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DIRECT SUMMANDS OF DIRECT PRODUCTS OF SLENDER MODULES

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Suppose $P = \prod_I G_i$ is a direct product of slender R-modules. If $|I|$ is non-measurable and A is a direct summand of P, then $A \cong \prod_J A_j$ where each A_j is isomorphic to a direct summand of a countable direct product of G_i's. If $R = Z$ and P is a torsion-free reduced abelian group, then, if each G_i has rank one, A is a direct product of rank one groups.

1. Introduction. If R is a ring and $M = \prod_1^\infty R_n$ with each $R_n \cong R$ as a R-module, then a R-module N is *slender* if: for any homomorphism f: $M \to N$, $f(R_n) = 0$ for almost all n. In Theorem 3.7 we will show that, if a R-module P equals $\prod_I G_i$ with $|I|$ non-measurable and each G_i slender, then any direct summand of P is isomorphic to $\prod_J A_j$ where each A_j is isomorphic to a direct summand of a countable direct product of G_i's. This theorem in a way does for direct products what Kaplansky's theorem does for direct sums of modules (i.e., the theorem which states that projective modules are direct sums of countably generated modules [**6**]). In Theorem 4.3 we prove that if V is a reduced vector group (a direct product of rank one torsion-free abelian groups) of non-measurable cardinality, then so is any direct summand of V. This answers Problem 74 in [**4**] for the non-measurable case.

2. Preliminaries. In this paper all groups are abelian, rings are associative with identity, modules are left unital, and homomorphisms are written on the left. Discussions of slender modules may be found in [**3**, **4**, **5**, and **7**]. For example, any torsion-free abelian group is a slender Z-module if it does not contain Q, Z^I with I infinite, or the p-adic integers for a prime p. This fact along with a good treatment of vector groups is contained in Chapter XIII of [**4**]. Unexplained terminology may be located in [**4**].

3. Slender modules. Throughout this section we shall consider the following situation. Let R be a ring and let R-module P equal $\prod_I G_i = A \oplus B$ where $|I|$ is non-measurable and each G_i is slender. By f_i, α, β we shall mean the projections of P to G_i, A, B respectively and we let $\alpha_i = f_i\alpha$.

Our first two lemmas are basic.

LEMMA 3.1.

(1) *If i is fixed, $\alpha_i(G_j) = 0$ for almost all j in I.*

(2) *If i is fixed and $\alpha_i(G_j) = 0$ for all j in $J \subseteq I$, then $\alpha_i(\prod_J G_j) = 0$.*

(3) *If $\alpha_i(G_j) = 0$ for all j in J and all i in $I \setminus J$, then $\prod_J G_j = \alpha(\prod_J G_j) \oplus \beta(\prod_J G_j)$.*

Proof. (1) follows from the definition of slender and (2) follows from Los' argument as given in the proof of Theorem 94.4 in [**4**] (see also the proof of Theorem 3 in [7] or Theorem 2.1 in [**5**]). The condition in (3) implies $\alpha(\prod_J G_j)$ is contained in $\prod_J G_j$ by (2). So $\beta(\prod_J G_j) \subseteq \prod_J G_j$ and (3) is true.

LEMMA 3.2. (1) *Let D be any direct summand of P and suppose $d_j \in D$ for each j in a set J. If, for each fixed i in I, $f_i(d_j) = 0$ for almost all j in J, then the element $d = \sum_I(\sum_J f_i(d_j))$ is in D. We define $d = \sum_J d_j$.*

(2) *Let A_j be a submodule of A for each j in a set J such that, for each $i \in I$, $f_i(A_j) = 0$ for almost all j in J. We define $\sum_J A_j = \{\sum_J a_j | a_j \in A_j\}$. Then $\sum_J A_j$ is in A and $\sum_J A_j \cong \prod_J A_j$ if, whenever $\sum_J a_j = 0$ with $a_j \in A_j$, then each $a_j = 0$.*

Proof. (1) We may suppose $D = B$ and just show $\alpha_i(d) = 0$ for arbitrary i. By the previous lemma $\alpha_i(\prod_K G_k) = 0$ for some K cofinite in I. For some subset L cofinite in J, $\sum_L d_j$ is in $\prod_K G_k$. Since $d = \sum_{J \setminus L} d_j + \sum_L d_j$ and the left sum is in B, $\alpha_i(d) = 0$. (2) By (1) $\sum_J A_j$ is in A and there is a natural isomorphism $\prod_J A_j \to \sum_J A_j$.

