PACIFIC JOURNAL OF MATHEMATICS
Vol. 117, No. 2, 1985

WEIGHTED REVERSE WEAK TYPE INEQUALITIES FOR THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

BENJAMIN MUCKENHOUPT

Kenneth Andersen and Wo-Sang Young recently obtained a sufficient condition and a different necessary condition on a pair of weight functions for which a reverse weak type norm inequality holds for the Hardy-Littlewood maximal function. It is shown here that their necessary condition is also sufficient. The consequences of the sufficiency theorem that they proved are strengthened by use of this result.

1. Introduction. Define the oriented cubes in R^n to be those with sides parallel to the coordinate axes, and for a set E in R^n let $|E|$ denote the Lebesgue measure of E. For f locally integrable on R^n, define the Hardy-Littlewood maximal function Mf by

$$(Mf)(x) = \sup \frac{1}{|Q|} \int_Q |f(t)|\,dt,$$

where the supremum is taken over all oriented cubes containing x. The local sufficiency result proved in [1] for a reverse weak type inequality for the Hardy-Littlewood maximal function is as follows.

THEOREM (1.1). *If $U(x)$ and $V(x)$ are nonnegative functions defined on an oriented cube Q_0 such that*

$$\frac{1}{|Q|} \int_Q U(x)\,dx \geq A \operatorname*{ess\,sup}_{x \in Q} V(x) \tag{1.2}$$

for all oriented cubes $Q \subset Q_0$, then

$$\int_{Q_0 \cap \{x:\,(Mf)(x) > \lambda\}} U(x)\,dx \geq \frac{A 2^{-n}}{\lambda} \int_{\{x:\,|f(x)| > \lambda\}} |f(x)| V(x)\,dx$$

for f supported on Q_0 and $\lambda \geq (1/|Q_0|) \int_{Q_0} |f(x)|\,dx$.

The local necessity result proved in [1] is the following; note that for a cube Q in R^n the notation SQ denotes the cube concentric with Q with sides S times as long.

THEOREM (1.3). *If $U(x)$ and $V(x)$ are nonnegative functions defined on an oriented cube Q_0 such that*

$$(1.4)\quad \int_{Q_0\cap\{x:\,(Mf)(x)>\lambda\}} U(x)\,dx \geq \frac{A}{\lambda}\int_{\{x:\,|f(x)|>\lambda\}} |f(x)|V(x)\,dx$$

for all f which are characteristic functions of measurable sets $E \subset Q_0$ and $\lambda \geq (1/|Q_0|)\int_{Q_0}|f(x)|\,dx$, then for oriented cubes $Q \subset Q_0$,

$$(1.5)\quad \frac{1}{|Q|}\int_{2Q\cap Q_0} U(x)\,dx \geq A2^{-n}\operatorname*{ess\,sup}_{x\in Q} V(x).$$

If $U(x)$ satisfies the doubling condition $\int_{2Q\cap Q_0}U(x)\,dx \leq C\int_Q U(x)\,dx$ for oriented $Q \subset Q_0$, conditions (1.2) and (1.5) are equivalent. It is easy to see, however, that (1.2) and (1.5) are not equivalent in general by the example $U(x) = V(x) = \chi_{(1/2)Q_0}(x)$ which satisfies (1.5) for $A = 1$ but does not satisfy (1.2) for any finite A.

Condition (1.2) resembles the well-known weight condition for A_1, $(1/|Q|)\int_Q U(x)\,dx \leq A\ \operatorname{ess\,inf}_{x\in Q} U(x)$, introduced on page 214 of [**4**], and (1.2) looks more "natural" than (1.5). It is, therefore, surprising that (1.5) is in fact a necessary and sufficient condition for the reverse weak type inequality (1.4). The main result of this paper is the following.

THEOREM (1.6). *If $U(x)$ and $V(x)$ are nonnegative functions defined on an oriented cube Q_0 such that*

$$(1.7)\quad \frac{1}{|Q|}\int_{2Q\cap Q_0} U(x)\,dx \geq A\operatorname*{ess\,sup}_{x\in Q} V(x)$$

for all oriented cubes $Q \subset Q_0$, then

$$(1.8)\quad \int_{Q_0\cap\{x:\,(Mf)(x)>\lambda\}} U(x)\,dx \geq \frac{A(300n)^{-n}}{\lambda}\int_{\{x:\,|f(x)|>\lambda\}} |f(x)|V(x)\,dx$$

for f supported on Q_0 and $\lambda \geq \lambda_0 = (1/|Q_0|)\int_{Q_0}|f(x)|\,dx$.

The following global version of Theorem (1.3) is proved in [**1**] as a corollary of Theorem (1.3).

