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THE 6π THEOREM ABOUT MINIMAL SURFACES

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We prove that if Γ is a real-analytic Jordan curve in R^3 whose total curvature does not exceed 6π, then Γ cannot bound infinitely many minimal surfaces of the topological type of the disk. This generalizes an earlier theorem of J. C. C. Nitsche, who proved the same conclusion under the additional hypothesis that Γ does not bound any minimal surface with a branch point. It should be emphasized that the theorem refers to arbitrary minimal surfaces, stable or unstable. This is the only known theorem which asserts that all members of a geometrically defined class of curves cannot bound infinitely many minimal surfaces, stable or unstable.

A result of Böhme [5] shows that for each integer n, there are curves meeting the hypotheses of our theorem which bound more than n minimal surfaces. Hence it will not be possible to improve the theorem by giving a fixed bound on the number of minimal surfaces bounded by Γ. The possibility remains open, however, to give such a bound (perhaps even 3) on the number of *immersed* minimal surfaces bounded by Γ.

In [2] and [3], an attack is begun on the "finiteness problem" for minimal surfaces which furnish *relative minima* for the area functional. This problem soon comes down to the study of one-parameter families of minimal surfaces terminating in a minimal surface with a branch point. The partial results obtained in those two papers form half the basis for the result of this paper. The other half is a calculation presented here, also concerning branch points. These two results enable us to remove the hypothesis about branch points from Nitsche's proof.

1. Introduction and notation. P is the unit disk, $\bar{P}$ its closure. A *minimal surface* is a map $u\colon \bar{P} \to R^3$ such that $\Delta u = 0$ and

$$(\partial u/\partial z)\cdot(\partial u/\partial z) = (\partial u/\partial z)^2 = 0.$$

We say u is *bounded by* the Jordan curve Γ in case u restricted to the circle S^1 is a reparametrization of Γ. A *branch point* of u is a zero of the analytic function $\partial u/\partial z$. It is an *interior* branch point or a *boundary* branch point according as it lies in P or on S^1. The *order* of a branch point is the order of the zero of $\partial u/\partial z$. We shall assume u is real-analytic in $\bar{P}$ and Γ is real-analytic. Sometimes, for convenience, we may allow other parameter domains than $\overline{P}$, in which case the above definitions undergo obvious modifications.

The *Dirichlet functional or Dirichlet integral* $E(u)$ is defined by $E(u) = \frac{1}{2}\int\int_P |\nabla u|^2\, dx\, dy$. On a suitable space of surfaces, E is Frechet differentiable, and its critical points are exactly the minimal surfaces bounded by Γ. At a minimal surface u, E is twice Frechet differentiable, and the second Frechet derivative $D^2E(u)$ is a bilinear operator on the space of "tangent vectors" k: $\overline{P} = \rightarrow R^3$ such that $\Delta k = 0$ and $k(e^{i\theta})$ is tangent to Γ at $u(e^{i\theta})$. This is the modern way of looking at the "second variation" of Dirichlet's integral. The kernel of the second variation is the kernel of this bilinear operator. The members of the kernel are characterized by the "kernel equation"

$$\partial k/\partial z \cdot \partial u/\partial z = 0.$$

For a proof, see [**2**].

If u is a minimal surface, there is always a three-dimensional family of kernel directions due to the action of the conformal group. These have the form $k = \mathrm{Re}(A\, \partial u/\partial z)$ where A is analytic. The condition that k be a tangent vector restricts the choice of A to a three-dimensional family. (See [**13**].)

If u is a minimal surface with branch points, there are in addition some kernel directions called "forced Jacobi fields" or "forced Jacobi directions". These have the form $\mathrm{Re}(A\, \partial u/\partial z)$, where A is now *meromorphic*, but with poles located among the branch points of u and of low enough orders so that $A\, \partial u/\partial z$ is analtyic. Again the condition that k be a tangent vector restricts the possible choices of A to a finite-dimensional family. For each interior branch point of order m, or boundary branch point of order $2m$, there are $2m$ forced Jacobi fields; this calculation is made explicitly in the appendix to [**6**]. Note that a boundary branch point must have even order, in order that the boundary be taken on monotonically.