Note. The ideas in the first two lemmas will be used repeatedly without reference in the sequel.

PROPOSITION 3.3. *Suppose J is a well-ordered set containing* 1 *and A has submodules A_j and A^j for each j in J such that*:

(1) *$A = A^1$ and $A^j = A_j \oplus A^{j+1}$ (where $A^{j+1} = 0$ if j is maximal in J),*

(2) *$A^k = \bigcap_{j<k} A^j$ if k is a limit element in J,*

(3) *for each i in I $f_i(A_j) = 0$ for almost all j in J,*

(4) *$\bigcap_J A^{j+1} = 0$.*

Then $A \cong \prod_J A_j$.

Proof. By (3) $\sum_J A_j$ is a submodule of A. We need to show $A = \sum_J A_j$ and that, if $\sum_J a_j = 0$ with $a_j \in A_j$, then each $a_j = 0$. By our suppositions it will suffice to show:

$$(*) \qquad A = \left(\sum_{j \leq m} A_j \right) \oplus A^{m+1} \text{ for each } m \text{ in } J.$$

Now $(*)$ is true for $m = 1$ by (1) and we assume it is true for all $m < k$. If $k - 1$ exists, $(*)$ is true for k by (1). Suppose k is a limit element in J. Let $a \in A$. By our assumption and (1) and (2) we may inductively choose $a_j \in A_j$ for each $j < k$ so that $a - (a_1 + \cdots + a_j) \in A^{j+1}$. Then $a - \sum_{j<k} a_j$ is in A^{i+1} for each $i < k$ and it is in A^k by (2). By (1) then $a - \sum_{j \leq k} a_j \in A^{k+1}$ for some $a_k \in A_k$ and $A = \sum_{j \leq k} A_j + A^{k+1}$. Suppose $\sum_{j \leq k} a_j + x = 0$ with $a_j \in A_j, x \in A^{k+1}$, and $a_i \neq 0$ for a minimal $i \leq k$. If $i = k$, $a_i = -x \in A^{i+1}$. If $i < k$, $a_i \in A^{i+1}$ by $(*)$ for $m = i$. Either case implies $a_i = 0$, a contradiction. So $(*)$ is true for $m = k$ and by induction for all m.

Our next two lemmas deal with a particular ordering of I.

LEMMA 3.4. *The set I can be ordered as an ordinal so that*:

(1) *for each j in I, if $\alpha_i(G_j) = 0$ for all $i < j$, then $\alpha_i(\prod_{k \geq j} G_k) = 0$ for all $i < j$,*

(2) *if j is a limit ordinal in I, then $\alpha_i(\prod_{k \geq j} G_k) = 0$ for all $i < j$.*

Proof. Let $1 \in I$ be arbitrary. Suppose the ordinals $< m$ have been identified with J a proper subset of I. Choose m from $I \setminus J$ so that $\alpha_k(G_m) \neq 0$ for minimal k in J if possible; otherwise let m from $I \setminus J$ be arbitrary. Continue in this manner until I is totally ordered as an ordinal. This ordering implies (1) and we now show (2). Since $\alpha_i(G_j) = 0$ for all $i < j$ if $j = 1$, assume it is true for all non-successor ordinals j less than limit ordinal s. Suppose $\alpha_n(G_s) \neq 0$ for some minimal $n < s$. Then $n - j$ is finite for $j = 1$ or j a limit ordinal $< s$. Let $K = \{i > n \mid \alpha_k(G_i) \neq 0$ for some $k \leq n\}$. Since $\alpha_k(\prod_{i \geq j} G_i) = 0$ for all $k < j$ and since $\oplus_j^n G_i$ is slender, K is finite. But s is in K and $s - n$ is finite by our ordering of I, a contradiction. Therefore (2) is true for s and by induction for all limit ordinals.