THEOREM (1.9). *If $U(x)$ and $V(x)$ are nonnegative functions defined on R^n and*

$$\int_{\{x:\,(Mf)(x)>\lambda\}} U(x)\,dx \geq \frac{A}{\lambda}\int_{\{x:\,f(x)>\lambda\}} |f(x)|V(x)\,dx$$

for all f which are characteristic functions of measurable sets $E \subset R^n$ and all $\lambda > 0$, then

$$\frac{1}{|Q|} \int_{2Q} U(x)\, dx \geq A 2^{-n} \operatorname*{ess\,sup}_{x \in Q} V(x)$$

for all oriented cubes $Q \subset R^n$.

The sufficiency of this condition will be proved here as the following corollary of Theorem (1.6). This and Theorem (1.13) generalize results by Stein in [**5**].

THEOREM (1.10). *If $U(x)$ and $V(x)$ are nonnegative functions defined on R^n and*

$$\frac{1}{|Q|} \int_{2Q} U(x)\, dx \geq A \operatorname*{ess\,sup}_{x \in Q} V(x) \tag{1.11}$$

for all oriented cubes $Q \subset R^n$, then

$$\int_{\{x:\, (Mf)(x) > \lambda\}} U(x)\, dx \geq \frac{A(300n)^{-n}}{\lambda} \int_{\{x:\, |f(x)| > \lambda\}} |f(x)| V(x)\, dx \tag{1.12}$$

for all measurable f and $\lambda > 0$.

The other results in [**1**] can be proved with weaker hypotheses by using Theorem (1.6) or Theorem (1.10) in their proofs. These strengthened versions are as follows.

THEOREM (1.13). *If $U(x)$ and $V(x)$ are nonnegative functions defined on an oriented cube Q_0 such that* (1.7) *holds for all oriented cubes $Q \subset Q_0$ with $A > 0$, f is supported on Q_0 and $\int_{Q_0} (Mf)(x) U(x)\, dx < \infty$, then*

$$\int_{Q_0} |f(x)| \left[\log^+ |f(x)|\right] V(x)\, dx < \infty.$$

For the next theorem define e_i as the vector in R^n with ith entry 1 and all other entries 0. Define $M_i f$, the Hardy-Littlewood maximal function in the ith variable by $(M_i f)(x) = \sup_{h \neq 0} (1/h) \int_0^h |f(x + te_i)|\, dt$. The following theorem generalizes a result in [**2**].

THEOREM (1.14). *If $U(x)$ is a nonnegative function on R^n, K is an integer, $1 \leq K \leq n$, for $h > 0$, $1 \leq i \leq K$ and almost every x in R^n*

$$\frac{1}{2h} \int_{-2h}^{2h} U(x + te_i)\, dt \geq A \operatorname*{ess\,sup}_{|t| \leq h} U(x + te_i) \tag{1.15}$$

with A independent of i, x and h, and $Tf(x) = (M_K \cdots M_1 f)(x)$, then for all $\lambda > 0$,

$$(1.16)\quad \int_{\{x:\, Tf(x)>\lambda\}} U(x)\,dx \geq \frac{300^{-K}A^K}{(K-1)!\lambda}\int_{\{x:\, |f(x)|>\lambda\}} |f(x)|\left[\log^+\left(\frac{|f(x)|}{\lambda}\right)\right]^{K-1} U(x)\,dx.$$

If it is also assumed that $\lim_{|x|\to\infty} Tf(x) = 0$ and $\int_{|x|<r} Tf(x)U(x)\,dx < \infty$ for every finite r, then

$$\int_{R^n} |f(x)|\left[\log^+|f(x)|\right]^K U(x)\,dx < \infty.$$

2. Proofs. The proof of Theorem (1.6) is based on the following version of the Whitney covering lemma. As usual, E° will denote the interior of the set E, and χ_E is the characteristic function of the set E.

LEMMA (2.1). *Given an open set G in R^n, its complement F and a closed cube Q such that $G \subset 3Q$, there exists a sequence $\{Q_j\}$ of closed cubes with sides parallel to those of Q such that*

(a) $Q_j \subset Q$,
(b) $UQ_j \supset G \cap Q^\circ$,
(c) $Q_i^\circ \cap Q_j^\circ = \varnothing$ *if* $i \neq j$,
(d) $\operatorname{diam} Q_j \leq \operatorname{dist}(Q_j, F) \leq 4 \operatorname{diam} Q_j$,
(e) $\Sigma_j \chi_{2Q_j}(x) \leq (33\sqrt{n})^n \chi_G(x)$.

