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THE ORDERING STRUCTURE ON BANACH SPACES

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Suppose X, Y are Banach spaces. The binary relation $X \prec Y$ means $X = \bigcap T^{**-1}[Y]$, where the intersection is taken over all bounded linear operators T: $X \to Y$. We use this definition to study sums of Banach spaces and the spaces $J(\omega_1)$, $C_{[0,\omega_1]}$.

Introduction. In this paper we use the binary relation defined by G. A. Edgar in [**5**] to investigate some relationships among some sequence Banach spaces.

Notation and terminology used in this paper follow [**5**] and match Diestel and Uhl [**1**], Dunford and Schwartz [**2**], Kelley [**8**], Lindenstrauss and Tzafriri [**9**]. If X is a Banach space, its dual will be denoted X^*, its bidual X^{**}. The subset of X^{**} canonically identified with X will simply be written X.

In [**5**], G. A. Edgar established a binary relation "$\prec$" for Banach spaces which is defined by

DEFINITION. Let X and Y be Banach spaces. Then $X \prec Y$ means

$$X = \bigcap T^{**-1}[Y]$$

where the intersection is taken over all bounded linear operators T: $X \to Y$.

Let $F(X, Y) = \bigcap T^{**-1}[Y]$. $F(X, Y)$ is called the Frame.

[**5**] points out that the definition can be rephrased as follows: $X \prec Y$ if and only if any $\alpha \in X^{**}$, such that $T^{**}(\alpha) \in Y$ for all bounded linear operators T: $X \to Y$, must be in X.

In this paper we use this ordering to consider the c_0-sum of Banach spaces, l_1-sum of Banach spaces and to compare them with c_0 space, $l_1(\Gamma)$ space respectively. We find out that under some natural conditions the ordering is preserved (see Propositions 3,5. More generally also see Proposition 4.) In this paper we also consider some function Banach spaces. We find out that the long James space, $J(\omega_1)$ is a predecessor of the continuous function space $C_{[0,\omega_1]}$ in this ordering (see Proposition 7), but $J(\omega_1)$ and $C_{[0,\omega_1]}$ both are not predecessors of l_∞ (see Propositions 8,11).

In [5] it was determined for which sets Γ is $l_1(\Gamma) \prec l_1$. We will show that $c_0(\Gamma) \prec c_0$ for all sets Γ, but $l_\infty(\Gamma) \prec l_\infty$ only for countable sets Γ. In [5] was investigated conditions under which $(\oplus_{\gamma \in \Gamma} X_\gamma)_1 \prec l_1$. We will investigate similar conditions for $(\oplus_{\gamma \in \Gamma} X_\gamma)_0 \prec c_0$ and $(\oplus_{\gamma \in \Gamma} X_\gamma)_0 \prec l_\infty$.

For any abstract set Γ, let $c_0(\Gamma)$ denote the Banach space of all scalar valued functions on Γ which vanish at infinity (i.e., for which $\{\gamma: |f(\gamma)| > \varepsilon\}$ is finite for any $\varepsilon > 0$) with the norm $\|f\| = \max_{\gamma \in \Gamma}\{|f(\gamma)|\}$.

1. Proposition. *Let Γ be any abstract set, then $c_0(\Gamma) \prec c_0$.*

Proof. Case (i). If Γ is finite, then $c_0(\Gamma)$ is reflexive, so according to Proposition 1 [5], $c_0(\Gamma) \prec c_0$ holds.

Case (ii). Assume Γ is infinite. Suppose $\alpha \in (c_0(\Gamma))^{**} = l_\infty(\Gamma)$, $\alpha = (\alpha(\gamma))_{\gamma \in \Gamma}$, but $\alpha \notin c_0(\Gamma)$. There is a positive number $\varepsilon_0 > 0$ such that $K_{\varepsilon_0} = \{\gamma: |\alpha(\gamma)| > \varepsilon_0\}$ is an infinite set. Choose distinct $\gamma_n \in K_{\varepsilon_0}$, $n = 1, 2, \ldots$, and define an operator T by $(Tx)(n) = x(\gamma_n)$, $n = 1, 2, \ldots$, where $x \in c_0(\Gamma)$ and $x = (x(\gamma))_{\gamma \in \Gamma}$.

