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ELIMINATION OF MALITZ QUANTIFIERS IN STABLE THEORIES

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The main result of this paper is (a slightly stronger form of) the following theorem: let T be a countable complete first-order theory which is stable. If, for some $\alpha \geq 1$, the Malitz quantifier Q_α^2 is eliminable in T then all Malitz quantifiers Q_β^m ($\beta \geq 0$, $m \geq 1$) are eliminable in T. This complements results of Baldwin-Kueker [1] and Rothmaler-Tuschik [3].

1. Introduction. In this paper we consider various logics extending first-order logic that are obtained by adding quantifiers asserting the existence of a large homogeneous set of n-tuples. For $\alpha \geq 0$, $m, n \geq 1$, $Q_\alpha^{m,n}$ is a quantifier binding $m \cdot n$ variables whose semantics is defined by

$$\begin{aligned}&\mathfrak{A} \models Q_\alpha^{m,n}\bar{x}_1 \cdots \bar{x}_m\varphi(\bar{x}_1,\ldots,\bar{x}_m)\\ &\qquad\text{iff there is a set } X \subset A^n \text{ of power } \geq \aleph_\alpha\\ &\qquad\text{which is homogeneous for } \varphi\text{, i.e., for all } n\text{-tuples}\\ &\qquad\bar{b}_1,\ldots,\bar{b}_m \in X\colon \mathfrak{A} \models \varphi[\bar{b}_1,\ldots,\bar{b}_m].\end{aligned}$$

The quantifiers $Q_\alpha^{m,n}$ were introduced by Baldwin and Kueker in [1] and denoted by $Q_\alpha^{*m,n}$. For $n = 1$, $Q_\alpha^{m,n}$ is the usual Malitz quantifier of order m in the $\aleph_\alpha$-interpretation, and $Q_\alpha^{1,1}$ is just the cardinality quantifier Q_α ("there are $\aleph_\alpha$ many"). We shall also consider another family of quantifiers: for $\alpha \geq 0$ and $n \geq 1$, E_α^n is defined by

$$\begin{aligned}&\mathfrak{A} \models E_\alpha^n x_1 \cdots x_n y_1 \cdots y_n \varphi(\bar{x}, \bar{y})\\ &\qquad\text{iff } \varphi \text{ is an equivalence relation on } A^n \text{ of index } \geq \aleph_\alpha.\end{aligned}$$

The statement on the right side is expressible by a sentence of $L(Q_\alpha^{2,n})$, but there is no $L(Q_\alpha^{1,n})$-sentence equivalent to it. So, being strictly stronger than $Q_\alpha^{1,n}$, E_α^n may be viewed as a weak form of $Q_\alpha^{2,n}$.

Adjoining a quantifier $Q_\alpha^{m,n}$ to an elementary logic L results in an increase of expressive power. There are, however, first-order theories that admit elimination of $Q_\alpha^{m,n}$: for each L-formula $\varphi(\bar{x}_1,\ldots,\bar{x}_m, \bar{y})$, where $l(\bar{x}_1) = \cdots = l(\bar{x}_m) = n$, there is another L-formula $\delta(\bar{y})$ such that, for all models $\mathfrak{A} \models T$ of power $\geq \aleph_\alpha$ and all $\bar{a} \in A$,

$$(\mathfrak{A}, \bar{a}) \models \delta \leftrightarrow Q_\alpha^{m,n}\bar{x}_1 \cdots \bar{x}_m\varphi.$$

Throughout this paper T always denotes a countable complete first-order theory with infinite models. For simplicity we assume that the language of T does not have function symbols.

If $Q_\alpha^{m,n}$ is eliminable in T, this has interesting consequences. Firstly, T remains complete as an $L(Q_\alpha^{m,n})$-theory. Secondly, if T is decidable and the elimination of $Q_\alpha^{m,n}$ can be carried out effectively, then it is also decidable whether a given $L(Q_\alpha^{m,n})$-sentence holds in some (all) model(s) of T. In many cases elimination procedures are known; for a survey see **[2]**.

Aside from investigating particular examples, it is natural to look for purely first-order properties of T which imply or characterize eliminability of certain quantifiers $Q_\alpha^{m,n}$. Another problem is to determine the relative strength of eliminability of various $Q_\alpha^{m,n}$. Regarding the $\aleph_0$-interpretation Baldwin and Kueker **[1]** gave a solution to both problems if stability of T is assumed.

