PACIFIC JOURNAL OF MATHEMATICS
Vol. 116, No. 2, 1985

AN ISOPERIMETRIC INEQUALITY FOR SURFACES STATIONARY WITH RESPECT TO AN ELLIPTIC INTEGRAND AND WITH AT MOST THREE BOUNDARY COMPONENTS

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Let M be a connected C^2 two dimensional submanifold with boundary of $\mathbf{R}^3$, with at most three boundary components. Let Φ be a positive even elliptic parametric integrand of degree two on $\mathbf{R}^3$ ([5]), and suppose that M is stationary with respect to Φ. In this paper we show that there is a constant $C(\Phi)$ such that M satisfies the isoperimetric inequality

$$L^2 \geq C(\Phi)A, \tag{1.1}$$

where L is the length of ∂M and A is the surface area of M. In the proof we also prove a lemma that M satisfies the inequality

$$\text{length}(\partial M) \geq C(\Phi) \text{ diameter } M. \tag{1.2}$$

In the case that M is simply connected (1.1) follows for $C(\Phi) = 4\pi$ from the fact that such a surface must have nonpositive Gauss curvature [4]. In the case that ∂M has two components and Φ is the parametric area integrand the inequality (1.1) with $C = 4\pi$ has been proven by Osserman and Schiffer, [9]. More generally, an inequality of the form (1.1) has been proven for area stationary k dimensional varifolds on $\mathbf{R}^n$ by Allard, [2]. For the case that M has two or three boundary components and Φ is different from the area integrand the results (1.1), (1.2) are new. We note that this result also allows us to obtain lower bounds on area for such a manifold M using (1.1) together with the techniques of [1], [9]. For a review of other results on the isoperimetric inequality see the paper by Osserman [7].

In many isoperimetric inequality proofs, the equation

$$2A = -2\int_M (x - c) \cdot H + \int_{\partial M} (x - c) \cdot \nu \tag{1.3}$$

plays a central role, where $c \in \mathbf{R}^3$, H is the mean curvature vector of M, and ν is the exterior normal of ∂M with respect to M. For example, see Osserman [7], pp. 1203–1204. In the present work a similar equation is used where H is replaced by a weighted combination of the principal curvatures of M with coefficients determined by $D^2\Phi$. A barrier argument is then used which makes use of the ellipticity of Φ.

2. THEOREM. *Suppose Φ is a positive, even, elliptic parametric integrand of degree* 2 *on* $\mathbf{R}^3$. *Then there is a constant $C(\Phi)$ with the following property. Suppose M is a bounded connected C^2 two dimensional submanifold with boundary of* $\mathbf{R}^3$, *stationary with respect to Φ. Suppose $\partial M = C_1 \cup C_2 \cup C_3$, where each C_i is connected. Then we have the isoperimetric inequality*

$$L^2 \geq C(\Phi)A \tag{2.1}$$

where $A =$ area M, $L =$ length ∂M. *Note*: *The case that M has two boundary components follows by setting $C_3 = \varnothing$.*

Proof. Define $L_\Phi\colon \mathbf{R}^3 \to \operatorname{Hom}(\mathbf{R}^3, \mathbf{R}^3)$ by requiring that $L_\Phi(n)(v) = \Phi(n)v - \nabla\Phi(n) \cdot v\, n$. By Allard [3] we have the following two formulae for the first variation of M with respect to Φ.

$$\delta(M; \Phi)(g) = \int_M Dg(x) \cdot L_\Phi(n(x))\, dH^2x \tag{2.2}$$

whenever $g\colon \mathbf{R}^3 \to \mathbf{R}^3$ has compact support in $\mathbf{R}^3$, where n is a normal vector field on M. Integrating by parts yields the formula

$$\begin{aligned}\delta(M; \Phi)(g) &\\ &= \sum_{i=1}^{2} \int_M k_i(x)\left\langle u_i(x)^2, D^2\Phi(n(x))\right\rangle g(x) \cdot n(x)\, dH^2x \\ &\quad + \int_{\partial M} \langle n_1(x), L_\Phi(n(x))\rangle \cdot g(x)\, dH^1x,\end{aligned} \tag{2.3}$$

where k_i, u_i are the principal curvatures and directions, respectively, to M and n_1 is the exterior normal of ∂M with respect to M. By our hypothesis that M be stationary,

$$\sum_{i=1}^{2} k_i(x)\left\langle u_i(x)^2, D^2\Phi(n(x))\right\rangle = 0 \tag{2.4}$$

for all $x \in M$, so that

$$\delta(M; \Phi)(g) = \int_{\partial M} \langle n_1, L_\Phi(n)\rangle \cdot g\, dH^1. \tag{2.5}$$