The definition of the Q_j's and the proof of (a)–(d) are essentially the proof of Theorem 1, p. 167 of [**6**]. The principal change is that the initial mesh $\mathcal{M}_0$ should be based on Q, i.e. $\mathcal{M}_0$ consists of Q and translations of Q parallel to its edges by integral multiplies of $S = |Q|^{1/n}$. The cubes in the mesh $\mathcal{M}_K$ then have side length $S2^{-K}$ and the layers Ω_K are defined by $\Omega_K = \{x\colon\ 2^{1-k}S\sqrt{n} < \operatorname{dist}(x, F) \leq 2^{2-K}S\sqrt{n}\}$. Let A be the set of all cubes q for which there is an integer K such that $q \in \mathcal{M}_K$ and $q \cap \Omega_K \neq \varnothing$. As shown in [**6**], we have

$$(2.2)\qquad \operatorname{diam}(q) \leq \operatorname{dist}(q, F) \leq 4\operatorname{diam}(q)$$

for q in A and

$$(2.3)\qquad G = \bigcup_{q\in A} q.$$

Since $G \subset 3Q$, we have $\operatorname{dist}(q, F) \leq \frac{1}{2}\operatorname{diam}(G) \leq \frac{3}{2}\operatorname{diam}(Q)$ for any cube q; this and (2.2) show that $\operatorname{diam}(q) \leq \frac{3}{2}\operatorname{diam}(Q)$ for $q \in A$. Because of the choice of $\mathscr{M}_0$, we conclude that every q in A is a subset of Q or has interior disjoint from Q. Let the Q_j be the members of A that are subsets of Q and which are not proper subsets of other members of A. Then properties (a) and (c) are immediate, property (d) follows from (2.2) and property (b) follows from (2.3).

The proof of property (e) is an adaptation of the proof of property (4) of the decomposition lemma on pages 15–16 of **[3]**. For this proof, fix an x in $\bigcup_{j\geq 1} 2Q_j$. If $2Q_j$ contains x, then $\operatorname{dist}(x, F) \leq \operatorname{dist}(x, Q_j) + \operatorname{diam}(Q_j) + \operatorname{dist}(Q_j, F)$. Since $x \in 2Q_j$, it follows that $\operatorname{dist}(x, Q_j) \leq (\frac{1}{2}) \operatorname{diam}(Q_j)$, and by (2.2) we have $\operatorname{dist}(Q_j, F) \leq 4 \operatorname{diam}(Q_j)$. Therefore,

$$\operatorname{dist}(x, F) \leq (\tfrac{11}{2}) \operatorname{diam}(Q_j). \tag{2.4}$$

Similarly, $\operatorname{dist}(x, F) \geq \operatorname{dist}(Q_j, F) - \operatorname{dist}(x, Q_j)$implies

$$\operatorname{dist}(x, F) \geq (\tfrac{1}{2}) \operatorname{diam}(Q_j). \tag{2.5}$$

Furthermore, if $y \in Q_j$, $\operatorname{dist}(x, y) \leq \operatorname{dist}(x, Q_j) + \operatorname{diam}(Q_j)$. This, the fact that $\operatorname{dist}(x, Q_j) \leq (\frac{1}{2}) \operatorname{diam}(Q_j)$ and (2.5) show that

$$\operatorname{dist}(x, y) \leq 3 \operatorname{dist}(x, F). \tag{2.6}$$

From (2.4), (2.5) and (2.6) it follows that all Q_j with x in $2Q_j$ lie in a sphere of radius $3\operatorname{dist}(x, F)$ about x, are disjoint and have diameter greater than or equal to $(\frac{2}{11})\operatorname{dist}(x, F)$. Since the sphere of radius $3\operatorname{dist}(x, F)$ has volume less than $[6\operatorname{dist}(x, F)]^n$, the number of Q_j with $x \in 2Q_j$ is bounded by $[6\operatorname{dist}(x, F)]^n/[(2/11\sqrt{n})\operatorname{dist}(x, F)]^n = (33\sqrt{n})^n$. Ths proves (e) for x in G. For x not in G, property (d) shows that the left side of (e) is 0. This completes the proof of Lemma (2.1).

To prove Theorem (1.6), fix a $\lambda \geq \lambda_0$ and let G be the set where $Mf(x) > \lambda$. If x is not in $3Q_0$, then any oriented cube that contains x and intersects Q_0 has sides with length greater than the sides of Q_0 and $Mf(x) < (1/|Q_0|)\int_{Q_0}|f(x)|\,dx = \lambda_0$. Therefore, $G \subset 3Q_0$ and we can apply Lemma (2.1) with this G and with Q taken to be Q_0. By property (e)

$$\begin{aligned}\int_{G\cap Q_0} U(x)\,dx &= \int_{Q_0} \chi_G(x) U(x)\,dx \\ &\geq (33\sqrt{n})^{-n} \sum_{j\geq 1} \int_{Q_0} \chi_{2Q_j}(x) U(x)\,dx;\end{aligned}$$

combining this and (1.7) shows that

$$\int_{G\cap Q_0} U(x)\,dx \geq A(33\sqrt{n})^{-n} \sum_{j\geq 1} |Q_j| \operatorname*{ess\,sup}_{x\in Q_j} V(x). \tag{2.7}$$