THEOREM 1.1. (a) *If T does not have the finite cover property (f.c.p.) then all quantifiers $Q_0^{m,n}$ for $m, n \geq 1$ are eliminable in T.*

(b) *If T is stable and E_0^1 is eliminable in T then T does not have the f.c.p.*

Thus Theorem 1.1. shows that in the stable case the following are equivalent:

(i) T does not have the f.c.p.,

(ii) all quantifiers $Q_0^{m,n}$ and E_0^n are eliminable in T;

(iii) any single quantifier $Q_0^{m,n}$ or E_0^n ($m \geq 2$) is eliminable in T.

Regarding $\aleph_\alpha$-interpretations for $\alpha \geq 1$ one has to look for a first-order property different from "not f.c.p." in order to characterize eliminability of $Q_\alpha^{m,n}$ or E_α^n. For example, put $T := \mathrm{Th}(A, R)$, where R is an equivalence relation on A with infinitely many equivalence classes and each class infinite. T is ω-categorical and ω-stable. Hence, by Theorem 7 of **[1]**, it admits elimination of all $Q_0^{m,n}$. Thus, by Theorem 1.1, T does not have the f.c.p. However, none of E_α^n or $Q_\alpha^{m,n}$ for $\alpha \geq 1$ is eliminable in T.

In **[3]** Rothmaler and Tuschik introduced the notion of a regular theory and proved

THEOREM 1.2. *If T is regular then all quantifiers $Q_\alpha^{m,n}$ ($\alpha \geq 0$, $m \geq 1$) are eliminable in T.*

It is also mentioned in their paper ("added in proof") that, for stable T, the converse of Theorem 1.2 is true. This suggests regularity as a substitute for "not f.c.p." when looking at all Malitz quantifiers $Q_\alpha^{m,n}$. Yet, when this result is compared to Theorem 1.1, two questions remain open:

(1) Does regularity of T imply eliminability of $Q_\alpha^{m,n}$ for $n > 1$?

(2) Is there a single quantifier whose eliminability implies that of all $Q_\alpha^{m,n}$?

It will turn out (see Corollary 3.4) that, in the stable case, question 1 may be answered positively. When using a slightly more general concept than regularity, which we call strong regularity, the stability assumption can be dropped (cf. Definition 2.3 and Theorem 2.4):

THEOREM 1.3. *If T is strongly regular then all quantifiers $Q_\alpha^{m,n}$ ($\alpha \geq 0$; $m, n \geq 1$) are eliminable in T.*

This theorem is also due to Rothmaler and Tuschik (compare the appendix of [3]).

With regard to question 2 the main result of the present paper shows that the situation closely parallels that of 1.1(b).

MAIN THEOREM (see 3.2). *Let T be stable and suppose some quantifier $Q_\alpha^{m,n}$ or E_α^n (where $\alpha \geq 1$ and $m \geq 2$) is eliminable in T. Then T is strongly regular.*

Hence, for stable T, the following are equivalent:

(i) T is strongly regular;

(ii) all quantifiers $Q_\alpha^{m,n}$ are eliminable in T,

(iii) any single quantifier E_α^n or $Q_\alpha^{m,n}$ ($m \geq 2$, $\alpha \geq 1$) is eliminable in T.

The main theorem partially answers problems 2 and 3 of [2]. It will be proved in §3. Section 2 contains definitions and known results. Moreover it is shown that the connections among eliminability of E_α^n for various α are exactly the same as for the cardinality quantifiers Q_α. We thus obtain a generalization of some of Tuschik's results in [5].

2. We shall sometimes abbreviate the statement that a quantifier Q is eliminable in a theory T by $EL_T(Q)$. If, for any generalized quantifier Q and Q', $L(Q)$ is included in $L(Q')$ then $EL_T(Q')$ implies $EL_T(Q)$. We shall collect some facts that are based on this simple observation.

PROPOSITION 2.1.

(i) *If $m \leq m'$ and $n \leq n'$ then $EL_T(Q_\alpha^{m',n'})$ implies $EL_T(Q_\alpha^{m,n})$,*

(ii) *If $n \leq n'$ then $EL_T(E_\alpha^{n'})$ implies $EL_T(E_\alpha^n)$,*

(iii) *If* $EL_T(E_\alpha^n)$ *then* $EL_T(Q_\alpha^{1,n})$,
(iv) $EL_T(Q_\alpha^{1,n})$ *if and only if* $EL_T(Q_\alpha^{1,1})$.