Let $(e_n^*)_{n=1}^\infty$ be the unit vectors in c_0^* and $(e_\gamma^*)_{\gamma \in \Gamma}$ be the unit vectors in $(c_0(\Gamma))^*$. For any $x \in c_0(\Gamma)$,

$$\langle T^* e_n^*, x \rangle = \langle e_n^*, Tx \rangle = x(\gamma_n) = \langle e_{\gamma_n}^*, x \rangle,$$

so $T^* e_n^* = e_{\gamma_n}^*$, $n = 1, 2, \ldots$, so

$$T^{**}(\alpha)(n) = \alpha T^*(n) = \langle \alpha, e_{\gamma_n}^* \rangle = \alpha(\gamma_n).$$

By the hypothesis $|\alpha(\gamma_n)| > \varepsilon_0$, $n = 1, 2, \ldots$. Therefore $T^{**}(\alpha) \notin c_0$, hence $c_0(\Gamma) \prec c_0$.

Let Γ be any abstract set, for each $\gamma \in \Gamma$, let X_γ be a Banach space. Define a space by

$$X = \{(x(\gamma))_{\gamma \in \Gamma} : x(\gamma) \in X_\gamma, \text{ and for any } \varepsilon > 0, \quad \{\gamma: \|x(\gamma)\| > \varepsilon\} \text{ is finite}\},$$

X has the norm $\|x\| = \max_{\gamma \in \Gamma}\|x(\gamma)\|$, X is a Banach space. X is called the c_0-sum of the X_γ's, written $X = (\oplus_{\gamma \in \Gamma} X_\gamma)_0$. The dual of X is denoted by

$$X^* = \left\{(f(\gamma))_{\gamma \in \Gamma} : f(\gamma) \in X_\gamma^* \text{ and } \|f\| = \sum_{\gamma \in \Gamma} |f(\gamma)| < +\infty\right\},$$

where X_γ^* is the dual of X_γ, for each $\gamma \in \Gamma$, X^* is called the l_1-sum of the X_γ^*'s, written $X^* = (\oplus_{\gamma \in \Gamma} X_\gamma^*)_1$. The bidual of X is denoted by

$$X^{**} = \left\{(\alpha(\gamma))_{\gamma \in \Gamma} : \alpha(\gamma) \in X_\gamma^{**} \text{ and } \|\alpha\| = \sup_{\gamma \in \Gamma} \|\alpha(\gamma)\| < \infty\right\},$$

where X_γ^{**} is the bidual of X_γ, for each $\gamma \in \Gamma$. X^{**} is called the l_∞-sum of X_γ^{**}'s, written $X^{**} = (\oplus_{\gamma \in \Gamma} X_\gamma^{**})_\infty$.

For each $\gamma \in \Gamma$ define a mapping J_γ: $X_\gamma \to X$ by

$$J_\gamma x(\gamma') = \begin{cases} x(\gamma) & \text{if } \gamma' = \gamma, \\ 0 & \text{if } \gamma' \neq \gamma, \end{cases}$$

which is called the coordinate embedding (see [5]). For each $\gamma \in \Gamma$ define a mapping P_γ: $X \to X_\gamma$ by $P_\gamma x = x(\gamma)$ which is called the coordinate projection (see [5]).

Godefroy and Talagrand [7] say that a Banach space X has *property* (X) if and only if any $\alpha \in X^{**}$ such that $\alpha(\sum f_n) = \sum \alpha(f_n)$, for every sequence $(f_n) \subset X^*$ with $\sum |f_n(x)| < \infty$ for all $x \in X$, must be in X. (The sum $\sum f_n$ is taken in the weak* topology of X^*.)

[5] proves that if X is a Banach space, the $X \prec l_1$ if and only if X has property (X).

2. Proposition. *Let Γ be a set with $l_1(\Gamma) \prec l_1$. For each $\gamma \in \Gamma$, let X_γ be a Banach space and X_γ have Property (X). Then*

$$\left(\bigoplus_{\gamma \in \Gamma} X_\gamma\right)_1 \prec \left(\bigoplus_{\gamma \in \Gamma} X_\gamma\right)_0.$$

Proof. By Proposition 10 [5], for each $\gamma \in \Gamma$, $X_\gamma \prec l_1$ and by Proposition 15 [5], $(\oplus_{\gamma \in \Gamma} X_\gamma)_1 \prec l_1$. Since $(\oplus_{\gamma \in \Gamma} X_\gamma)_0$ is not reflexive, it follows from Proposition 2 [5] that $l_1 \prec (\oplus_{\gamma \in \Gamma} X_\gamma)_0$. So $(\oplus_{\gamma \in \Gamma} X_\gamma)_1 \prec (\oplus_{\gamma \in \Gamma} X_\gamma)_0$.

