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A UNIFIED APPROACH TO CARLESON MEASURES AND A_p WEIGHTS

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In the present note, for each p ($1 < p < \infty$), we find a condition on the pair (μ ω) (where μ is a measure on R^{n+1}_+ and ω a weight) for the Poisson integral to be a bounded operator from $L^p(R^n; \omega(x)\,dx)$ into weak-$L^p(R^{n+1}_+, \mu)$.

Our Theorem I includes, on the one hand, the results of Carleson [1] and Fefferman-Stein [2] concerning the boundedness of the Poisson integral and, on the other hand, Muckenhoupt's results concerning A_p-weights.

1. Introduction. Given a function f on R^n, set

$$\mathscr{M}f(x,t) = \sup_Q \left\{ \frac{1}{|Q|} \int_Q |f| \right\} \qquad (x \in R^n, t \geq 0),$$

where the supremum is taken over the cubes Q in R^n centered at x with sides parallel to the axes and has side length at least t.

The operator $\mathscr{M}$ is the maximal operator which "controls" the Poisson integral

$$Pf(x,t) = \int_{R^n} f(y) P(x-y,t)\,dy \qquad (x \in R^n, t \geq 0),$$

where

$$P(x,t) = \frac{c_n t}{\left(|x|^2 + t^2\right)^{(n+1)/2}}$$

is the Poisson Kernel.

The following question arises:

For a given positive measure on R^{n+1}_+ ($= R^n \times [0,\infty)$), when can we assert that $\mathscr{M}$ is bounded from $L^p(R^n)$ into $L^p(R^{n+1}_+, \mu)$ and from $L^1(R^n)$ into weak-$L^1(R^{n+1}_+, \mu)$?

Carleson [1] showed that this is true if and only if μ satisfies the growth condition, called the "Carleson condition",

$$(1) \qquad \mu(\tilde{Q}) \leq C|Q| \quad \text{for each cube } Q \text{ in } R^n,$$

where $\tilde{Q}$ denotes the cube in R^{n+1}_+ with the cube Q as its base.

Afterwards, Fefferman and Stein [2] proved that $\mathcal{M}$ is bounded from the weighted space $L^p(R^n, \omega(x)\,dx)$ into $L^p(R^{n+1}_+, \mu)$ and from $L^1(R^n, \omega(x)\,dx)$ into weak-$L^1(R^{n+1}_+, \mu)$ if the following condition is satisfied:

$$\mu^*(x) = \sup_{x\in Q} \frac{\mu(\tilde{Q})}{|Q|} \le C\omega(x) \quad \text{a.e.} \tag{2}$$

In fact, from (2), the weak type $(1,1)$ inequality is obtained, and the rest follows by interpolation with the trivial result for $p = \infty$.

Here we find the exact condition on the pair $(\mu\ \omega)$ for $\mathcal{M}$ to be a bounded operator from $L^p(R^n; \omega(x)\,dx)$ into weak-$L^p(R^{n+1}, \mu)$. The results of Carleson and Fefferman-Stein mentioned above are particular cases of our Theorem I (below), and so are Muckenhoupt's results concerning A_p weights.

Throughout this note μ will always denote a positive measure on R^{n+1}_+, ω a nonnegative weight in R^n and, finally, C will denote a positive constant, not necessarily the same at each occurrence.

2. Definition. Let $1 < p < \infty$.

Given ω we shall denote by $C_p(\omega)$ the set of measures μ on R^{n+1}_+ such that

$$\sup_Q \frac{\mu(\tilde{Q})}{|Q|}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'/p}\,dx\right)^{p/p'} = C < +\infty, \tag{3}$$

where the supremum is taken over all cubes Q in R^n. $C_1(\omega)$ will denote the set of measures μ such that

$$\mu^*(x) = \sup_{x\in Q} \frac{\mu(\tilde{Q})}{|Q|} \le C\omega(x) \quad \text{a.e}, \tag{4}$$

and $C_\infty(\omega)$ the set of measures μ such that

$$\mu(\tilde{Q}) \le C\int_Q \omega(x)\,dx, \quad \text{for all cubes } Q. \tag{5}$$