Proof. Observe, for instance:
for (i):

$$\models Q_\alpha^{m,n} x_{11} \cdots x_{mn} \varphi(\bar{x}_1,\ldots,\bar{x}_m) \leftrightarrow \exists u Q_\alpha^{m,n+1} \bar{x}_1 y_1 \cdots \bar{x}_m y_m \left(\varphi(\bar{x}_1,\ldots,\bar{x}_m) \wedge \bigwedge_{1 \le i \le m} u = y_i\right);$$

for (iii):

$$\models Q_\alpha^{1,n} \bar{x} \varphi(\bar{x}) \leftrightarrow E_\alpha^n \bar{x}\bar{y}[(\varphi(\bar{x}) \wedge \varphi(\bar{y}) \wedge \text{“}\bar{x} = \bar{y}\text{”}) \vee (\neg\varphi(\bar{x}) \wedge \neg\varphi(\bar{y}))];$$

for (iv):

$$\models Q_\alpha^{1,1} x \varphi(x) \leftrightarrow Q_\alpha^{1,n} \bar{x} \bigwedge_{1 \le i \le n} \varphi(x_i);$$

$$\models Q_\alpha^{1,n} \bar{x} \varphi(\bar{x}) \leftrightarrow Q_\alpha^{1,1} y(\text{“}y \in \text{field}(\varphi)\text{”}).$$

If $\varphi(x_1,\ldots,x_n, y_1,\ldots,y_n)$ is a first-order formula then eq(φ) stands for the first-order sentence expressing that φ is an equivalence relation on n-tuples. The number of equivalence classes is denoted by ind(φ).

For m, n, $k < \omega$ and a formula $\delta(\bar{x}_1,\ldots,\bar{x}_m)$, where $l(\bar{x}_1) = \cdots = l(\bar{x}_m) = n$, $H_k^{m,n} \bar{x}_1 \cdots \bar{x}_m \delta$ stands for the first-order sentence asserting the existence of a homogeneous set of n-tuples for δ that contains at least k such n-tuples.

The following lemma serves as a basic tool for eliminability investigations. In its general form it is due to Tuschik, for a proof see [**2**]. We state the lemma for the quantifiers $Q_\alpha^{m,n}$ and E_α^n: it shows that if they are eliminable in some theory T, then this can be done in a very simple way.

LEMMA 2.2 (*Definability lemma*).
(a) $EL_T(Q_\alpha^{m,n})$ iff for each first-order formula $\delta(\bar{x}_1,\ldots,\bar{x}_m, \bar{z})$ there is a number $k < \omega$ such that

$$T \cup \{Q_\alpha y(y = y)\} \models \forall \bar{z}(H_k^{m,n} \bar{x}_1 \cdots \bar{x}_m \delta \rightarrow Q_\alpha^{m,n} \bar{x}_1 \cdots \bar{x}_m \delta).$$

(b) $EL_T(E_\alpha^n)$ iff for each first-order formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = l(\bar{y}) = n$, there is $k < \omega$ such that

$$T \cup \{Q_\alpha y(y = y)\} \models \forall \bar{z}(\text{eq}(\varphi) \wedge (\varphi) \ge k \rightarrow E_\alpha^n \bar{x}\bar{y}\varphi).$$

DEFINITION 2.3. Let L be a first-order language.
(i) A structure $\mathfrak{A}$ for L is called (m, n)-*singular* if there are
(a) an L-formula $\varphi(\bar{x}_1,\ldots,\bar{x}_m, \bar{y})$ with $l(\bar{x}_i) = n$ for $1 \le i \le m$,

(b) $\bar{a} \in A$ and $C \subset A^n$, where $\aleph_0 \leq |C| < |A|$,
such that C is maximally homogeneous for φ in $(\mathfrak{A}, \bar{a})$.

(ii) *T is (m, n)-regular* if it has no (m, n)-singular models.

(iii) *T is regular* if it is $(m, 1)$-regular for all m;

T is strongly regular if it is (m, n)-regular for all $m, n \geq 1$.