3. Proposition. *Let Γ be an abstract set and for each $\gamma \in \Gamma$, X_γ be a Banach space with $X_\gamma \prec l_1(\Gamma)$. Then $(\oplus_{\gamma \in \Gamma} X_\gamma)_1 \prec l_1(\Gamma)$.*

The proof is very similar to the proof of Proposition 15 [5].

4. Proposition. *Let Γ be an abstract set for each $\gamma \in \Gamma$, let X_γ, Y_γ be Banach spaces such that $X_\gamma \prec Y_\gamma$. Then $X \prec Y$ where $X = (\oplus_{\gamma \in \Gamma} X_\gamma)_0$, $Y = (\oplus_{\gamma \in \Gamma} Y_\gamma)_0$.*

Proof. Suppose $\alpha = (\alpha(\gamma))_{\gamma \in \Gamma}$, $X^{**} = (\oplus_{\gamma \in \Gamma} X_\gamma^{**})_\infty$, but $\alpha \notin X$. There are two cases we have to consider:

1. If there is a $\gamma_0 \in \Gamma$ such that $\alpha(\gamma_0) \notin X_{\gamma_0}$. By the hypothesis there is a linear bounded operator T_{γ_0}: $X_{\gamma_0} \to Y_{\gamma_0}$ such that $T_{\gamma_0}^{**}\alpha(\gamma_0) \notin Y_{\gamma_0}$. Let

$Tx = J_{\gamma_0} T_{\gamma_0} P_{\gamma_0}(x)$. This is a bounded linear operator from X into Y, where P_γ is a coordinate projection from X into X_γ, J_γ is a coordinate embedding from Y_γ into Y. So $T^{**} = J_{\gamma_0}^{**} T_{\gamma_0}^{**} P_{\gamma_0}^{**}$. So $T^{**}\alpha = J_{\gamma_0}^{**} T_{\gamma_0}^{**}\alpha(\gamma_0) \notin Y$, since $T_{\gamma_0}^{**}\alpha(\gamma_0) \notin Y_{\gamma_0}$.

2. Suppose $\alpha(\gamma) \in X_\gamma$ for all $\gamma \in \Gamma$ and there is $\varepsilon_0 > 0$ such that the set $\{\gamma: \|\alpha(\gamma)\| > \varepsilon_0\}$ is infinite. Suppose $\{\gamma_k\}_{k=1}^\infty \subset \{\gamma: \|\alpha(\gamma)\| > \varepsilon_0\}$. For each k choose $f_{\gamma_k} \in X_{\gamma_k}^*$, $\|f_{\gamma_k}\| = 1$ such that $\langle f_{\gamma_k}, \alpha(\gamma_k)\rangle = \|\alpha(\gamma_k)\| > \varepsilon_0$, and choose $y(\gamma_k) \in Y_{\gamma_k}$, $\|y(\gamma_k)\| = 1$. Now we define a linear bounded operator T from X into Y by $(Tx)(\gamma) = T_\gamma x(\gamma)$, for all $\gamma \in \Gamma$, where $T_\gamma x(\gamma) = 0$, if $\gamma \neq \gamma_k$;

$$T_\gamma x(\gamma) = \langle f_\gamma, x(\gamma)\rangle y(\gamma), \quad \text{if } \gamma = \gamma_k,\ k = 1, 2, \ldots.$$

Since the range of T_γ is a finite dimensional space which is reflexive, so

$$T_{\gamma_k}^{**}\alpha(\gamma_k) = \langle f_{\gamma_k}, \alpha(\gamma_k)\rangle y(\gamma_k), \qquad k = 1, 2, \ldots,$$

the others $T_\gamma^{**} = 0$.

Let $(\overline{T}\alpha)(\gamma) = T_\gamma^{**}(\alpha(\gamma))$, $\gamma \in \Gamma$. Now $\overline{T}$ is weak*-weak* continuous on X^{**}. By definition $\overline{T}|_X = T$. So $\overline{T} = T^{**}$. According to the definition

$$\begin{aligned}\|T^{**}(\alpha)(\gamma_k)\| &= |\langle f_{\gamma_k}, \alpha(\gamma_k) y(\gamma_k)\rangle| \\ &= \|\alpha(\gamma_k)\| > \varepsilon_0, \qquad k = 1, 2, \ldots.\end{aligned}$$

So $T^{**}\alpha \notin Y$.