PROPOSITION. *Let $1 \le p \le q \le \infty$. If $\mu \in C_p(\omega)$ then $\mu \in C_q(\omega)$.*

Proof. This is evident for $1 < p < q < \infty$ by Hölder's inequality. If $\mu \in C_p(\omega)$ $(1 < p < \infty)$, from (3) we get

$$1 = \frac{1}{|Q|}\int_Q \omega^{1/p}\omega^{-1/p}$$

$$\le \left(\frac{1}{|Q|}\int_Q \omega\right)^{1/p}\left(\frac{1}{|Q|}\int_Q \omega^{-p'/p}\right)^{1/p'} \le \left(\frac{1}{|Q|}\int_Q \omega\right)^{1/p}\left(C\frac{|Q|}{\mu(\tilde{Q})}\right)^{1/p}$$

and, therefore,

$$\mu(\tilde{Q}) \leq C\int_Q \omega.$$

To finish the proof, let $\mu \in C_1(\omega)$ and let Q be any cube in R^n. Then

$$\left(\frac{\mu(\tilde{Q})}{|Q|}\right)^{p'/p} \omega(x)^{-p'/p} \leq C \quad \text{for a.e. } x \in Q,$$

and integrating over Q we obtain (3).

REMARK. In general, $C_p(\omega)$ is properly contained in $C_q(\omega)$ if $1 \leq p < q \leq \infty$. However, if ω belongs to the class A_p of Muckenhoupt, i.e.

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{-p'/p}\right)^{p/p'} \leq C,$$

then it is obvious that $C_p(\omega) = C_q(\omega), p \leq q \leq \infty$.

Moreover, in this case, $\mu \in C_p(\omega)$ implies $\mu \in C_{p-\varepsilon}(\omega)$ for some $\varepsilon > 0$ (since $\omega \in A_{p-\varepsilon}$ (see [**4**])).

3. **The results.** The relation between the class $C_p(\omega)$ and the boundedness of the maximal operator $\mathscr{M}$ is given by the following

THEOREM I. *Let $1 \leq p < \infty$. Then, the inequality*

(6) $$\mu(\{(x,t) \in R_+^{n+1} : \mathscr{M}f(x,t) > \alpha\}) \leq \frac{C}{\alpha^p}\int_{R^n} |f|^p\omega \qquad (f \in L^p(\omega))\ (\alpha > 0)$$

holds if and only if $\mu \in C_p(\omega)$.

Particular cases are:

A. If $\omega(x) \equiv 1$, then the classes $C_p(\omega)$ are the same for all p $(1 \leq p \leq \infty)$ and consist of all measures μ such that

$$\mu(\tilde{Q}) \leq C|Q| \quad \text{for each cube } Q \text{ in } R^n,$$

which is Carleson's condition (1). In this case, Theorem I gives us Carleson's result, mentioned in the introduction.

B. Let us consider now the measures μ on R_+^{n+1} of the form

$$d\mu(x) = v(x)\,dx \quad \text{concentrated in } R^n \times \{0\}.$$

Then $\mu \in C_p(\omega)$ means that

(7) $$\sup_Q \left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'/p}\,dx\right)^{p/p'} < \infty,$$

i.e. $\mu \in C_p(\omega)$ if and only if (v, ω) satisfies the A_p condition (see [**4**]).

Since $\mathscr{M}f(x,0) = f^*(x)$ $(x \in R^n)$ (where f^* denotes the Hardy-Littlewood maximal function of f), we obtain

THEOREM (*Muckenhoupt* [4]). *Let* $1 < p < \infty$. *The following statements are equivalent*:

(i) (v, ω) *satisfies the* A_p *condition* (7).