(iv) $(\mathfrak{A}, \mathfrak{B})$ is a *generalized Vaughtian pair of index* (m, n) if $\mathfrak{A} \prec \mathfrak{B}$, $A \neq B$, and for some formula $\varphi(\bar{x}_1, \ldots, \bar{x}_m, \bar{y})$ there are $\bar{a} \in A$ and $C \subset A^n$ such that C is an infinite maximally homogeneous set for φ in $(\mathfrak{B}, \bar{a})$.

A straightforward application of the arguments in [**3**] to sequences rather than elements yields

THEOREM 2.4 (*Rothmaler, Tuschik*).

(i) *Let $\varphi(\bar{x}_1, \ldots, \bar{x}_m, \bar{y})$ be an L-formula T has a generalized Vaughtian pair of index (m, n) for φ iff T has an (m, n)-singular model of power $\aleph_1$ for φ iff T has any (m, n)-singular model for φ.*

(ii) *If T is (m, n)-regular then $Q_\alpha^{m,n}$ is eliminable in T for all $\alpha \geq 0$.*

Clearly, (ii) entails Theorems 1.2 and 1.3. In general, a formula $\varphi(\bar{x}_1, \ldots, \bar{x}_m)$ where $m > 1$ may have maximal homogeneous sets of different cardinality in a model $\mathfrak{A}$. However, if $m = 1$ then there is just one, namely $\varphi^{\mathfrak{A}}$. Similarly, if $\delta(\bar{x}, \bar{y})$ is an equivalence relation on A^n, then a maximally homogeneous set for $\neg\delta$ is just a set of representatives, and, clearly, all such sets have the same cardinality.

DEFINITION 2.5. *T is n-regular for equivalence relations* if whenever $\mathfrak{A} \models T$, $\bar{a} \in A$ and $(\mathfrak{A}, \bar{a}) = \mathrm{eq}(\delta(\bar{x}_1, \bar{x}_2, \bar{z}))$, where $l(\bar{x}_1) = l(\bar{x}_2) = n$, then either $\mathrm{ind}(\delta^{\mathfrak{A}}) < \omega$ or $\mathrm{ind}(\delta^{\mathfrak{A}}) = |A|$.

COROLLARY 2.6. *T is n-regular for equivalence relations if and only if E_α^n is eliminable in T for all $\alpha \geq 0$.*

Proof. The direction from left to right is shown exactly like Theorem 2.4(ii). For the other one use the remark preceding Definition 2.5.

The remainder of this section is devoted to generalizing Tuschik's result on the relative strength of eliminability of Q_α to the quantifiers E_α^1. Recall that, by Lemma 2.1(iv), $EL_T(Q_\alpha^{1,n})$ if and only if $EL_T(Q_\alpha)$; so the situation does not change if Q_α is replaced by $Q_\alpha^{1,n}$ for some $n > 1$. The same is true for E_α^1 and E_α^n although this is not so obvious at first sight (see Proposition 3.1).

PROPOSITION 2.7.
(i) $EL_T(E_\alpha^1)$ *for some* $\alpha > 0$ *implies* $EL_T(E_0^1)$,
(ii) $EL_T(E_1^1)$ *implies* $EL_T(E_\alpha^1)$ *for all* α.

Proof. (i) By the definability Lemma 2.2, for each formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ there is a number k such that, for all $\mathfrak{A} \models T$ of power $\geq \aleph_\alpha$ and all $\bar{a} \in A$, the following holds:

$$\text{if} \quad \operatorname{ind}(\varphi^{(\mathfrak{A},\bar{a})}) \geq k \quad \text{then} \quad \operatorname{ind}(\varphi^{(\mathfrak{A},\bar{a})}) \geq \aleph_\alpha.$$

Suppose there are $\mathfrak{B} \models T$, $\bar{b} \in B$ such that $(\mathfrak{B}, \bar{b}) \models \mathrm{eq}(\varphi)$ and $k \leq \operatorname{ind}(\varphi^{(\mathfrak{B},\bar{b})}) < \aleph_0$. By hypothesis $|B| < \aleph_\alpha$. But $(\mathfrak{B}, \bar{b})$ has an elementary extension $(\mathfrak{C}, \bar{b})$ of power $\aleph_\alpha$. Now $k \leq \operatorname{ind}(\varphi^{(\mathfrak{B},\bar{b})}) = \operatorname{ind}(\varphi^{(\mathfrak{C},\bar{b})}) < \aleph_0 < \aleph_\alpha$, a contradiction.

