial/\partial t_i}$, where $t = (t_1, \ldots, t_n)$ is the variable in $\mathbf{R}^n$, the unknown f is in $C^\infty(\mathbf{R}^n, C(E))$ and must satisfy an initial condition $f(0) = T$, $T \in C(E)$. From the results of §3, (i) is integrable if and only if F satisfies the Frobenius compatibility condition

$$\frac{\partial F_i}{\partial t_j} - \frac{\partial F_j}{\partial t_i} + [F_i, F_j] = 0 \qquad \forall i, j.$$

We assume in the following that this condition is satisfied. Our goal in this section is to see whether such systems can be reduced to autonomous systems (i.e. $F \in L(\mathfrak{g}, C(E))$) by an equivalence of type (3.6) (i.e. an equivalence involving a Γ-periodic function). We begin by the reduction of linear systems.

7.2. PROPOSITION. *Any linear integrable systems of type* (i) *is linearly equivalent to a linear autonomous systems*

$$\frac{\partial f}{\partial t_i} = G_i \circ f(t), \qquad G_i \in L(E),\ i = 1,\dots,n.$$

Proof. From (3.3) there exists a linear multiplier A over $G \times \mathfrak{X}$ such that $F = dA$. Set $I_k = A(k,0)$, $k \in \Gamma$, and introduce the linear representation S of Γ constructed from I during the proof of (7.1). Since the linear multipliers A and S over $G \times \mathfrak{X}$ both extend I, they are linearly equivalent (2.2). Therefore, the given system is equivalent to the system associated with the multiplier S (3.6). Since this last system has constant coefficients, (7.2) follows.

REMARK. For $n = 1$, (7.2) is known as Floquet's theorem [**1**].

We now carry general systems into "normal form". First, we need some notation: given a linear representation I^1 of Γ, we denote by s (resp. u) the semi-simple part (resp. the unipotent part) of I^1.

7.3. LEMMA. *Given a formal representation I of* Γ, *let s be the semi-simple part of I^1, then, up to equivalence, one has $s_k \circ I_l = I_l \circ s_k$, $\forall k, l \in \Gamma$.*

Proof. We denote by $\mathrm{Ad}^n I^1$ the linear representation of Γ on $C^n(E) = L(\otimes_{n,s} E, E)$ defined by

$$\mathrm{Ad}^n I^1_k(T) = I^1_k \circ T \circ I^1_{-k} \qquad \forall k \in \Gamma,\ T \in C^n(E).$$

Obviously $\mathrm{Ad}^n s$ (resp. $\mathrm{Ad}^n u$) is the semisimple part (resp. the unipotent part) of $\mathrm{Ad}^n I^1$. We decompose $C^n(E) = C^n_+ \oplus C^n_0$, where C^n_+ is the sum of the eigenspaces of $\mathrm{Ad}^n s$ corresponding to non trivial eigenfunctions, and C^n_0 is the eigenspace of the trivial eigenfunction. C^n_+ and C^n_0 are stable under $\mathrm{Ad}^n I^1$.

Let us set $i^n_k = I^n_k \circ I^1_{k^{-1}}$, $k \in \Gamma$. Writing $I_{k+k'} = I_k \circ I_{k'}$, at order 2, we get

$$i^2_{k+k'} = \mathrm{Ad}^2 I^1_k\left(i^2_{k'}\right) + i^2_k, \qquad \forall k, k' \in \Gamma.$$

Therefore $i^2 \in Z^1(\Gamma, \mathrm{Ad}^2 I^1)$. We decompose $i^2 = i^2_+ + i^2_0$, where $i^2_+ \in Z^1(\Gamma, \mathrm{Ad}^2 I^1|_{C^2_+})$ and $i^2_0 \in Z^1(\Gamma, \mathrm{Ad}^2 I^1|_{C^2_0})$. Since $\mathrm{Ad}^2 I^1|_{C^2_+}$ does not contain the trivial representation, i^2_+ is a coboundary. Therefore, by a standard argument using an equivalence of type $(I + A^2)$, $A^2 \in C^2(E)$, we can assume $i_+^2 = 0$, i.e. $I_k^2 \in C_0^2$, $\forall k \in \Gamma$. Writing now $I_{k+k'} = I_k \circ I_{k'}$ at order 3, we get