(ii) $\displaystyle\int_{\{f^* > \alpha\}} v(x)\,dx \le \frac{C}{\alpha^p}\int |f|^p \omega(x)\,dx \qquad (f \in L^p(\omega))\ (\alpha > 0).$

REMARK. In addition, Muckenhoupt showed that (i) is not in general sufficient for

$$\int f^*(x)^p v(x)\,dx \le C\int |f(x)|^p \omega(x)\,dx.$$

Therefore, in Theorem I we cannot substitute the weak type inequality (6) for the corresponding strong type inequality. However, if we add the hypothesis "$\omega \in A_p$", and use the remark in §2 and Marcinkiewicz's interpolation theorem, then the strong type inequality follows.

Another way of deriving the same result is shown in Corollary II.

C. For $p = 1$ the theorem gives us the result of Fefferman-Stein, already named in the introduction.

For the class $C_\infty(\omega)$ we have the following result.

THEOREM II. *If* $\mu \in C_\infty(\omega)$, *then*

$$\mu(\{(x,t) \in R_+^{n+1} : \mathscr{M}f(x,t) > \alpha\}) \le C\int_{\{f^* > 4^{-n}\alpha\}} \omega(x)\,dx. \tag{8}$$

From the distribution inequality (8), the following result is immediate.

COROLLARY I. *Let* $1 < p < \infty$. *If* $\mu \in C_\infty(\omega)$, *then*

$$\int |\mathscr{M}f|^p\,d\mu \le C\int |f^*|^p\,\omega.$$

Since f^* is bounded in $L^p(\omega)$ if and only if $\omega \in A_p$ $(1 < p < \infty)$ we have:

COROLLARY II. *Let* $1 < p < \infty$ *and* $\omega \in A_p$. *The following statements are equivalent*

(i) $\mu \in C_\infty(\omega)$

(ii) $\int_{R_+^{n+1}} |\mathscr{M}f|^p\,d\mu \le C\int |f|^p \omega$.

4. Proof of Theorem I. Assume first that (6) is verified and let $1 < p < \infty$. For any cube Q of R^n, and for all $(x, t) \in \tilde{Q}$ it is easy to see that

$$\frac{1}{|Q|}\int_Q |f| \le 2^n \mathscr{M} f(x, t);$$

therefore

$$\begin{aligned}\mu(\tilde{Q}) &\le \mu\left(\left\{(x, t) \in R_+^{n+1} : \mathscr{M} f(x, t) \ge \frac{2^{-n}}{|Q|}\int_Q |f|\right\}\right)\\ &\le C|Q|^p\left(\int_Q |f|\right)^{-p}\int_{R^n} |f|^p \omega(x)\, dx.\end{aligned}$$

Taking $f = \chi_Q \omega^{-p'/p}$ in the last inequality, we obtain $\mu \in C_p(\omega)$. For the case $p = 1$, let $x \in R^n$ be a Lebesgue point of ω^{-1}, and take an arbitrary cube Q such that $x \in Q$. $\chi_Q \omega^{-1} \in L^1(\omega)$ and therefore $\chi_Q \omega^{-1} \in L^1$, because otherwise it would be $\mathscr{M}(\chi_Q \omega^{-1})(x, t) = +\infty$ for all $(x, t) \in R_+^{n+1}$, contradicting (6). Then like in the previous case, taking $f = \chi_{Q'} \omega^{-1}$, where Q' is any cube with $x \in Q' \subset Q$, we have

$$\frac{\mu(\tilde{Q})}{|Q|} \le C\left(\frac{1}{|Q'|}\int_{Q'} \omega^{-1}\right)^{-1}.$$

Now, we let Q' tend to x and it follows that

$$\mu(\tilde{Q})/|Q| \le C\omega(x),$$

which implies $\mu \in C_1(\omega)$.