Now given $j \geq 1$, we have $\operatorname{dist}(Q_j, F) \leq 4\operatorname{diam}(Q_j)$. Therefore, there is an x in F such that $|x - c_j| \leq (\frac{9}{2})\operatorname{diam}(Q_j)$, where c_j is the center of Q_j. Since $9\sqrt{n}\,Q_j$ contains the sphere of radius $(\frac{9}{2})\operatorname{diam}(Q_j)$ about c_j, it follows that x is in $9\sqrt{n}\,Q_j$, and since $Mf(x) \leq \lambda$, we have

$$\frac{1}{|9\sqrt{n}\,Q_j|}\int_{Q_j}|f(x)|dx \leq \lambda.$$

This shows that the right side of (2.7) is bounded below by

$$A(33\sqrt{n})^{-n}\sum_{j\geq 1}|Q_j|\operatorname*{ess\,sup}_{Q_j}V(x)\frac{1}{\lambda|9\sqrt{n}\,Q_j|}\int_{Q_j}|f(x)|dx.$$

This is bounded below by

$$\frac{A(300n)^{-n}}{\lambda}\sum_{j\geq 1}\int_{Q_j}|f(x)|V(x)\,dx.$$

By property (b) of Lemma (2.1), this is bounded below by

$$\frac{A(300n)^{-n}}{\lambda}\int_{G\cap Q_0}|f(x)|V(x)\,dx.$$

Finally since $Mf(x) \geq |f(x)|$ almost everywhere, this is bounded below by the right side of (1.8). This completes the proof of Theorem (1.6).

The proof of Theorem (1.10) is a minor modification of the sufficiency proof of Corollary 1 of [**1**]. Let Q_1 be a fixed oriented closed cube. For $t > 0$ let $f_t(x) = f(x)$ if $|f(x)| \leq t$ and $x \in tQ_1$, $f_t(x) = t\operatorname{sgn} f(x)$ if $|f(x)| > t$ and $x \in tQ_1$, and $f_t(x) = 0$ if $x \notin tQ_1$. Given $\lambda > 0$ and R such that $R > t$ and

$$\frac{1}{|(R-t)Q_1|}\int_{R^n}|f_t(x)|dx \leq \lambda, \tag{2.8}$$

define $U_R(x) = U(x)$ if $x \in RQ_1$ and $U(x) = \infty$ if $x \notin RQ_1$. We will apply Theorem (1.6) to f_t with Q_0, U and V taken respectively as $2RQ_1$, U_R and V. Hypothesis (1.7) is satisfied if $2Q \subset RQ_1$ by (1.11) while if $2Q \not\subset RQ_1$, hypothesis (1.7) is satisfied because the left side is infinite. Inequality (2.8) implies that λ is greater than the required λ_0. Theorem (1.6) then implies that

$$\begin{aligned}\int_{2RQ_1\cap\{x:\,(Mf_t)(x)>\lambda\}} & U_R(x)\,dx \\ & \geq \frac{A(300n)^{-n}}{\lambda}\int_{\{x:\,|f_t(x)|>\lambda\}}|f_t(x)|V(x)\,dx.\end{aligned} \tag{2.9}$$

Now (2.8) implies that $(Mf_t)(x) \leq \lambda$ for $x \notin RQ_1$; therefore, (2.9) is equivalent to

$$\int_{\{x:\,(Mf_t)(x)>\lambda\}} U(x)\,dx \geq \frac{A(300n)^{-n}}{\lambda} \int_{\{x:\,|f_t(x)|>\lambda\}} |f_t(x)|V(x)\,dx.$$

Now let t approach ∞ and use the monotone convergence theorem to complete the proof of Theorem (1.10).

Theorem (1.13) is proved from Theorem (1.6) in the same way that Theorem 2 of [**1**] is proved from Theorem 1 of [**1**]. Theorem (1.14) is proved from Theorem (1.10) in the same way that Theorem 3 and Corollary 2 of [**1**] are proved from Theorem 2 of [**1**].

References

[1] K. F. Andersen and W.-S. Young, *On the reverse weak type inequality for the Hardy maximal function and the weighted classes* $L(\log L)^k$, Pacific J. Math., **112** (1984), 257–264.

[2] H. A. Favo, E. A. Gatto and C. Gutiérrez, *On the strong maximal function and Zygmund's class* $L(\log^+ L)^n$, Studia Math., **69** (1980), 155–158.

[3] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math., **124** (1970), 9–36.

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165** (1972), 207–226.

[5] E. M. Stein, *Note on the class* $L \log L$, Studia Math., **32** (1969), 305–310.

[6] _____, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.

Received October 3, 1983. Research supported in part by N. S. F. Grant MCS 80-03098.

RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08903