$$\begin{aligned} i_{k+k'}^3 &= \mathrm{Ad}^3 I_k^1\left(i_{k'}^3\right) + i_k^3 + I_k^2 \circ \sum_{i_1+i_2=3} \left(i_{k'}^{i_1} \circ i_{k'}^{i_2}\right) \circ \sigma_3 \\ &= \mathrm{Ad}^3 I_k^1\left(i_{k'}^3\right) + i_k^3 + d_k^3. \end{aligned}$$

Obviously, $d_k^3 \in C_0^3$, $\forall k \in \Gamma$. Writing $i^3 = i_0^3 + i_+^3$, we therefore conclude that

$$i_{+k+k'}^3 = \mathrm{Ad}^3 I_k^1\left(i_{+k'}^3\right) + i_{+k}^3, \qquad \forall k, k' \in \Gamma.$$

Therefore $i_+^3 \in Z^1(\Gamma, \mathrm{Ad}^3 I^1|_{C_+^3})$, and since this representation does not contain the trivial representation, i_+^3 is a coboundary. Using an equivalence of type $(I + A^3)$, $A^3 \in C^3(E)$, we can assume $i_+^3 = 0$, i.e. $I_k^3 \in C_0^3$, $\forall k \in \Gamma$.

It is now clear that repeating the preceding arguments, we shall construct an equivalent formal representation satisfying the desired relation. □

REMARK. Note that (7.3) is true for any abelian group Γ with exactly the same proof.

7.4. PROPOSITION. (1) *Given a multiplier A over $G \times \mathfrak{X}$, let I be the associated formal representation of Γ, and let s be the semi-simple part of I^1. Up to equivalence, one has $s_k \circ A(t, x) = A(t, x) \circ s_k$, $\forall k \in \Gamma$, $t, x \in G$.*

(2) *Given an integrable system of type* (i), *let A be the associated multiplier* (3.3) *and let us keep the notation of* (1). *Up to equivalence* (i) *reduces to a system which contains only "resoning terms", i.e.*

$$s_k \circ F_i(t) = F_i(t) \circ s_k, \qquad \forall k \in \Gamma,\ t \in G.$$

Proof. (1) From Lemma 7.3, up to equivalence, we can assume that $s_k \circ I_l = I_l \circ s_k$, $\forall k, l \in \Gamma$.

Let S^1 be a linear representation of $\mathbf{R}^n$ extending I^1 (see (7.1)). We now are going to refine the proof of (2.5) in order to get our result: with the notation of (2.5) we introduce

$$F(t) = \int_\Gamma I_k^1 f(t+k) I_k^{-1}\, dk, \qquad t \in G.$$

Since s is linear and commutes with I and I^1, we get that s commutes with F. Moreover, $F(t+k) = I_{-k}^1 \circ F(t) \circ I_k$, $\forall t \in G$, $k \in \Gamma$, and $F^1(t) = \mathrm{Id}_E$, $\forall t \in G$. Following (2.5), we introduce $b(t) = F(t)^{-1} \circ S_{-t}^1 \circ F(0)$, which also commutes with s, and satisfies $b(t+k) = I_k^{-1} \circ b(t)$, $\forall t \in G$, $k \in \Gamma$, $b(t) \in D(E)$, $\forall t \in G$, and $b(0) = \mathrm{Id}_E$. The multiplier A' over $G \times \mathfrak{X}$ defined by $A'(t,x) = b(x)^{-1} \circ b(x-t)$, $t, x \in G$, commutes with s, and since it extends I, it is equivalent to A (2.2).

(2) We set $F = dA$ and use (1). □

REMARK. (7.4) is proved for $n = 1$ in [**1**].

Given $\alpha = (\alpha_1, \dots, \alpha_m) \in C^\infty(G, \mathbf{C}^m)$ and $p = (p_1, \dots, p_m) \in \mathbf{N}^m$, we set $\alpha^p = \alpha_1^{p_1} \cdots \alpha_m^{p_m}$ and $|p| = p_1 + \cdots + p_m$.