Note that since (2.3) is linear in g, and Φ is even, we need not assume, due to the existence of partitions of unity, that M is orientable. Further, by using a suitable cutoff, since M is bounded we can apply the formula to the vector field $g(x) = x$. Noting that $Dg(x) \cdot L_\Phi(n) = 2\Phi(n)$; we derive from (2.5) the equation

$$(2.6) \qquad 2\int_M \Phi(n)\,dH^2 = \sum_{i=1}^{3} \int_{C_i} \langle n_1, L_\Phi(n)\rangle \cdot x\,dH^1$$
$$= \sum_{i=1}^{3} \int_{C_i} \langle n_1, L_\Phi(n)\rangle \cdot (x - a_i)\,dH^1$$
$$+ \sum_{i=1}^{3} \int_{C_i} \langle n_1, L_\Phi(n)\rangle \cdot a_i\,dH^1$$

for any $a_i \in \mathbf{R}^3$, $i = 1, 2, 3$. We choose a_i to be the center of mass of C_i, i.e.

$$(2.7) \qquad \int_{C_i} (x - a_i)\,dH^1 = 0 \in \mathbf{R}^3.$$

Defining

$$\lambda = \frac{\sup\|L_\Phi(u)\|}{\inf \Phi(w)},$$

where the indicated sup and inf are over unit vectors $\mathbf{u}, \mathbf{w}$ of $\mathbf{R}^3$, we derive from (2.6)

$$(2.8) \qquad 2A \le \lambda \sum_{i=1}^{3} \int_{C_i} |x - a_i|dH^1x + \lambda \sum_{i=1}^{3} |a_i|L_i,$$

where $L_i = \text{length}\, C_i$. Using (2.7) and a Wirtinger inequality argument (for details see Osserman [7], p. 1204) we can derive

$$(2.9) \qquad \int_{C_i} |x - a_i|dH^1x \le \frac{L_i^2}{2\pi}.$$

Combining (2.8) and (2.9) we obtain

$$(2.10) \qquad 2A \le \frac{\lambda}{2\pi}\left(L_1^2 + L_2^2 + L_3^2\right) + \lambda \sum_{i=1}^{3} |a_i|L_i.$$

Suppose $L_1 \ge L_2, L_3$ and choose coordinates so that $a_1 = 0$. Then from (2.10) we derive

$$\frac{4\pi}{\lambda}A \le L_1^2 + L_2^2 + L_3^2 + 2\pi\left(|a_2|L_2 + |a_3|L_3\right)$$
$$\le C\left(L_1^2 + L_2^2 + L_3^2\right) + 2\pi\left(|a_2|L_2 + |a_3|L_3\right)$$

for any $C \geq 1$,

$$= C\left(L^2 - 2L_1L_2 - 2L_2L_3 - 2L_1L_3\right) + 2\pi\left(|a_2|L_2 + |a_3|L_3\right).$$

Let $r = L_1/2\pi$, $d = \max\{|a_i - a_j|\} \geq \max\{|a_2|, |a_3|\}$. Then for any $C \geq 1$,

$$L^2 - \frac{4\pi}{\lambda C}A \geq 4\pi(L_2 + L_3)\left(r - \frac{d}{2C}\right) + 2L_2L_3. \tag{2.11}$$

It now remains only to prove that for some $C = C(\Phi)$ large enough, we always have the bound

$$d \leq 2Cr. \tag{2.12}$$

The proof of (2.12) will be contained in the lemma of §3.