Now we assume $\mu \in C_p(\omega)$ and we have to prove (6). Only the case $1 < p < \infty$ will be considered, since the modifications needed to deal with the case $p = 1$ are rather straightforward. Let $f \in L^p(\omega)$, $\alpha > 0$, and

$$\begin{aligned}\Omega_\alpha &= \left\{(x, t) \in R_+^{n+1} : \mathscr{M} f(x, t) > \alpha\right\},\\ \Omega'_\alpha &= \left\{x \in R^n : f^*(x) > \alpha\right\}.\end{aligned}$$

Let $x_0 \in R^n$ be fixed. It is obvious that if $(x_0, t) \in \Omega$ and $t' < t$, then $(x_0, t') \in \Omega_\alpha$ and $x_0 \in \Omega'_\alpha$, and we define

$$\begin{aligned}t(x_0; \alpha) &= \sup\{t : (x_0, t) \in \Omega_\alpha\}\\ &= \sup\left\{t : \frac{1}{|Q(x_0; t)|}\int_{Q(x_0; t)} |f| > \alpha\right\}\end{aligned} \tag{9}$$

(where $Q(x_0; t)$ denotes the cube centered at x_0 with side length t).

LEMMA I. *If* $\alpha > (C/\mu(R_+^{n+1}))^{1/p}\|f\|_{L^p(\omega)}$, *then* $t(x_0; \alpha) < \infty$ *for every* $x_0 \in \Omega'_\alpha$.

Take Lemma I for granted, and consider the following two possibilities:

(a) $\mu(R_+^{n+1}) = \infty$.

(b) $\mu(R_+^{n+1}) < \infty$.

In case (a), no matter how $\alpha > 0$ is chosen, we have $t(x_0; \alpha) < \infty$ for every $x_0 \in \Omega'_\alpha$.

We shall need the following covering lemma of Besicovitch type.

LEMMA II. *Let* A *be a bounded set in* R^n. *For each* $x \in A$ *a cube* $Q(x)$ *centered at* x *is given. Then one can choose, from among the given cubes* $\{Q(x)\}_{x\in A}$, *a sequence* $\{Q_k\}$ *(possibly finite) such that:*

(i) *The set* A *is covered by the sequence, i.e.* $A \subset \bigcup Q_k$.

(ii) *The sequence* $\{Q_k\}$ *can be distributed in* N *(a number that depends only on* n*) families of disjoint cubes.*

A proof of the Lemma II can be found in [3, Chapter I.1].

Let K be any bounded measurable set of R^n. For each $x \in \Omega'_{2^{-n}\alpha} \cap K$ we take the cube $Q(x; t(x; 2^{-n}\alpha))$.

We can apply Lemma II, obtaining $\{Q_k\}$ from

$$\{Q(x; t(x; 2^{-n}\alpha))\}_{x\in\Omega'_{2^{-n}\alpha}\cap K}$$

such that $\Omega'_{2^{-n}\alpha} \cap K \subset \bigcup Q_k$ and we have $\{Q_k\}$ distributed in N (depending only on the dimension) families of disjoint cubes.

Purely geometrical considerations show that $\{\tilde{Q}_k\}$ consist also of N subfamilies of disjoint elements and $\Omega_\alpha \cap (K \times [0, \infty)) \subset \bigcup Q_k$.

For each subfamily, say $\{Q_i\}$, we have

$$\mu(\bigcup \tilde{Q}_i) = \sum_i \mu(\tilde{Q}_i) = \sum_i \frac{\mu(\tilde{Q}_i)}{|Q_i|^p}|Q_i|^p.$$