7.5. PROPOSITION. *Given an integrable system of type* (i), *assume there are no resonance relations, i.e. the eigenfunctions* $\alpha_1 \cdots \alpha_m$ *of the semi-simple part* s *of the corresponding formal representation* I *of* Γ *satisfy* $\alpha^p \neq \alpha_j$, $\forall j = 1, \dots, m$, $\forall p \in \mathbf{N}^m$ *such that* $|p| \geq 2$; *then* (i) *is equivalent to a linear autonomous system*

$$\frac{\partial f}{\partial t_i} = G_i \circ f(t), \qquad i = 1, \dots, n, \ t \in G, \ G_i \in L(E).$$

Proof. With the assumption of (7.5), only linear elements of $C(E)$ can commute with s. Therefore, using (7.4), we reduce (i) to a linear system. We then apply (7.2). □

We are now going to make (7.4) precise by reducing systems of type (i) to a canonical form. First, we need a lemma, which concerns extensions of representations of Γ to representations of G.

7.6. LEMMA. *Given a formal representation of* Γ *with unipotent linear part, there exists a unique formal representation* S *of* G, *with unipotent linear part, such that* $S|_\Gamma = I$. *Assuming that* $T \in C(E)$ *and satisfies* $T \circ I_k = I_k \circ T$, $\forall k \in \Gamma$, *then* $T \circ S_t = S_t \circ T$, $\forall t \in G$.

Proof. We begin with the case $\Gamma = Z$, $G = \mathbf{R}$. Given an element X of $C(E)$, let us recall how the flow S of X can be constructed (see [**5**] or [**11**] for details): S is the solution of $dS_t/dt = X \circ S_t$, $S_0 = \mathrm{Id}_E$. Setting $S_t^1 = \exp tX^1$, $e \in \mathbf{R}$, we introduce $s_t = S_{-t}^1 \circ S_t$, which is the solution of

$$\frac{ds_t}{dt} = -X_1 \circ s_t + S_{-t}^1 \circ X \circ S_t^1 \circ s_t, \qquad s_0 = \mathrm{Id}_E.$$

Obviously, $s_t^1 = \mathrm{Id}_E$, $\forall t \in \mathbf{R}$, and

$$s_t^n = \sum_{p=2}^{n} \int_0^t \left(S_{-u}^1 \circ X^p \circ S_u^1 \sum_{i_1 + \cdots + i_p = n} s_u^{i_1} \otimes \cdots \otimes s_u^{i_n} \circ \sigma_n \right) du.$$

Now, setting $J = I_1$, we solve the equation $S_1 = J$, where the unknown is the vector field X. Since we are looking for a unipotent S^1, X will be a vector field with nilpotent linear part. We first get $J^1 = \exp X^1$, which is known to have a unique nilpotent solution X^1. We recall that X^1 commutes with any $T \in L(E)$ commuting with J. Assuming we have found $X^1, \ldots, X^{n-1}$, we now compute X^n. Let us define a linear representation $\mathrm{Ad}_n S^1$ of G on $C^n(E)$ by

$$\mathrm{Ad}_n S^1_t(T) = S^1_{-t} \circ T \circ S^1_t, \qquad t \in \mathbf{R},\ T \in C^n(E).$$

From the beginning of the proof, we see that

$$(J^1)^{-1} \circ J^n - \sum_{p=2}^{n-1} \int_0^1 du \left(\mathrm{Ad}_p S^1_u(X^p) \sum_{i_1 + \cdots + i_p = n} s_u^{i_1} \otimes \cdots \otimes s_u^{i_p} \right) \circ \sigma_n$$
$$= \int_0^1 \mathrm{Ad}_n S^1_u(X^n)\, du.$$

Therefore, X^n can be computed if the linear mapping $\int_0^1 \mathrm{Ad}_n S^1_u\, du$ is invertible in $L(C^n(E))$. But since S^1 is unipotent, $\mathrm{Ad}_n S^1$ is also unipotent, and therefore $\int_0^1 \mathrm{Ad}_n S^1_u\, du$ is invertible. This shows the existence and unicity of S when $\Gamma = Z$ and $G = \mathbf{R}$.