5. Corollary. *Let Γ be any abstract set, for each $\gamma \in \Gamma$, let X_γ be a Banach space such that $X_\gamma \prec c_0$. Then*

$$X = \Big(\bigoplus_{\gamma\in\Gamma} X_\gamma\Big)_0 \prec c_0.$$

Proof. By the hypothesis and Proposition 4 we have

$$\Big(\bigoplus_{\gamma\in\Gamma} X_\gamma\Big)_0 \prec \Big(\bigoplus_{\gamma\in\Gamma} c_0(\mathbf{N})\Big)_0 = c_0(\mathbf{N}\times\Gamma),$$

where $\mathbf{N}$ indicates the integers. According to Proposition 1

$$c_0(\mathbf{N}\times\Gamma) \prec c_0$$

is true for any Γ. So the Proposition holds.

[5] proves that if X is a Banach space, then the following are equivalent.

(a) $X \prec l_\infty$

(b) If $\alpha \in X^{**}$ is weak* continuous on all bounded weak* separable subset of X^*, then $\alpha \in X$.

We use this characterization to prove next proposition.

6. PROPOSITION. *Let Γ be an abstract set, for each $\gamma \in \Gamma$, let X_γ be a Banach space such that $X_\gamma \prec l_\infty$. Then $X = (\oplus_{\gamma \in \Gamma} X_\gamma)_0 \prec l_\infty$.*

Proof. Suppose $\alpha \in X^{**} = (\oplus_{\gamma\in\Gamma} X_\gamma^{**})_\infty$, $\alpha = (\alpha(\gamma))_{\gamma\in\Gamma}$, $\alpha(\gamma) \in X_\gamma^{**}$, $\gamma \in \Gamma$ and $\sup_{\gamma\in\Gamma} \|\alpha(\gamma)\| < +\infty$. Suppose $\alpha \in F(X, l_\infty)$. We can claim that for any $\gamma \in \Gamma$, $\alpha(\gamma)$ is weak* continuous on any bounded weak* separable set $B(\gamma)$ in X_γ^*. In fact, suppose $B(\gamma) = \{f(\gamma): f(\gamma) \in X_\gamma^*\}$. Let $A = \{P_\gamma^* f(\gamma): f(\gamma) \in B(\gamma)\}$, where P_γ^* is the adjoint operator of the coordinate projection $P_\gamma: X \to X_\gamma$, $\gamma \in \Gamma$. A is also bounded weak* separable in X^*. Since α is weak* continuous on A, then $\alpha(\gamma)$ is also weak* continuous on $B(\gamma)$, for all $\gamma \in \Gamma$. So by Proposition 6 [5] and the hypothesis we get $\alpha(\gamma) \in X_\gamma$, for all $\gamma \in \Gamma$. We will show that $\alpha \in X$. Suppose $\alpha \notin X$. So there is $\varepsilon_0 > 0$ such that the set $\{\gamma: \|\alpha(\gamma)\| > \varepsilon_0\}$ is infinite. Let $\{\gamma: \|\alpha(\gamma)\| > \varepsilon_0\} \supseteq \{\gamma_k\}_{k=1}^\infty$. By the Hahn-Banach Theorem we choose $g_{\gamma_k} \in S(x_{\gamma_k}^*)$ such that $\alpha(\gamma_k)(g_{\gamma_k}) = g_{\gamma_k}(\alpha(\gamma_k)) = \|\alpha(\gamma_k)\|$, where $S(X_{\gamma_k}^*)$ is the sphere of $X_{\gamma_k}^*$, $k = 1, 2, \ldots$. Let $h_{\gamma_k} = P_{\gamma_k}^* g_{\gamma_k}$, $k = 1, 2, \ldots$, and $A_1 = \{h_{\gamma_k}\}_{k=1}^\infty \cup \{0\}$. A_1 is bounded weak* separable and

$$h_{\gamma_k} \underset{k\to\infty}{\overset{W^*}{\to}} 0.$$

$$|\alpha(h_{\gamma_k})| = |\alpha(\gamma_k) g_{\gamma_k}| = \|\alpha(\gamma_k)\| > \varepsilon_0 \underset{k\to\infty}{\to} 0 = \alpha(0).$$

Contradiction. So $\alpha \in X$.