By a *one-parameter family* of minimal surfaces u^t (we prefer this notation to $u(t)$), we mean a real-analytic function of two variables t and z, defined for (t, z) in some set $I \times \overline{P}$, where I is a real interval of the form $[0, t_0]$ for some $t_0 > 0$, such that for each t, $u^t = u(t, \cdot)$ is a minimal surface. The work of Böhme, Tomi, and Tromba has reduced the question, whether a (real-analytic) Jordan curve can bound infinitely many minimal surfaces, to the study of the possible existence of one-parameter families of minimal surfaces, all with the same boundary. The strongest of these theorems, which produces analytic one-parameter families as defined above, is due to Böhme [**4**].

We do not want to consider one-parameter families which are trivial in the sense of being induced by the conformal group. We therefore impose the following additional condition on the meaning of the phrase "one-parameter family": That $u_t = \partial u/\partial t = t^a h$ for some tangent vector h to u^0, where h is not a conformal direction.

2. One-parameter families of branched minimal surfaces. We define a one-parameter family u^t of minimal surfaces to be a *forced Jacobi family* if for every t (in some interval which is the domain of definition of u^t), u_t is a forced Jacobi vector of u^t. Thus necessarily each u^t is a branched minimal surface.

2.1. THEOREM. *Let u^t be a one-parameter family of minimal surfaces bounded by the same real-analytic Jordan curve. Then u^t is not a forced Jacobi family.*

Proof. A forced Jacobi vector has the form $\mathrm{Re}(A\,\partial u/\partial z)$ for some meromorphic function A. Our first objective, preliminary to the main computation, is to show that if a forced Jacobi family exists, we may assume that for t in some interval $[0, t_0]$ each u^t has a branch point of the same order m at 0, or else that each u^t has a branch point of order m at 1, such that A has a pole at this branch point.

Suppose 0 is a branch point of u^0. Then for small t, and for some neighborhood U of 0, the set of branch points of u^t in U is given by finitely many functions $q_i(t)$ which are analytic in some rational power of t; this follows from the theorems on the structure of analytic varieties discussed in §3 of [2], since the branch points are the simultaneous zeroes of the three analytic functions which are the coordinates of $\partial u/\partial z$. We may bring any specified branch point of u^t to origin by a conformal transformation depending on t. This process does not change the property that u_t is forced Jacobi, but only adds a conformal direction to u_t, i.e. adds an analytic function to A. In the case of a boundary branch point of u^0, we first note that for some branch point of u^0 and some one of the branch points $q_i(t)$ which converge to that branch point as $t \to 0$, we have that $c_i(t)$ is in the closed disk $\overline{P}$ for $t \geq 0$ and A has a pole at $q_i(t)$; if one of these $q_i(t)$ lies in the open disk P for $t > 0$, we can simply restrict the allowed interval of t-values and assume we are dealing with an interior branch point. Otherwise, we may assume that $q_i(t)$ remains a boundary branch point for positive t; then we can bring it to 1 by a rotation.

It will be convenient to treat the boundary branch point case and the interior branch point case simultaneously. This can be arranged by supposing that the branch point is at $z = 0$; in the boundary branch point case that means that the parameter domain is the circle of radius 1 centered at $z = -1$.