Assume now that $T \in D(E)$ satisfies $T \circ I_k = I_k \circ T$, $\forall k \in \Gamma = Z$. Then setting $V_t = T \circ S_t \circ T^{-1}$, we obtain a formal representation of $\mathbf{R}$ satisfying $V^1 = S^1$ and $V|_\Gamma = I$. By unicity we conclude that $V = S$, and therefore $T \circ S_t = S_t \circ T$, $\forall t \in \mathbf{R}$.

Now, assume $\Gamma = Z^n$ and $G = \mathbf{R}^n$. Let $e_1, \ldots, e_n$ be the canonical basis of G and define $I_i(k) = I_{ke_i}$, $k \in Z$. From the preceding results, there exists a unique formal representation S_i of $\mathbf{R}$ such that $S_{i|Z} = I_i$. Since I_i and I_j commute, so do S_i and S_j, therefore we define a formal representation S of G by

$$S(t_1 e_1 + \cdots + t_n e_n) = S_1(t_1) \circ \cdots \circ S_n(t_n).$$

Obviously, $S|_\Gamma = I$. □

7.7. PROPOSITION. (1) *Let A be a multiplier over $G \times \mathfrak{X}$ such that the associated formal representation I of Γ has unipotent linear part. Then there exists a formal representation S of G such that the multipliers A and S are equivalent.*

(2) *Given an integrable system of type* (i), *let A be the associated multiplier* (3.3) *and assume that A satisfies the hypothesis of* (1); *then there exist $G_1, \ldots, G_n$ in $C(E)$ with nilpotent linear part such that* (i) *is equivalent to the autonomous system*

$$\frac{\partial f}{\partial t_i} = G_i \circ f(t), \qquad \forall t \in G,\ i = 1, \ldots, n.$$

Proof. (1) From (7.6) there exists a formal representation S of G such that $S|_\Gamma = I$. Since S and A are multipliers over $G \times \mathfrak{X}$ which both extend I, they are equivalent (2.2).

(2) Apply (3.3) and (1). □

7.8. PROPOSITION. (1) *Let A be a multiplier over $G \times \mathfrak{X}$, I the associated formal representation of Γ, s the semi-simple part of I^1. We continue to denote by I^1 (resp. s) any extension of I^1 to G (resp. its semi-simple part). There exists a formal representation S of G such that*:

$s_k \circ S_t = S_t \circ s_k$, $\forall t \in G$, $k \in \Gamma$;

A is equivalent to the multiplier $A'(t,x) = s_x \circ S_t \circ s_{(t-x)}$, $\forall t, x \in G$.

(2) *Given an integrable system of type* (i), *let A be the associated multiplier* (3.3), *and keep the notation of* (1). *There exists $G_1, \ldots, G_n$ in $C(E)$ with nilpotent linear part such that*:

$s_k \circ G_i = G_i \circ s_k$, $\forall k \in \Gamma$, $i = 1, \ldots, n$; $[G_i, G_j] = 0$, $\forall i, j$;

(i) *is equivalent to the system*

$$\frac{\partial f}{\partial t_i} = ds_i \circ f(t) + (s_t \circ G_i \circ s_{-t}) \circ t(t),$$

where ds is the differential of the linear representation s.

Before proving (7.8), let us make a few remarks: first, if there are no resonance relations for the eigenfunctions of s (see (7.5) for the definition), (7.8) is nothing but a very complicated way of proving (7.5)! On the other hand, when there exist resonance relations, (7.8) gives some minimal canonical reduction of systems of type (i) much more precise than (7.4).