Suppose ω_1 is the first uncountable ordinal number. Let $[0, \omega_1]$ have the order topology. $C_{[0,\omega_1]}$ consists of all continuous functions on $[0, \omega_1]$ with max norm. The dual of $C_{[0,\omega_1]}$ consists of all functions f on $[0, \omega_1]$ satisfying $\|f\| = \Sigma_{\gamma\in[0,\omega_1]} |f(\gamma)| < +\infty$, written $C_{[0,\omega_1]}^* = l_{1_{[0,\omega_1]}}$. The bidual of $C_{[0,\omega_1]}$ consists of all functions α on $[0, \omega_1]$ satisfying $\|\alpha\| = \sup_{\gamma\in[0,\omega_1]} |\alpha(\gamma)| < \infty$, written $C_{[0,\omega_1]}{}^{**} = l_{\infty[0,\omega_1]}$.

Let $[0, \omega_1]$ have the order topology. $J(\omega_1)$ is the James-type Banach space which consists of all of the functions x on the ordinal space $[0, \omega_1]$ satisfying

(i) $x(0) = 0$;

(ii) x is continuous on $[0, \omega_1]$;

(iii) $\|x\|_J = \sup(\Sigma_{i=1}^n |x(\gamma_i) - x(\gamma_{i-1})|^2)^{1/2} < +\infty$;

the sup is taken over all finite sequences $\gamma_0 < \gamma_1 < \cdots < \gamma_n$ in $[0, \omega_1]$ (see [6]). $C_{[0, \omega_1]}$ consists of all of the continuous functions x on the ordinal space $[0, \omega_1]$ with the norm $\|x\| = \max_{\gamma \in [0, \omega_1]} |x(\gamma)|$.

7. PROPOSITION. $J(\omega_1) \prec C_{[0, \omega_1]}$.

Proof. According to [6] we know that $J^{**}(\omega_1)$ can be identified with the set of all x on $[0, \omega_1]$ satisfying (i) and (iii). Suppose $\alpha \in J^{**}(\omega_1)$ but $\alpha \notin J(\omega_1)$. Define a bounded linear operator from $J(\omega_1)$ into $C_{[0, \omega_1]}$ by

$$Tx = x, \quad \text{for all } x \in J(\omega_1).$$

So $T^{**}(\alpha) = \alpha \notin C_{[0, \omega_1]}$. This completes the proof.

8. PROPOSITION. $J(\omega_1) \not\prec l_\infty$.

In order to prove Proposition 8, we need a lemma.

9. LEMMA. *Suppose A is a weak* separable bounded subset in $l_{1_{[0, \omega_1]}}$. Then there is a $\beta \in [0, \omega_1)$ so that $f = 0$ on (β, ω_1) for all $f \in A$.*

Proof. Suppose $\{f_n\}$ is weak* dense in A. Since $f_n \in l_{1_{[0, \omega_1]}}$, there are at most countably many points $\{\gamma_{n_j}\}$, $j = 1, 2, \ldots$, so that $f_n(\gamma_{n_j}) \neq 0$ for $n = 1, 2, \ldots$. Since ω_1 is uncountable ordinal number, so

$$\sup_j \left\{ \{\gamma_{n_j}\} \setminus \omega_1 \right\} = \beta_n < \omega_1, \qquad n = 1, 2, \ldots.$$

Let $\beta = \sup_n \{\beta_n\}$. For the same reason we get

$$\beta < \omega_1.$$

Suppose $f \in A$, $\{f_\delta\}$ is a subnet of $\{f_n\}$ and $f_\delta \xrightarrow{W^*} f$. We will show that $f = 0$ on (β, ω_1). Suppose t is an arbitrary point in (β, ω_1). Then

(1) If t has an immediate predecessor, the characteristic function $\chi_{\{t\}} \in C_{[0, \omega_1]}$, so

$$f(t) = \langle f, \chi_{\{t\}} \rangle = \lim_\delta \langle f_\delta, \chi_{\{t\}} \rangle = f_\delta(t) = 0$$

since $f_\delta = 0$ on (β, ω_1) for all δ.