(ii) By (i), $EL_T(E_0^1)$. Assuming that E_α^1 is not eliminable in T for some $\alpha > 1$, we can conclude that there are a model $\mathfrak{A} \models T$, such that $|A| \geq \aleph_\alpha$, and a singular equivalence relation on A defined by some formula $\varphi(x, y, \bar{z})$ in an expansion $(\mathfrak{A}, \bar{a})$. Put $T' := \mathrm{Th}(\mathfrak{A}, \bar{a}, P^{\mathfrak{A}})$, where the new predicate symbol P is interpreted in $\mathfrak{A}$ by a set of representatives for φ. By Vaught's two-cardinal theorem, T' has a singular model of power $\aleph_1$, whence E_1^1 is not eliminable in T.

Thus $EL_T(E_1^1)$ is the strongest notion, $EL_T(E_0^1)$ the weakest, and $EL_T(E_\alpha^1)$ for $\alpha > 1$ is somewhere in between. Under additional assumptions we can say more:

COROLLARY 2.8 (*GCH*). *If* $\aleph_\alpha$ *is regular and* $EL_T(E_{\alpha+1}^1)$ *then* $EL_T(E_1^1)$.

Proof. Use Chang's two-cardinal theorem in the proof of 2.7(ii).

THEOREM 2.9. *Let* T *be stable. If* $EL_T(E_\alpha^1)$ *for some* $\alpha > 1$ *then* $EL_T(E_1^1)$.

Proof. Again, the hypothesis together with the negation of the claim imply that there is $\mathfrak{A} \models T$, and some formula $\varphi(x, y, \bar{z})$ defines a singular equivalence relation in some expansion $(\mathfrak{A}, \bar{a})$. We may assume that $|A| = \aleph_1$.

In order to finish the proof by contradiction, we have to verify that there is a model $\mathfrak{B} \models T$ of power $\aleph_\alpha$ such that, for some $\bar{b} \in B$, φ defines a singular equivalence relaton in $(\mathfrak{B}, \bar{b})$. We would like to apply Shelah's two-cardinal theorem [4, p. 287] to $T' = \mathrm{Th}(\mathfrak{A}, \bar{a}, P^{\mathfrak{A}})$ as above, but due to new dependencies caused by $P^{\mathfrak{A}}$, T' may not be stable any longer. However, Shelah's method of "imaginary elements" provides a means to overcome this difficulty.

First, we may assume w.l.o.g. that the singular equivalence relation in $\mathfrak{A}$ is defined (without parameters) by a two-place relation symbol R, where $R \in L$ and L is the language of $\mathfrak{A}$. Now let $L^+ := L \cup \{P, E\}$ and $\mathfrak{A}^+$ be the following model for L^+:

(a) as domain $A \cup X$, where $X \cap A = \varnothing$ and X contains exactly one element for each equivalence class of $R^{\mathfrak{A}}$,

(b) $P^{\mathfrak{A}^+} = A$ and the relations of L are restricted to $P^{\mathfrak{A}^+}$,

(c) $E^{\mathfrak{A}^+} = \{(a, x) \mid a \in A, x \in X, a \text{ is in the } R^{\mathfrak{A}}\text{-class coded by } x\}$. Put $T^+ := \mathrm{Th}(\mathfrak{A}^+)$. Some obvious facts about T^+ are the following:

(1) each $\mathfrak{A} \models T$ has, up to isomorphism, a unique extension $\mathfrak{A}^+ \models T^+$;

(2) if $\mathfrak{B} \models T^+$ then $\mathfrak{B}^{-1} \models T$, where $\mathfrak{B}^{-1} = P^{\mathfrak{B}} \upharpoonright L$.

Hence $(\mathfrak{B}^{-1})^+ = \mathfrak{B}$.

Moreover, we have

LEMMA 2.10. *T^+ is stable.*

From this, Theorem 2.9 is proved as follows: apply Shelah's two-cardinal theorem to T^+. $(\mathfrak{A}^+, (\neg P)^{\mathfrak{A}^+})$ is a model of type $(\aleph_1, \aleph_0)$, so there exists $\mathfrak{B} \models T^+$, where $\mathfrak{B}$ is of type $(\aleph_\alpha, \aleph_0)$. Now $\mathfrak{B}^{-1}$ is the required model of T.