DEFINITION 3.5. Suppose I is an ordinal and J is a subset of I containing 1. For each j in J let j' be the successor of j in J (if j is maximal in J let $j' = I$). For each j in J set $I_j = \{i \in I \mid j \leq i < j'\}$. Then $\{I_j\}$, $j \in J$, partitions I. Now let $P_j = \prod_{I_j} G_i$ whence $P = \prod_J P_j$. Also let $P^j = \prod_{i \geq j} G_i$. Then $P^j = P_j \oplus P^{j'}$ (if j is maximal in J set $P^{j'} = 0$).

LEMMA 3.6. *Suppose I is an ordinal, $1 \in J \subseteq I$, and, for each $j \in J$, $\alpha_i(P^j) = 0$ for all i in I less than j. Then $P_j = \alpha(P_j) \oplus \beta(P_j)$ for each j and $A \cong \prod_J \alpha(P_j)$.*

Proof. Let $j \in J$ be arbitrary. By Lemma 3.1, $P^j = \alpha(P^j) \oplus \beta(P^j)$ and $P^{j'} = \alpha(P^{j'}) \oplus \beta(P^{j'})$. Therefore $P_j = \alpha(P_j) \oplus \beta(P_j)$. We now let $A_j = \alpha(P_j)$ and $A^j = \alpha(P^j)$ and apply Proposition 3.3 to show $A \cong \prod_J A_j$. Since $1 \in J$ and $\alpha(P^j) = \alpha(P_j) \oplus \alpha(P^{j'})$, (1) is true. Suppose k is in J and a limit element therein. Then

$$A^k = \alpha(P^k) \subseteq \bigcap_{j<k} \alpha(P^j) = \bigcap_{j<k} A^j = \alpha\Big(\bigcap_{j<k} A^j\Big)$$
$$\subseteq \alpha\Big(\bigcap_{j<k} P^j\Big) = \alpha(P^k) = A^k.$$

So (2) is true. Let $i \in I$ be fixed. From the map $\alpha_i\colon P = \prod_J P_j \to G_i$ we see that $f_i(A_j) = \alpha_i(P_j) = 0$ for almost all j. Hence (3) is true. Since J is unbounded or $P^{j'} = 0$ for a maximal j in J, $\bigcap A^{j'} \subseteq \bigcap P^{j'} = 0$ and (4) is true. Therefore $A \cong \prod_J A_j$.

THEOREM 3.7. *Suppose R-module P equals $\prod_I G_i = A \oplus B$ with $|I|$ non-measurable and each G_i slender. Then $A \cong \prod_J A_j$ where each A_j is isomorphic to a direct summand of a countable direct product of G_i's.*

Proof. Let I be ordered as in 3.4 and let J consist of 1 and all limit ordinals in I. For each $j \in J$ define P^j and P_j as in Definition 3.5 and set $A_j = \alpha(P_j)$. By 3.4, for each j in J, $\alpha_i(P^j) = 0$ for all i in I less than j. The theorem now follows from 3.6 and the fact that each P_j is a countable product of G_i's.

4. Vector groups. A vector group is an abelian group of the form $V = \prod_I R_i$ where each R_i is torsion-free of rank one. Some twenty years ago (see **[1]**) it was shown that, if V is reduced, $|I|$ is non-measurable, and $R_i \cong R_j$ or $\mathrm{Hom}(\mathrm{Hom}(R_i, R_i), R_j) = 0$ for each i and j, then any direct summand of V is a vector group. We now remove the restrictions on the types of the R_i's. We thereby solve Problem 74 in **[4]** for the non-measurable case.