After the conformal transformation to bring the branch point to origin, u is analytic not in t but in the new parameter $s = t^\gamma$ for some positive rational number γ. We have $u_t = \mathrm{Re}(A\,\partial u/\partial z) = u_s \gamma t^{\gamma-1}$, so $u_s = t^{1-\gamma}\gamma^{-1}\mathrm{Re}(A\,\partial u/\partial z)$. Restricting the range of values of t to avoid the problematic point $t = 0$, we see that u_s is still a forced Jacobi direction. We now drop the letter s, using t instead for the new parameter. We have shown that we can assume that 0 is a branch point for all small t, that A has a pole at 0, and that either 0 is an interior branch point for all small t or a boundary branch point for all small t. Let m be the order of the branch point for $t = 0$. For each $i \le m$, consider the set of simultaneous zeroes of z^i and the three components of the vector function $\partial u/\partial z$. This consists, by the structure theorems for analytic varieties discussed in [2], §3, locally of finitely many analytic arcs in (t, z) space. But these arcs must lie on $z = 0$, since they are zeroes of z^i. Hence either this set consists of the isolated point $z = t = 0$, or of (a portion of) the t-axis. Let m be the greatest value for which it consists of the t-axis. Then for t positive, the order of the branch point at 0 is exactly m. Again choosing a slightly different origin of t, we see that we can assume the order of the branch point is m for $t \ge 0$. We next wish to show that we can assume the order of the pole of A is independent of t also. Let U be an analytic function with $u = \mathrm{Re}(U)$; all such differ by an imaginary constant. If the constant is suitably chosen, we have $U_t = A\,\partial u/\partial z$, so $U_t \partial u/\partial \bar{z} = A(\partial u/\partial z)(\partial u/\partial \bar{z}) = \frac{1}{2}WA$, where W is the area element of u. Thus A can be written as a quotient of functions, each of which is real-analytic in z and t, say $A = F/W$. The zero of W at origin we have just seen has order $2m$, independent of t. Hence the order J of the pole of A at origin is $2m$ minus the order of the zero of F at origin. This order is certainly constant on $t > 0$ for sufficiently small t; again changing the origin of t, we may assume it is constant on $t \ge 0$. We have now achieved our first objective: we have shown that we can assume 0 is a branch point for all small t, either an interior branch point for all small t or a boundary branch point for all small t; that the order m of this branch point is independent of t; and that A has a pole of order J at the origin, with J independent of t.

As above let U be analytic with $u = \mathrm{Re}(U)$; let $K = U_t$ and $k = \mathrm{Re}(K) = u_t$. We have $K = A\,\partial u/\partial z$ for a certain meromorphic A.

We now give the main argument. We distinguish two cases. Case 1, A has a pole at origin of order $J < m$. Case 2, $J = m$. In case 1, we have some power z^j with $1 \le j \le m$ in the expansion of K, hence $\partial k/\partial z = \partial k/\partial z$ has a power of z^{j-1}. On the other hand, since $\partial u/\partial z = O(z^m)$, upon differentiating with respect to t we obtain $\partial k/\partial z = O(z^m)$, contradiction.

Now consider case 2. We have $A = az^{-m} + O(z^{-m+1})$ where $a \neq 0$. We have $du/dz = bz^m + O(z^{m+1})$, where b is a vector. We may assume that the normal to u^0 is directed along the Z-axis, and that the image of the x-axis in the parameter domain is directed along the X-axis in XYZ-space. In that case b is a scalar multiple of $(1, -i, 0)$. Changing the parameter domain by a scale factor, we may assume $b = (1, -i, 0)$. Then $K = A\,\partial u/\partial z = a(1, -i, 0) + O(z)$. Since K is real-analytic in z and t, this equation is true for all sufficiently small t. We have $u_t(0) = \operatorname{Re}(K(0)) = \operatorname{Re}[a(1, -i, 0)] \neq 0$. The fact that $u_t(0) \neq 0$ is the only use we will make of the fact that A has a pole of order m.