Proof of Lemma 2.10 (*sketch*). The first step is to show that T^+ is a conservative extension of T: for each L^+-formula $\varphi(x_1, \ldots, x_n)$ there is an L-formula $\pi(\bar{x})$ such that

$$(*)\qquad \begin{gathered} T^+ \models \forall \bar{x}\Big(\bigwedge_{i \le n} Px_i \to (\varphi \leftrightarrow \pi)\Big), \\ T^+ \models \forall \bar{x}\Big(\pi(\bar{x}) \to \bigwedge_{i \le n} Px_i\Big), \\ \text{for all } \mathfrak{B} \models T^+\text{: } \mathfrak{B} \models \pi[\bar{c}] \quad \text{iff} \quad \mathfrak{B}^{-1} \models \pi[\bar{c}]. \end{gathered}$$

π is defined by induction on L^+-formulas. The idea is that quantification over elements in $(\neg P)$ can be replaced by quantification over the corresponding R-equivalence classes.

Now assume T^+ is not stable in λ. Then, for some $\mathfrak{M} \models T^+$, there is a subset $N \subset M$, $|N| \le \lambda$ and $\mathfrak{M}$ realizes λ^+ L^+-types over N. W.l.o.g. $N \subset P^M$. By $(*)$ one can conclude that $\mathfrak{M}^{-1}$ realizes λ^+ L-types over N, whence T is not stable in λ.

3. In this section we prove the main theorem of this paper. We need another fact about the quantifiers E_α^n.

PROPOSITION 3.1. *For all $n > 1$:*
(i) *$L(E_0^n)$ is included in $L(E_0^1)$;*
(ii) *for all $\alpha \geq 0$, $EL_T(E_\alpha^1)$ implies $EL_T(E_\alpha^n)$.*

Proof. Let $\varphi(\bar{x}, \bar{x}_2)$ be a first-order formula with $l(\bar{x}_1) = l(\bar{x}_2) = n$. We define

$$\pi(y_1, y_2) := \forall \bar{w}\left[\exists \bar{v}_1 \varphi(y_1 \frown \bar{v}_1, \bar{w}) \leftrightarrow \exists \bar{v}_2 \varphi(y_2 \frown \bar{v}_2, \bar{w})\right].$$

Let $\mathfrak{A}$ be a model of $T \cup \{\mathrm{eq}(\varphi)\}$ and let $K(\varphi^{\mathfrak{A}})$ be the set of all $\varphi^{\mathfrak{A}}$-equivalence classes. For $b \in A$ put

$$\varphi_b^{\mathfrak{A}} := \left\{(\bar{a}_1, \bar{a}_2) \mid \bar{a}_1, \bar{a}_2 \in A^{n-1};\ \mathfrak{A} \models \varphi[b \frown \bar{a}_1, b \frown \bar{a}_2]\right\}.$$

Then:

(1) $\varphi_b^{\mathfrak{A}}$ is an equivalence relation on A^{n-1} and $K(\varphi_b^{\mathfrak{A}})$ consists essentially of all those $k \in K(\varphi^{\mathfrak{A}})$ that contain some n-tuple $a_1, \ldots, a_n$ with $a_1 = b$. Therefore, $\mathrm{ind}(\varphi_b^{\mathfrak{A}}) \leq \mathrm{ind}(\varphi^{\mathfrak{A}})$.

(2) $K(\varphi^{\mathfrak{A}}) = \bigcup_{b \in A} K(\varphi_b^{\mathfrak{A}})$.

(3) $\pi^{\mathfrak{A}}$ is an equivalence relation on A. For $b \in A$ the $\pi^{\mathfrak{A}}$-equivalence class of b is completely determined by the set $K(\varphi_b^{\mathfrak{A}})$. Hence, $\mathrm{ind}(\pi^{\mathfrak{A}}) \leq 2^{\mathrm{ind}(\varphi^{\mathfrak{A}})}$.