If V above is reduced, it is a direct product of slender Z-modules; so the results in §3 apply to it. Since 2^μ is non-measurable for any non-measurable cardinal μ, V above has non-measurable cardinality if and only if I has; so we equate these two properties henceforth.

LEMMA 4.1. *If $V = A \oplus B$ is a reduced vector group and $|V|$ is non-measurable, then there is a decomposition $V = \prod_I R_i$ where each R_i has rank one and type t_i and, if f_i, α are the projections to R_i, A, respectively, and $\alpha_i = f_i\alpha$, then $\alpha_i(R_j) = 0$ for each i and j unless $i = j$ or $t_i > t_j$.*

Proof. Write $V = \prod_I S_i$ with each S_i of rank one and type t_i. Let t be a type and set $V_t = \prod_{t_i = t} S_i$ and $V^t = \prod_{t_i > t} S_i$. By Lemma 96.1 in **[4]** V^t and $V_t \oplus V^t$ are fully invariant subgroups of V. So $V^t = A^t \oplus B^t$ with A^t in A and B^t in B. Also $V_t \oplus V^t = A_t \oplus B_t \oplus V^t$ with $A_t = A \cap (V_t \oplus B^t)$ and $B_t = B \cap (V_t \oplus A^t)$. If ϕ is the projection $\prod_I S_i \to V_t$, then $V_t = \phi(A_t) \oplus \phi(B_t)$ and each summand is a vector group by Theorem 1 in **[1]** (also exercise 10, p. 171, Vol. II of **[4]**). Thus, if $I_t = \{i \in I \mid t_i = t\}$, then V_t has a decomposition $\prod_{I_t} R_i$, each R_i of rank one, where, for each i in I_t and x_i in R_i, there is a y_i in V^t such that $x_i = (x_i - y_i) + y_i$ with one term in A and the other in B. Now, for some set T of distinct types t, $V = \prod_T V_t = \prod_T(\prod_{I_t} R_i) = \prod_I R_i$. By full invariance $\prod_{t_i > t} R_i = V^t$ for each t. The conclusion of the lemma follows.

Our next lemma deals with a countable set of types.

LEMMA 4.2. *Let I be the natural numbers and let $T_1 = \{t_i\}$, $i \in I$, be a set of types (not necessarily distinct). Let $I_1 = \{i \in I \mid t_i$ is maximal in $T_1\}$. For each $n > 1$ let $T_n = \{t_i \mid i \notin I_1 \cup \cdots \cup I_{n-1}\}$ and $I_n = \{i \in I \mid t_i$ is maximal in $T_n\}$. Either* (1) *I contains an infinite chain $i_1 < i_2 < \cdots$ such that, for each n, $t_i \not> t_{i_n}$ whenever $i_1 \leq i \leq i_n$ or* (2) *$I = \bigcup_1^\infty I_n$ and each I_n is finite.*

Proof. Suppose (2) is not true. Then, for some least k, I_n is finite for $n < k$ and either I_k is infinite or T_k contains a chain of types not bounded above by an element in T_k. Let i_1 be an element in I such that $i < i_1$ for all i in I_n, $n < k$. Now i_1 satisfies the requirement in (1) and we assume $i_1, \ldots, i_m$ satisfy it. By our choice of k and i_1 there is a $j > i_m$ such that $t_i \not> t_j$ whenever $i_1 \leq i \leq i_m$. Let i_{m+1} be the least such j. Then $t_i \not> t_{i_{m+1}}$ for $i_1 \leq i \leq i_{m+1}$. Induction completes the proof.

THEOREM 4.3. *If $V = A \oplus B$ is a reduced vector group and $|V|$ is non-measurable, then A and B are vector groups.*

Proof. A proof for A will suffice. Write $V = \prod_I R_i$ as in Lemma 4.1 and let t_i, α, α_i be as defined there. Let I be ordered as in 3.4 for $P = V$ and $G_i = R_i$. Thus for each j in I, if $\alpha_i(R_j) = 0$ for all $i < j$, then

$\alpha_i(\prod_{k \geq j} R_k) = 0$ for all $i < j$ and, by the proof of Theorem 3.7, we may assume I is the natural numbers. We now let I_n and T_n be as defined in 4.2 and treat the cases given there.