Away from branch points, and for small t, we may represent u^t in the "normal bundle" of u^0. That is, for each z_0 which is not a branch point of u^0, we can find a neighborhood $\Delta \times I$ of $(z_0, 0)$ in (z, t) space and real-analytic functions Φ and ψ defined on $\Delta \times I$ such that

$$u^t \circ \Phi = u^0 + \psi N \quad \text{in } \Delta \times I \tag{1}$$

where $N = u_x^0 \times u_y^0 / |u_x^0 \times u_y^0|$ is the unit normal to u^0. Differentiating with respect to t, and remembering $k = u_t$, we have

$$k \circ \Phi + (u_x \circ \Phi)\operatorname{Re}(\Phi_t) + (u_y \circ \Phi)\operatorname{Im}(\Phi_t) = \psi_t N.$$

Remembering $k = \operatorname{Re}(A\,du/dz)$ we have

$$\operatorname{Re}\left(A \circ \Phi \,\frac{du}{dz} \circ \Phi\right) + (u_x \circ \Phi)\operatorname{Re}(\Phi_t) + (u_y \circ \Phi)\operatorname{Im}(\Phi_t) = \psi_t N \quad \text{in } \Delta \times I.$$

The left-hand side is a vector tangent to u^t at $\Phi(z)$. The right-hand side is normal to u^0 at z. Since $\Phi(z) = z$ when $t = 0$, the only possibility is that both vectors are 0. Thus ψ_t is identically zero. Integrating with respect to t we find that ψ is constant. The constant is zero if Δ touches the boundary, since all the u^t have the same boundary curve. Analytic continuation along a chain of neighborhoods avoiding the branch points shows that in any case the constant is zero; so ψ is identically zero and (1) simplifies to

$$u^t \circ \Phi = u^0 \quad \text{in } \Delta \times I.$$

In particular all the u^t occupy the same two-dimensional subset of R^3, and locally away from branch points, u^t is a reparametrization of u^0. However, the argument is far from finished, because we haven't yet used the hypothesis that the branch point is a true one.

An example is instructive at this point, even though not strictly needed for the proof. Consider the minimal surfaces

$$u^t = \mathrm{Re}\big[(1, -i, 0)(z^{m+1} - t)/(1 - tz^{m+1})\big]$$

which form an example of a forced Jacobi family, but with non-Jordan boundary, namely the circle S^1 traced out $m + 1$ times. In that example, Φ may be constructed as above—it turns out to be

$$\Phi(z) = \big[(z^{m+1} + t)/(1 + tz^{m+1})\big]^{1/(m+1)},$$

which is not analytic in the whole unit disk, but has "branches" like $(z^2 - t)^{1/2}$.

This kind of behavior depends on the non-Jordan nature of the boundary. According to [7], since the boundary is a Jordan curve, the branch point is a true branch point. (See especially Remark 6.22.2 of this reference for the case of a boundary branch point.) Also according to [7], in the vicinity of a true interior branch point there is an arc of transversal self-intersection. That is, there are two analytic arcs in the parameter domain terminating at the origin whose images are the same, and along which the self-intersection is transversal. According to [8], the same is true of a boundary branch point.

Since $u_t(0) \neq 0$, the branch point does not remain fixed in space as t changes. Since the surfaces u^t occupy the same two-dimensional subset of R^3, the point $u^t(0)$ (which is the image of the branch point) lies on u^0. However, $u^t(0)$ is distinguished from all the points of u^0 near the branch point by the facts that (a) it lies on a line of self-intersection of the surface and (b) the self-intersection at that point is not transversal, since the normals at a branch point all point the same direction. This is a contradiction and completes the proof of the theorem.

3. The 6π theorem.

3.1. THEOREM. *Let Γ be a real-analytic Jordan curve in R^3 with total curvature $\leq 6\pi$. Then Γ cannot bound infinitely many minimal surfaces.*

REMARK. J. C. C. Nitsche has proved a similar theorem, with the same conclusion, but with the additional hypothesis that Γ does not bound any minimal surface with a branch point.