3. LEMMA. *Suppose Φ satisfies the hypotheses of the theorem of* §2. *Then there is a constant $C(\Phi)$ with the following property. Suppose M is a bounded connected C^2 two dimensional submanifold with boundary of $\mathbf{R}^3$, stationary with respect to Φ. Then M satisfies the inequality*

$$\operatorname{length}(\partial M) \geq C(\Phi)\operatorname{diam}(M). \tag{3.1}$$

Proof. We begin by using a barrier argument to prove (2.12). Since M is stationary, by (2.4) we have

$$-\frac{k_1}{k_2} = \frac{\left\langle u_2^2, D^2\Phi(n)\right\rangle}{\left\langle u_1^2, D^2\Phi(n)\right\rangle}. \tag{3.2}$$

By the ellipticity of Φ, this places upper and lower bounds

$$\frac{1}{1+\varepsilon} \leq -\frac{k_1}{k_2} \leq 1 + \varepsilon \tag{3.3}$$

for some $\varepsilon = \varepsilon(\Phi)$. We now construct a hypersurface N with principal curvatures c_1 and c_2 satisfying

$$0 \leq -\frac{c_1}{c_2} \leq \frac{1}{1+\varepsilon}. \tag{3.4}$$

We construct N in such a way that either (2.12) holds or by a rigid translation of N we must be able to achieve an interior point of tangent contact between M and N, in such a way as to contradict (3.3) and (3.4).

Since C_i is a closed connected curve we have $2\operatorname{diam} C_i \leq L_i \leq L_1$, so that

$$\partial M \subset \bigcup_{i=1}^{3} B(a_i, \operatorname{diam} C_i) \subset \bigcup_{i=1}^{3} B(a_i, \pi r). \tag{3.5}$$

We assume each a_i lies in the xy plane, so that by the convex hull property [8] we know that $M \subset \{(x, y, z): |z| \leq \pi r\}$. By definition of d, we know that for one of the a_i, say a_j, the other two a_i are not in $B(a_j, d/2)$. For sake of exposition we assume without loss of generality that $j = 1$. We define hypersurfaces $N(\theta)$, each identical to within a rigid motion. For each $\theta \in [0, 2\pi]$, $N(\theta)$ will be the inside half of a torus of minor radius $s > \pi r$ and major radius $R = d/4$, $R > s$:

$$N(\theta) = a_1 + (R\cos\theta + (R + s\cos u)\cos v,\\ R\sin\theta + (R + s\cos u)\sin v, s\sin u)$$

for $-\pi \leq v \leq \pi$, $\pi/2 < u < 3\pi/2$. For each θ, $N(\theta)$ has principal curvatures

$$c_1 = \frac{\cos u}{R + s\cos u}, \qquad c_2 = \frac{1}{s}$$

(see [6]), so that

$$0 < -\frac{c_1}{c_2} \leq \frac{s}{d/4 - s}.$$

Since $2R = d/2$, $s > \pi r$, and a_2, a_3 are not in $B(a_1, d/2)$ we have that as θ ranges over $[0, 2\pi]$, N never intersects ∂M and ∂N never intersects M. Further, we can choose an initial value θ_0 such that $N(\theta_0) \cap M = \varnothing$. Thus since M is connected there is a first value $\theta_1 > \theta_0$ for which $N(\theta_1) \cap M \neq \varnothing$. Since θ_1 is the first such value, the intersection must include an interior point p of both surfaces such that $T_pN(\theta_1) = T_pM$. Now if

$$\frac{\pi r}{d/4 - \pi r} < \frac{1}{1 + \varepsilon}, \tag{3.6}$$

we can then choose $s > \pi r$ such that

$$-\frac{c_1}{c_2} < \frac{1}{1 + \varepsilon} \leq -\frac{k_1}{k_2}.$$

Orienting the normal of $T_pN(\theta_1)$ positive in the direction of decreasing θ, from this we conclude that there are directions in T_pM such that the corresponding normal curvature in M is nonpositive while the normal curvature in the same direction in $N(\theta)$ is positive. This contradicts the assumption that θ_1 is the first $\theta > \theta_0$ for which $N(\theta) \cap M \neq \varnothing$. From this we conclude that M cannot be connected if (3.6) holds, and so (2.12) is proven with $C = \pi(4 + 2\varepsilon)$.

This establishes the isoperimetric inequality. To finish the proof of the lemma, we note that length$(\partial M) \geq L_1 = 2\pi r$, and by the convex hull property and (3.5) $\operatorname{diam}(M) \leq 2\pi r + d$. Thus, by (2.12), we have

$$\operatorname{diam}(M) \leq 2(\pi + C)r \leq \frac{\pi + C}{\pi} \operatorname{length}(\partial M). \tag{3.7}$$

4. Acknowledgements. This work was carried out at Duke University, where the author benefited greatly from the advice and encouragement of William K. Allard. The author would also like to acknowledge the anonymous reviewer, who pointed out that the inequality (2.12) could be expressed in the form of the lemma of §3.

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Received August 2, 1983.

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