(2) If t has no immediate predecessor, then there exist $t_n < t$, $t_n \to t$ and $t_n > \beta$, $n = 1, 2, \ldots$, $\chi_{(t_n, t]} \in C_{[0, \omega_1]}$. (Since $[0, \omega_1]$ is a well-ordered set, so any element in $[0, \omega_1]$ has an immediate successor, so $\chi_{(t_n, t]} \in C_{[0, \omega_1]}$.) So

$$\langle f, \chi_{(t_n, t]} \rangle = \lim_\delta \langle f_\delta, \chi_{(t_n, t]} \rangle = 0$$

i.e., $\Sigma_{t_n<\gamma\le t} f(\gamma) = \langle f, \chi_{(t_n, t]}\rangle = 0$ for all n. Since

$$\bigcap_n (t_n, t] = \{t\} \quad \text{and} \quad |f\cdot\chi_{(t_n,t]}| \le |f| \in l_{1_{[0,\omega_1]}}.$$

Using the dominated convergence theorem we get

$$f(t) = \sum_{\gamma=t} f(t) = \lim_{n\to\infty} \sum_{t_n<\gamma\le t} f(\gamma) = \lim_{n\to\infty} \langle f, \chi_{(t_n,t]}\rangle = 0,$$

f is arbitrary in A. This completes the proof.

10. COROLLARY. *Suppose A is weak* separable bounded subset in $l_{1_{[0,\omega_1]}}$. Suppose a net $\{f_\delta\}\subseteq A, f\in A$. If*

$$f_\delta \underset{\delta}{\overset{W^*}{\to}} f,$$

then $\lim_\delta f_\delta(\omega_1) = f(\omega_1)$.

Proof. According to Lemma 9 there is a $0\le\beta<\omega_1$ so that $f=0$ on (β,ω_1) and $f_\delta = 0$ on (β,ω_1) for all δ. Since $\chi_{(\beta,\omega_1]}\in C_{[0,\omega_1]}$, so

$$f_\delta(\omega_1) = \langle f_\delta, \chi_{(\beta,\omega_1]}\rangle, \qquad f(\omega_1) = \langle f, \chi_{(\beta,\omega_1]}\rangle.$$

Therefore $\lim_\delta f_\delta(\omega_1) = f(\omega_1)$.

Proof of Proposition 8. Recall that according to Proposition 2 [**6**] and Proposition 3 [**6**] the transfinite sequence $\{h_\gamma = \chi_{(\gamma,\omega_1]}\}_{\gamma<\omega_1}$ is a transfinite basis for the Banach space $J(\omega_1)$; the transfinite sequence $\{e_\gamma\}_{\gamma\in(0,\omega_1]}$ is a basis for the Banach space $J(\omega_1)^*$ which is the dual of $J(\omega_1)$, where $e_\gamma(x) = x(\gamma)$, for all $x\in J(\omega_1)$, for all $\gamma\in(0,\omega_1]$.

Suppose $f\in J(\omega_1)^*$. Then $f = \Sigma_{\gamma\in(0,\omega_1]}c_\gamma(f)e_\gamma$, where $c_\gamma\in J(\omega_1)^{**}$ are the coordinate functionals, $J(\omega_1)^{**}$ is the bidual of $J(\omega_1)$. Since $\langle e_\gamma, h_\beta\rangle = \chi_{(\beta,\omega_1]}(\gamma)$, $\beta\in[0,\omega_1)$, so $\Sigma_{\gamma\in(0,\omega_1]}c_\gamma(f) = \langle f, h_0\rangle$ converges. So there are at most countably many coefficients which are not zero, say $c_{\gamma_j}(f)\ne 0, j = 1,2,\ldots$. Since ω_1 is the first uncountable ordinal number so $\beta_f = \sup_j\{\{\gamma_j\}\setminus\omega_1\} < \omega_1$. So $c_\gamma(f) = 0$, for any $\gamma\in(\beta_f,\omega_1)$.

Let A be a bounded weak* separable set in $J(\omega_1)^*$. Then there is a $\beta<\omega_1$ so that $c_\gamma(f) = 0$ for any $f\in A$ and for any $\gamma\in(\beta,\omega_1)$.

Indeed, suppose $\{f_n\}_{n=1}^\infty$ is a weak* dense in A. So $\beta = \sup_n\{\beta_{f_n}\} < \omega_1$. Suppose $f\in A$, $\{f_\delta\}$ is a subnet of $\{f_n\}_{n=1}^\infty$ and

$$f_\delta \underset{\delta}{\overset{W^*}{\to}} f,$$

and $f_\delta = \Sigma_{\gamma\in(0,\omega_1]}c_\gamma(f_\delta)e_\gamma$, for any δ.

Suppose $t\in(\beta,\omega_1)$.