Proof. Let u be a minimal surface bounded by Γ. By the work of Böhme [4] (see also [2], §3), either u is isolated (in C^k topology for every large k) or there is an analytic one-parameter family of minimal surfaces

u^t defined for $0 \leq t \leq t_0$ for some $t_0 > 0$, such that u_t is not a conformal direction. If every minimal surface bounded by Γ is isolated, then by the compactness of the set of minimal surfaces bounded by Γ in C^k topology [9], there are finitely many minimal surfaces bounded by Γ. Hence, if the theorem is false, there is a one-parameter family u^t of minimal surfaces bounded by Γ. According to Theorem 2.1, it cannot be the case that for all t, u_t is a forced Jacobi vector. Let $\Phi = u_t \cdot N$. By differentiating $(\partial u/\partial z)^2$ with respect to t, we find $(\partial k/\partial z)(\partial u/\partial z) = 0$, where $k = u_t$. This is the kernel equation of $D^2E(u)$. Hence u_t is in $\operatorname{Ker} D^2E(u)$. It follows from Theorem 1.2 of [2] that Φ satisfies $\Delta\Phi - 2KW\Phi = 0$, so 2 is an eigenvalue of $\Delta\Phi - \lambda KW\Phi = 0$; here we have chosen the origin of t so that u_t^0 is not forced Jacobi, hence Φ is not identically zero, by Theorem 1.2 of [2].

By the hypothesis on the total curvature of Γ, together with the Gauss-Bonnet-Sasaki-Nitsche formula (see [2], §2), we have

$$6\pi > 2\pi + 2\pi M + \oint -KW\,dx\,dy \tag{1}$$

where M is the sum of orders of interior branch points, plus half the orders of the boundary branch points, and the integral is over the parameter domain. Note that strict inequality holds in (1) even if the total curvature of Γ is exactly 6π, since the geodesic curvature (which occurs in the Gauss-Bonnet-Sasaki-Nitsche formula) can equal the total curvature of Γ, for u a minimal surface, only if Γ lies in a plane [11], and in this case the theorem is known. If there is a boundary branch point, its order is necessarily even; hence M is an integer.

We shall now prove that either Φ^0 is identically zero or $\lambda_{\min}(u^0)$, the least eigenvalue of $\Delta\Phi - \lambda KW\Phi = 0$, is 2. Since u^0 is not isolated, we have $\lambda_{\min} \leq 2$, assuming Φ^0 is not identically zero. Since Φ is an eigenfunction for eigenvalue 2, Φ is orthogonal to the first eigenfunction. (For eigenfunctions, orthogonality in H_0^1 inner product and orthogonality in inner product $\iint -KW\Phi\psi\,dx\,dy$ are equivalent.) Hence, Φ cannot have just one sign, since the first eigenfunction does have only one sign (as discussed in [2], §2). It is not difficult to verify the well-known fact that $D^+ = \{z \in D\colon \Phi(z) > 0\}$, where D is the parameter domain, and similarly D^- are composed of finitely many connected domains bounded by finitely many analytic arcs. Since by (1) we have $\iint -KW\,dx\,dy < 4\pi$, there must exist one of these connected domains Q for which $\iint - KW\,dx\,dy < 2\pi$. In this domain, Φ has only one sign and vanishes on the boundary. Hence $\lambda_{\min}(Q) = 2$, where $\lambda_{\min}(Q)$ is the least eigenvalue of $\Delta\Phi - \lambda KW\Phi = 0$ in Q, $\Phi = 0$ on ∂Q. But this contradicts the theorem of

Barbosa and do Carmo, according to which $\int\int_Q - KW\,dx\,dy < 2\pi$ implies $\lambda_{\min}(Q) > 2$. See [2], §2 for discussion of Barbosa-do Carmo's theorem, and [1] for proof. (It is easy to check that the theorem holds for domains with corners, provided it holds for domains with smooth boundaries.)