Now, using (9), Hölder's inequality (applied to $(f\omega^{1/p})\omega^{-1/p}$) and the hypothesis we obtain

$$\begin{aligned}\mu(\bigcup \tilde{Q}_i) &\le 2^{np}\sum_i \frac{\mu(\tilde{Q}_i)}{|Q_i|^p}\frac{\left(\int_{Q_i}|f|\right)^p}{\alpha^p}\\ &\le 2^{np}\sum_i \frac{\mu(\tilde{Q}_i)}{|Q_i|^p}\frac{1}{\alpha^p}\left(\int_{Q_i}|f|^p\omega\right)\left(\int_{Q_i}\omega^{-p'/p}\right)^{p/p'}\\ &\le \frac{C}{\alpha^p}\int_{R^n}|f|^p\omega,\end{aligned}$$

and, therefore,

$$\mu(\Omega_\alpha \cap (K \times [0,\infty))) \le \frac{NC}{\alpha^p} \int_{R^n} |f|^p \omega.$$

Since this estimate is independent of K, we obtain

$$\mu(\Omega_\alpha) \le \frac{NC}{\alpha^p} \int |f|^p \omega.$$

In case (b), (6) is proved (as above) for all

$$\alpha > 2^n \left(\frac{C}{\mu(R_+^{n+1})} \right)^{1/p} \|f\|_{L^p(\omega)}.$$

But for $\alpha \le 2^n (C/\mu(R_+^{n+1}))^{1/p} \|f\|_{L^p(\omega)}$, we have

$$\frac{1}{\alpha^p} \int_{R^n} |f|^p \omega \ge \frac{2^{-np}}{C} \mu(R_+^{n+1}) \ge \frac{2^{-np}}{C} \mu(\Omega_\alpha)$$

and (6) follows.

Proof of Lemma I. We suppose that μ is not identically zero (otherwise, the theorem is trivial).

If $t(x_0; \alpha) = +\infty$, then

$$\begin{aligned} \alpha &\le \limsup_{t \to \infty} \frac{1}{|Q(x_0;t)|} \int_{Q(x_0;t)} |f| \\ &\le \limsup_{t \to \infty} \frac{1}{|Q(x_0;t)|} \left(\int_{Q(x_0;t)} \omega^{-p'/p} \right)^{1/p'} \|f\|_{L^p(\omega)} \\ &\le \limsup_{t \to \infty} \left\{ \frac{C}{\mu(\tilde{Q}(x_0;t))} \right\}^{1/p} \|f\|_{L^p(\omega)} = \left\{ \frac{C}{\mu(R_+^{n+1})} \right\}^{1/p} \|f\|_{L^p(\omega)}, \end{aligned}$$

and, therefore, the lemma is proved.

This finishes the proof of Theorem I.

Proof of Theorem II. Maintaining the same notations, we suppose, first, that $t(x; 2^{-n}\alpha) < \infty$ for every $x \in \Omega'_{2^{-n}\alpha}$.

Then, let K any bounded measurable set of R^n and let $\{Q_i\}$ be one of the N subfamilies of disjoint elements whose unions cover $\Omega'_{2^{-n}\alpha} \cap K$.

If $y \in Q_i$, then it is easy to see that $\alpha < 4^n f^*(y)$ and, therefore,

$$\cup Q_i \subset \{x : f^*(x) > 4^{-n}\alpha\}$$

and, from the hypothesis we have

$$\mu\Big(\bigcup_i \tilde{Q}_i\Big) = \sum_i \mu(\tilde{Q}_i) \leq C\sum_i \int_{Q_i} \omega(x)\, dx$$

$$= C\int_{\cup Q_i} \omega(x)\, dx \leq C\int_{\{f^* > 4^{-n}\alpha\}} \omega(x)\, dx.$$

From this, (8) follows immediately.

If $t(x_0; 2^{-n}\alpha) = +\infty$ for some $x_0 \in \Omega'_{2^{-n}\alpha}$, then it is immediate that $\{x\colon f^*(x) > 4^{-n}\alpha\} = R^n$, and in this case we get

$$\mu(\Omega_\alpha) \leq \mu\big(R_+^{n+1}\big) \leq C\int_{R^n} \omega(x)\, dx.$$

Therefore, Theorem II is proved.

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