Proof. (1) Up to equivalence ((7.3)), we can assume that $s_k \circ I_l = I_l \circ s_k$, $\forall k, l \in \Gamma$. Therefore, setting $J_k = I_k \circ s_{-k}$, $k \in \Gamma$, we obtain a formal representation of Γ with unipotent linear part. Applying (7.6), there exists a formal representation S of G such that $S|_\Gamma = J$. Moreover, $s_k \circ S_t = S_t \circ s_k$, $\forall k \in \Gamma$, $t \in G$. Let us now define $b'(t) = S_{-t} \circ s_{-t}$; we have $b'(t+k) = I_{-k} \circ b'(t)$, $\forall k \in \Gamma$, $t \in G$. Consequently, $A'(t,x) = b'(x)^{-1} \circ b'(x-t)$, $t, x \in G$, defines a multiplier over $G \times \mathfrak{X}$, and it is obvious that A' extends I. Therefore A and A' are equivalent (2.2).

Computing A', one obtains $A'(t,x) = s_x \circ S_t \circ s_{(t-x)}$, $\forall t, x \in G$.

(2) Computing dA', one obtains (2). □

8. Examples of non linear induced representations of $G = \mathrm{SU}(1,1)$. In this section $G = \mathrm{SU}(1,1)$, $\mathfrak{X} = \{x \in \mathbf{C} \mid |x| = 1\}$. Following **[20]** we introduce an action of G on $\mathfrak{X}$ by

$$g \cdot x = \frac{\bar{\alpha}x - \bar{\beta}}{\alpha - \beta x}, \qquad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \ |\alpha|^2 - |\beta|^2 = 1.$$

We introduce the linear scalar multiplier $m(g, x) = \bar{\alpha} + \beta x$ and define the continuous series of (linear) representations of G on $C^\infty(\mathfrak{X})$ by

$$\begin{cases} T_g^l(f)(x) = |m(g, x)|^{2l} f(g^{-1}x), \\ \qquad\qquad g \in G, f \in C^\infty(\mathfrak{X}), x \in \mathfrak{X}, l \in \mathbf{C}. \\ U_g^l(f)(x) = |m(g, x)|^{2l-1} m(g, x) f(g^{-1}x), \end{cases}$$

We define π: $G \to \mathfrak{X}$ by $\pi(g) = g \cdot 1$. Using π we can identify $\mathfrak{X}$ and G/Γ, where Γ is the following closed subgroup of G:

$$\Gamma = \left\{ g = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in G \middle| \alpha - \beta \in \mathbf{R} \right\}.$$

We denote by Γ_0 the connected component of $\mathbf{1}$ ($\mathbf{1} = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$) in Γ, i.e.

$$\Gamma_0 = \left\{ g = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in G \middle| \alpha - \beta \in \mathbf{R}^+ \right\}.$$

Γ/Γ_0 can be identified with the subgroup $\{\mathbf{1}, -\mathbf{1}\}$ of Γ.

It is easily seen that the multiplier defining T^l (resp. U^l) extends the linear character $I_l(\gamma) = |\bar{\alpha} + \beta|^{2l}$ (resp. $J_l(\gamma) = |\bar{\alpha} + \beta|^{2l-1}(\bar{\alpha} + \beta)$) of Γ. Therefore, both T^l and U^l are linear C^∞-induced representations of G ((4.7)). We are now going to solve the following problem: find and classify all non linear induced representations of G having T^l or U^l, $l \neq 0$, as linear part. This problem will be solved in two steps: first, we find and classify all the formal representations of Γ in $\mathbf{C}$ having I_l or J_l as linear part, then we construct the corresponding multipliers over $G \times \mathfrak{X}$. Given such a multiplier A, the corresponding representation V^A is induced ((6.8)), and we finally obtain all the formal representations of G induced by formal representations of Γ in $\mathbf{C}$ with non trivial linear part.