Now suppose u^t is an arbitrary one-parameter family of minimal surfaces bounded by Γ. Let $k = u_t = t^a h$, $\Phi = u_t \cdot N$. It is impossible that Φ is identically zero in both z and t, for then, by Theorem 1.2 of [2], k is forced Jacobi, which contradicts Theorem 2.1. Hence for t small but positive, u_t is not a forced Jacobi direction, and Φ is not identically zero. We claim that u^t is immersed for t positive. If not, then $M \geq 1$ in (1), so $\int\int - KW\,dx\,dy < 2\pi$. Hence, by Barbosa-do Carmo's theorem, $\lambda_{\min}(u^t) > 2$. But since Φ^t is not identically zero, 2 is an eigenvalue, contradiction. Hence u^t is immersed for t positive. Then u^0 cannot have an interior branch point, by Theorem 5.1 of [2]. Now there are three possibilities: u^0 is immersed, or it has a boundary branch point and h^0 is forced Jacobi, or it has a boundary branch point and h^0 is not forced Jacobi. The last possibility contradicts Theorem 7.2 of [3]. Suppose the second possibility holds. Then by Theorem 8.1 of [3], the boundary branch point of u^0 has order ≥ 4, so M in (1) satisfies $M \geq 2$, contradiction. Hence u^0 is immersed.

We have now shown the following: Every one-parameter family of minimal surfaces bounded by Γ consists, for small t, of immersed surfaces with $\lambda_{\min} = 2$ for t positive. It is also true that $\lambda_{\min} = 2$ for $t = 0$, as we now prove: Since h^0 is not forced Jacobi, we have $\Phi = t^n \psi$ for some integer n, where ψ^0 is not identically zero. Since for t positive, Φ has only one sign (since $\lambda_{\min} = 2$), it follows that ψ has only one sign, even for $t = 0$; since ψ is an eigenfunction for the eigenvalue 2, it follows that $\lambda_{\min} = 2$ when $t = 0$. Thus every one-parameter family of minimal surfaces bounded by Γ consists entirely of immersed surfaces with $\lambda_{\min} = 2$. We may now employ Tomi's argument [12] (see also the proof of Theorem 5.1 of [2]) to reach a contradiction. That completes the proof of Theorem 3.1.

REFERENCES

[1] J. Barbosa and M. do Carmo, *Stable minimal surfaces*, Bull. Amer. Math. Soc., **80** (1974), 581–583.

[2] M. Beeson, *Some results of finiteness in Plateau's problem, Part I.*, Math. Zeitschrift, **175** (1980), 103–123.

[3] ______, *Some results on finiteness in Plateau's problem, Part II*, Math. Zeitschrift, **181** (1982), 1–30.

[4] R. Böhme, *Die Zusammenhangskomponenten der Lösungen analytischer Plateauprobleme*, Math. Zeitschrift, **133** (1973), 31–40.

[5] ______, *Plateau problems with many solutions.*

[6] R. Böhme and A. J. Tromba, *The index theorem for classical minimal surfaces*, Annals of Math., **113** (1981), 447–499.

[7] R. Gulliver, R. Osserman, and H. Royden, *A theory of branched immersions of surfaces*, Amer. J. Math., **95** (1973), 750–812.

[8] R. Gulliver and F. Lesley, *On boundary branch points of minimizing surfaces*, Arch. Rat. Mech. Anal., **52** (1973), 20–25.

[9] S. Hildebrandt, *Über das Randverhalten von Minimalflächen*, Math. Ann., **165** (1966), 1–18.

[10] J. C. C. Nitsche, *Contours bounding only finitely many solutions of Plateau's problem*, in the proceedings of a conference dedicated to I. N. Vekua, published in the USSR. A summary of this article appeared in Minimal Submanifolds and Geodesics, Proc. Japan-US seminar, Tokyo, 1977, North-Holland, Amsterdam (1979).

[11] J., C. C. Nitsche, *A new uniqueness theorem for minimal surfaces*, Arch. Rat. Mech. Anal., **52** (1973), 319–329.

[12] F. Tomi, *On the finite solvability of Plateau's problem*, in: Geometry and Topology, Rio de Janeiro, July 1976, pp. 679–695. Springer Lecture Notes in Mathematics 597, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[13] A. J. Tromba, *On the number of simply connected minimal surfaces spanning a curve*, Memoirs of the Amer. Math. Soc., **12**, No. 194 (1977).

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