Now we claim

$$(*) \quad \begin{aligned} &T \models E_0^n \bar{x}_1 \bar{x}_2 \varphi \leftrightarrow \\ &\qquad \left[\mathrm{eq}(\varphi) \wedge \left(\exists z E_1^{n-1} \bar{v}_1 \bar{v}_2 \varphi(z \frown \bar{v}_1, z \frown \bar{v}_2) \vee E_0^1 y_1 y_2 \pi(y_1, y_2)\right)\right]. \end{aligned}$$

Let $\mathfrak{A} \models E_0^n \bar{x}_1 \bar{x}_2 \varphi$. Then $\mathrm{ind}(\varphi^{\mathfrak{A}}) = |K(\varphi^{\mathfrak{A}})| \geq \aleph_0$. Now either some $K(\varphi_b^{\mathfrak{A}})$ is infinite or otherwise (2) implies $K(\pi^{\mathfrak{A}})$ to be infinite.

For the other direction, suppose $\varphi^{\mathfrak{A}}$ is an equivalence relation of finite index. By (1), $\mathrm{ind}(\varphi_b^{\mathfrak{A}}) < \aleph_0$ for each $b \in A$. By (3), $\mathrm{ind}(\pi^{\mathfrak{A}}) \leq 2^{\mathrm{ind}(\varphi^{\mathfrak{A}})} < \aleph_0$. So the right side of $(*)$ is false in $\mathfrak{A}$.

From $(*)$ it follows by induction on n that (i) of the proposition holds. Also, (ii) is clear for $\alpha = 0$. Now assume that, for some $\alpha \geq 1$, $EL_T(E_\alpha^1)$ holds and $EL_T(E_\alpha^n)$ does not, where $n > 1$ is minimal with this property. By 2.7(i) and 3.1(i) above, $EL_T(E_0^n)$. Hence, there is a model $\mathfrak{B} \models T$, $|B| = \aleph_\alpha$ and $\varphi(\bar{x}_1, \bar{x}_2)$ defines a singular equivalence relation on B^n. Therefore $\mathfrak{B} \models E_0^n \bar{x}_1 \bar{x}_2 \varphi$ and we may apply $(*)$ again.

Case 1. For some $b \in B$, $\mathrm{ind}(\varphi_b^{\mathfrak{B}}) \geq \aleph_0$. Since $\mathrm{ind}(\varphi_b^{\mathfrak{B}}) \leq \mathrm{ind}(\varphi^{\mathfrak{B}}) < \aleph_\alpha$, there is a definable singular equivalence relation on B^{n-1}, whence E_α^{n-1} is not eliminable in T.

Case 2. $\operatorname{ind}(\varphi_b^{\mathfrak{B}})$ is finite for all $b \in B$. But then $\operatorname{ind}(\pi^{\mathfrak{B}}) \geq \aleph_0$. Also, $\operatorname{ind}(\pi^{\mathfrak{B}}) \leq (\operatorname{ind}(\varphi^{\mathfrak{B}}))^{<\omega} = \operatorname{ind}(\varphi^{\mathfrak{B}}) < \aleph_\alpha$ and, therefore, $EL_T(E_\alpha^1)$ fails.

In either case this contradicts our original assumption and the proof is complete.

REMARK. To prove (*), regularity of $\aleph_0$ is needed for $\to$ and inaccessibility for $\leftarrow$. So the stronger statement in (i) also holds for all α such that $\aleph_\alpha$ is a strongly inaccessible cardinal; we do not know if it is true, e.g., for $\alpha = 1$.

THEOREM 3.2. *Let T be stable. Suppose some quantifier $Q_\alpha^{m,n}$ or E_α^n (where $\alpha \geq 1$ and $m \geq 2$) is eliminable in T. Then T is strongly regular and, hence, all $Q_\beta^{m,n}$ for $m, n \geq 0$ and $\beta \geq 0$ are eliminable in T.*

Proof. By 2.1 and 2.7(i) we have

(1) $$EL_T(E_\alpha^1), \qquad EL_T(E_0^1).$$

As T is stable it follows from Baldwin's and Kueker's Theorem 1.1 that

(2) T does not have the finite cover property.

Now assume, for contradiction, that T is not strongly regular. That is, for some m and k, T has an (m, k)-singular model. So let Y be a λ-powered maximally homogeneous set of k-tuples for $\varphi(\bar{x}_1, \ldots, \bar{x}_m, \bar{y})$ in $(\mathfrak{A}, \bar{a})$. Suppose $|A| = \aleph_\beta$, $\aleph_0 \leq \lambda < \aleph_\beta$. For notational simplicity we suppress $\bar{y}$ and $\bar{a}$.