8.1. PROPOSITION. *Assume $l \neq 0$. There exists a formal representation I of Γ with linear part I_l (resp. J_l) if and only if $l = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ (resp. $l = \frac{1}{2}, \frac{1}{4}, \ldots$). Up to equivalence this representation is unique and given by the (formal development of) formula*:

$$n \in \mathbf{N}, \ n \geq 2, l = \frac{1}{n-1}, \ z \in \mathbf{C}, \qquad I_\gamma(z) = \frac{|\bar{\alpha} + \beta|^{2l} z}{\left[1 + l^{-1}\operatorname{Im}(\alpha\beta) z^{1/l}\right]^l}$$

(*resp*.

$$n - 1 \in 2\mathbf{N}^*,\ l = \frac{1}{n-1}, \qquad I_\gamma(z) = \frac{|\bar{\alpha} + \beta|^{2l-1}(\bar{\alpha} + \beta)z}{[1 + l^{-1}\operatorname{Im}(\alpha\beta)z^{1/l}]^l}).$$

Proof. Let $\underline{\gamma}$ be the Lie algebra of Γ. We introduce the generators

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} i & i \\ -i & -i \end{bmatrix}$$

of γ, with commutation rule $[A, N] = -2N$. Let us denote by π the representation dI_l (or dJ_l) of $\underline{\gamma}$. One has $\pi_A(z) = 2lz$, $\pi_N(z) = 0$, $\forall z \in \mathbf{C}$. Assume next that there exists a formal representation S of $\underline{\gamma}$ with linear part π. Then, up to equivalence, we can suppose that $S_A = \pi_A$. Now the problem of finding such an S is equivalent to the following problem: given the vector field $\tilde{A} = -2lz(d/dz)$, it is possible to find a vector field $\tilde{N}$ with vanishing linear part such that $[\tilde{A}, \tilde{N}] = -2\tilde{N}$. Since $[\tilde{A}, z^n(d/dz)] = -2l(n-1)z^n(d/dz)$, we find that, necessarily, $l = 1/(n-1)$, $n = 2, 3, \ldots$ For such values of l, $\tilde{N} = \lambda z^n(d/dz)$ satisfies our condition. Up to equivalence we can assume that $\tilde{N} = z^n(d/dz)$. Now defining $S_A(z) = 2z/(n-1)$, $S_N(z) = -z^n$, $z \in \mathbf{C}$, we get S. Using the Iwasawa decomposition and some computation, we get $S = dI$, where I is the formal representation of Γ_0 defined by the (formal development of) formula

$$\gamma = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \Gamma_0, \qquad z \in \mathbf{C},$$

$$I_\gamma(z) = \frac{(\bar{\alpha} + \beta)^{2/(n-1)}z}{(1 + (n-1)\operatorname{Im}(\alpha\beta)z^{n-1})^{1/(n-1)}}.$$

We can extend I to Γ by

$$I_\gamma(z) = \frac{|\bar{\alpha} + \beta|^{2/(n-1)}z}{(1 + (n-1)\operatorname{Im}(\alpha\beta)z^{n-1})^{1/(n-1)}},$$

and obviously the linear part of I is $I_{1/(n-1)}$. It remains now to extend I to Γ in such a way that its linear part is $J_{1/(n-1)}$ if possible. Noticing that $I(-\mathbf{1})$ must commute with the flow of $\pi_A = dI_A$, we obtain $I(-\mathbf{1}) = J_{1/(n-1)}(-\mathbf{1}) = -\mathrm{Id}_{\mathbf{C}}$. But since $I(-\mathbf{1})$ must commute with I_γ, $\forall \gamma \in \Gamma$, we obtain that $n - 1$ must be even. Assuming that it is, we define

$$I_\gamma(z) = \frac{|\bar{\alpha} + \beta|^{2/(n-1)-1}(\bar{\alpha} + \beta)z}{(1 + (n-1)\operatorname{Im}(\alpha\beta)z^{n-1})^{1/(n-1)}}, \qquad \gamma \in \Gamma,\ z \in \mathbf{C}.$$