Case 1. There is an infinite sequence $i_1 < i_2 < \cdots$ in I such that, for each n, $t_i \not> t_{i_n}$ for $i_1 \leq i \leq i_n$. Since $\oplus_{i<i_1} R_i$ is slender, for some m, $\alpha_i(\prod_{k \geq i_m} R_k) = 0$ for all $i < i_1$. By our choice of i_n's and by 4.1 we must have, for each $n \geq m$, $\alpha_i(R_{i_n}) = 0$ for all $i < i_n$. Therefore, from the way I was ordered, for each $n \geq m$, $\alpha_i(\prod_{k \geq i_n} R_k) = 0$ for all $i < i_n$. Let $J = \{1, i_m i_{m+1}, \dots\}$ and define P^j and P_j (with $G_i = R_i$) as in 3.5. By 3.6 then $A \cong \prod_J \alpha(P_j)$ and each $\alpha(P_j)$ is a direct summand of P_j. Since each P_j is a finite rank vector group, so is each $\alpha(P_j)$. Therefore V is a vector group.

Case 2. $I = \bigcup_1^\infty I_n$ and each I_n is finite. We may assume I is infinite. For each n set $K_n = I_1 \cup \cdots \cup I_n$ and let $V_n = \oplus_{K_n} R_i$ and $V^n = \prod_{I \setminus K_n} R_i$. V_n is fully invariant in V and equals $A_n \oplus B_n$ with A_n in A and B_n in B. Also $A = A_n \oplus A^n$ where $A^n = A \cap (B_n \oplus V^n)$. We now find subgroups C_j and C^j in A for $j \in J = (1, 2, \dots)$ such that:

(a) $A = C^1$ and $C^j = C_j \oplus C^{j+1}$,

(b) $C_1 \oplus \cdots \oplus C_j = A_{m_j}$ for some m_j,

(c) $C^{j+2} \subseteq V^j$.

Let $C^1 = A$, $C_1 = A_1$, and $C^2 = A^1$. The conditions are satisfied for $j = 1$ by these C's and we assume they are satisfied for $j \leq k$ by the C's up to C_k and C^{k+1}. Now $A = A_{m_k} \oplus C^{k+1}$ and $V = V_k \oplus V^k$. Since A_{m_k} and V_k are slender, from a consideration of projections: $V \to A \to A_{m_k}$ and $V \to A \to V_k$ we see that, for some large n, $\alpha(V^n)$ is in C^{k+1} and V^k. For this n then $A = A_n \oplus A^n$ where $A_n \supseteq A_{m_k}$ and

$$A^n = A \cap (B_n \oplus V^n) \subseteq \alpha(V^n) \subseteq C^{k+1} \cap V^k.$$

Let $C_{k+1} = A_n \cap C^{k+1}$, $C^{k+2} = A^n$, and $m_{k+1} = n$. Now $C^{k+1} = C_{k+1} \oplus C^{k+2}$, $A_{m_k} \oplus C_{k+1} = A_{m_{k+1}}$, and $C^{k+2} \subseteq V^k$, as desired. Induction completes the sequences. Next we apply Proposition 3.3 to the subgroups C_j and C^j with $j \in J$. Conditions (1) and (2) are clearly satisfied. Since $C_j \subseteq C^j$, (3) follows from (c) as does (4). So $A \cong \prod_J C_j$. Since each C_j is a finite rank vector group, A is a vector group.

REMARK. This theorem cannot be improved. That is: a countably infinite direct product of rank two torsion-free groups can equal the direct sum of two indecomposable subgroups. An example of such a group can be constructed by modifying an infinite direct "sum" example of Corner (as found in **[2]** or Theorem 91.1 in **[4]**). This is explained more fully in **[8]**.

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