Let $\delta(\bar{u}, \bar{y}_1, \ldots, \bar{y}_{m-1})$ be the following formula, where $l(\bar{u}) = l(\bar{y}_i) = k$ $(1 \leq i \leq m)$:

$$\delta := \bigwedge_{1 \leq i < m} \text{“}\bar{u} \neq \bar{y}_i\text{”} \wedge \bigwedge_{\substack{f:\{1,\ldots,m\} \\ \to \{\bar{u}, \bar{y}_1, \ldots, \bar{y}_{m-1}\} \\ \bar{u} \in \operatorname{range}(f)}} \varphi(\bar{x}_1 \backslash f(1), \ldots, \bar{x}_m \backslash f(m)).$$

Put $\Sigma := \{\delta(\bar{u}, \bar{b}_1, \ldots, \bar{b}_{m-1}) \mid \bar{b}_1, \ldots, \bar{b}_{m-1} \in Y\}$.

As Y is infinite, Σ is consistent in $\mathfrak{A}$, i.e., for every finite $\Sigma_0 \subset \Sigma$: $\mathfrak{A} \models \exists \bar{u} \bigwedge_{\pi \in \Sigma_0} \pi$. Also, Σ is omitted in $\mathfrak{A}$ since Y maximally homogeneous.

Now, much like Baldwin and Kueker made use of the f.c.p.-theorem, we shall apply another result of Shelah:

THEOREM 3.3 ([4, p. 80]). *Suppose T does not have the f.c.p. and some $\mathfrak{M} \models T$ omits a Δ-m-type of cardinality λ, where Δ is finite. Then there are*

$p < \omega$, a formula $\gamma(\bar{x}_1, \bar{x}_2, \bar{z})$ with $l(\bar{x}_1) = l(\bar{x}_2) = p$ and $\bar{c} \in M$ such that

(i) $(\mathfrak{M}, \bar{c}) \models \mathrm{eq}(\gamma)$;

(ii) $\aleph_0 \leq \mathrm{ind}(\gamma^{(\mathfrak{M}, \bar{c})}) \leq \lambda$.

By (2) and the previous remarks, the hypotheses of 3.3 are satisfied with $\mathfrak{M} = \mathfrak{A}$, $m = k$ and $\Delta = \{\delta\}$.

Now the conclusion of 3.3 implies that, for some number $p < \omega$, $EL_T(E_\beta^p)$ fails. Although p may be very large (as a proof of 3.3 would show), it follows from Proposition 3.1(ii) that

$$E_\beta^1 \text{ is not eliminable in } T. \tag{3}$$

But this leads to a contradiction: by (3) and 2.7(ii), $EL_T(E_1^1)$ fails. On the other hand, as T is stable, (1) together with Theorem 2.9 imply $EL_T(E_1^1)$.

COROLLARY 3.4. *Let T be stable. Then the following are equivalent:*

(i) *T is strongly regular;*

(ii) *T is regular;*

(iii) *T is 1-regular for equivalence relations.*

Proof (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): If $\varphi(x, y)$ defines a singular equivalence relation on A in some model $\mathfrak{A}$, then $\neg\varphi$ shows that $\mathfrak{A}$ is a (2, 1)-singular model.

(iii) $\Rightarrow$ (i): By (the proofs of) 3.3 and 3.1 (i).

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REFERENCES

[1] J. Baldwin and D. Kueker, *Ramsey-quanatifiers and the finite cover property*, Pacific J. Math., **90** (1980), 11–19.

[2] A. Baudisch, D. Seese, H. P. Tuschik and M. Weese, *Decidability and quantifier elimination*, Chapter XIX of "Higher Model Theory" (to appear).

[3] P. Rothmaler and H. P. Tuschik, *A two-cardinal theorem for homogeneous sets and the elimination of Malitz quantifiers*. Trans. Amer. Math. Soc., **269** (1982), 273–283.

[4] S. Shelah, *Classification Theory*, North Holland, 1978.

[5] H. P. Tuschik, *Elimination of cardinality quantifiers*, ZMLG **28** (1982), 75–81.

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