The linear part of I is $J_{1/(n-1)}$. □

8.2. COROLLARY. *Non linear scalar multipliers over $G \times \mathfrak{X}$ with non trivial linear part are given, up to equivalence, by the (formal) development of the formulae:*

$$l = 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \quad g \in G, \ x \in \mathfrak{X}, \ z \in \mathbf{C},$$

$$A_l(g, x)(z) = \frac{|\bar{\alpha} + \beta x|^{2l} z}{\left[1 + l^{-1}\operatorname{Im}(\alpha\beta x) z^{1/l}\right]^l}$$

$$l = \tfrac{1}{2}\tfrac{1}{4}, \ldots, \quad g \in G, \ x \in \mathfrak{X}, \ z \in \mathbf{C},$$

$$B_l(g, x)(z) = \frac{|\bar{\alpha} + \beta x|^{2l-1}(\bar{\alpha} + \beta x) z}{\left[1 + l^{-1}\operatorname{Im}(\alpha\beta x)\, z^{1/l}\right]^l}.$$

The formal representations V^{A_l} and V^{B_l} (see (1.1)) *describe all formal representations of G induced (in the sense of* 6) *by formal representations of Γ in $\mathbf{C}$ with non trivial linear part. The linear part of V^{A_l} is T_l, and the linear part of V^{B_l} is U_l.*

Proof. We have to extend the formal representations of Γ given by (8.1) to multipliers on $G \times \mathfrak{X}$. Since the proof is rather combinatorial, we omit details: the formulae of (8.2) can be obtained using the formula given in the proof of (2.4), with an adapted section s of the projection π and use of the Iwasawa decomposition. □

8.3. REMARK. It can be seen that formulae (8.2) are particular cases of formula (4.8).

REFERENCES

[1] Arnold, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, Ed. Mir, Moscou (1978).

[2] Bourbaki, *Groupes et algèbres de Lie*, Chapter I, Hermann, Paris (1960).

[3] F. Brickell and R. S. Clark, *Differentiable Manifolds*, Van Nostrand Reinhold Company-London, 1970.

[4] M. Flato, C. Fronsdal and D. Sternheimer, *Difficulties with Massless particles*, Preprint Université de Dijon.

[5] M. Flato, G. Pinczon and J. Simon, *Non-linear representations of Lie groups*, Ann. Scient. Ec. Norm. Sup. 4e serie t. 10, (1977), p. 405.

[6] M. Flato and J. Simon, *Non linear equations and covariance*, Letters Math. Phys., **3** (1979), p. 279.

[7] _____, *On a linearization program of non linear field equations*, Phys. Lett. 94B, (1980), p. 518.

[8] _____, *Linearization of relativistic non linear wave equations*, J. Math. Phys., **21** (4), (1980), p. 913.

[9] A. Guichardet, *Cohomologie des groupes topologiques et des algèbres de Lie*, F. Nathan. Paris 1980.

[10] Y. Kossmann-Schwarzbach, in S. Sternberg editor (Jerusalem colloquium, to appear in Math. Phys. Studies, Reidel (1984)) and references quoted there.

[11] G. Pinczon, *Séminaire sur les representations non Linéaires*, (1981), Lecture notes Université de Dijon.

[12] G. Pinczon and J. Simon, *Non linear representations of inhomogeneous groups*, Lett. Math. Phys., **2** (1978), p. 499.

[13] _____, *Sur la 1-cohomologie des groupes de Lie semi-simples*, C. R. Acad. Sci. Paris, A-279, (1974), p. 455.

[14] _____, *Models of non-linear representations and examples of linearization techniques*, J. Math. Phys., **24** (3) (1983), p457.

[15] J. Simon, *Nonlinear representations of the Poincaré group and global solutions of relativistic wave equations*, to be published in Pacific J. Math.

[16] H. Stetkaer, *On angular momentum Helmhotz theorems and cohomology of Lie algebras*, Trans. Amer. Math. Soc., **214** (1975), p. 349.

[17] E. Taflin, *Analytic linearization, Hamiltonian formalism and Infinite sequence of constants of motion for the Burgers equations*, Phys. Rev. Lett. **47-20** (1981), p. 1425.

[18] _____, *Analytic linearization of the K. D. V. equation*, Preprint, Université de Genève.

[19] _____, *Formal linearization of non-linear massive representations of the connected Poincaré group*, Preprint, Université de Genève.

[20] N. Ja. Vilenkin, *Fonctions spéciales et théorie de la représentation des groupes*, Dunod. Paris, (1969).

[21] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I*. Springer Verlag, Berlin (1972).

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