If t has an immediate predecessor $t-1$, then $\chi_{\{t\}} \in J(\omega_1)$. $\chi_{\{t\}} = h_{t-1} - h_t$, so $\langle f, \chi_{\{t\}} \rangle = c_t(f)$, $\langle f_\delta, \chi_{\{t\}} \rangle = c_t(f_\delta) = 0$ so

$$c_t(f) = \langle f, \chi_{\{t\}} \rangle = \lim_\delta \langle f_\delta, \chi_{\{t\}} \rangle = \lim_\delta c_t(f_\delta) = 0.$$

If t has no immediate predecessor, let $t_n < t$, $t_n \to t$ and $t_n > \beta$, $n = 1, 2, \ldots$, and t_n non-limits, $\chi_{(t_1, t_n)}, \chi_{(t_n, t]} \in J(\omega_1)$, $n = 1, 2, \ldots$.

$$\sum_{\gamma \in (t_1, t_n)} c_\gamma(f) = \langle f, \chi_{(t_1, t_n)} \rangle = \lim_\delta \langle f_\delta, \chi_{(t_1, t_n)} \rangle = 0.$$

Similarly $\sum_{\gamma \in (t_n, t]} c_\gamma(f) = 0$, for all n. Since $\sum_{\gamma \in (0, \omega_1]} c_\gamma(f)$ converges, so

$$\lim_{n \to \infty} \sum_{\gamma \in (t_1, t_n)} c_\gamma(f) = \sum_{\gamma \in (t_1, t)} c_\gamma(f).$$

So

$$\begin{aligned} 0 = \sum_{\gamma \in (t_1, t]} c_\gamma(f) &= \sum_{\gamma \in (t_1, t)} c_\gamma(f) + c_t(f) \\ &= \lim_{n \to \infty} \sum_{\gamma \in (t_1, t_n)} c_\gamma(f) + c_t(f) = c_t(f). \end{aligned}$$

This shows that there is $\beta < \omega_1$ such that $c_\gamma(f) = 0$ for all $f \in A$ and all $\gamma \in (\beta, \omega_1)$.

Since $c_{\omega_1}(f) = \langle f, \chi_{(\beta, \omega_1]} \rangle$, $c_{\omega_1}(f) = \langle f_\delta, \chi_{(\beta, \omega_1]} \rangle$, for all δ. So $\lim_\delta c_{\omega_1}(f_\delta) = c_{\omega_1}(f)$.

Let

$$\alpha(\gamma) = \begin{cases} 0, & \text{if } \gamma < \omega_1, \\ 1, & \text{if } \gamma = \omega_1. \end{cases}$$

$\alpha(0) = 0$ and $\|\alpha\|_{J^{**}} = 1 < +\infty$, so $\alpha \in J(\omega_1)^{**}$. But $\alpha \notin J(\omega_1)$. From

$$f_\delta \underset{\delta}{\overset{W^*}{\to}} f$$

in A it follows that $\lim_\delta c_{\omega_1}(f_\delta) = c_{\omega_1}(f)$. So $\lim_\delta \alpha(f_\delta) = \alpha(f)$. So α is weak* continuous on A. A is any bounded weak* separable set in $J(\omega_1)^*$. By Proposition 6 [5] $\alpha \in F(J(\omega_1), l_\infty)$. This means that

$$F(J(\omega_1), l_\infty) \setminus J(\omega_1) \neq \varnothing.$$

So $J(\omega_1) \not\prec l_\infty$.

11. Corollary. $C_{[0, \omega_1]} \not\prec l_\infty$.

Proof. Suppose $C_{[0, \omega_1]} \prec l_\infty$. By Proposition 7 we have

$$J(\omega_1) \prec C_{[0, \omega_1]}.$$

So $J(\omega_1) \prec l_\infty$. This contradicts Proposition 8.

12. COROLLARY. *Suppose* Γ *is an abstract set. Then* $l_\infty(\Gamma) \prec l_\infty$ *if and only if* $\overline{\overline{\Gamma}} \leq \aleph_0$.

Proof. If $\overline{\overline{\Gamma}} > \aleph_0$, then $C_{[0,\omega_1]} \prec l_\infty(\Gamma)$. If $l_\infty(\Gamma) \prec l_\infty$, then $C_{[0,\omega_1]} \prec l_\infty$, this contradicts Corollary (11), so $l_\infty(\Gamma) \not\prec l_\infty$.

The other direction is easy to check.

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