PACIFIC JOURNAL OF MATHEMATICS
Vol 117, No 2, 1985

DETERMINING INCOMPRESSIBILITY OF SURFACES IN ALTERNATING KNOT AND LINK COMPLEMENTS

WILLIAM MENASCO

In this paper we carry out the next necessary step in the study of closed incompressible surfaces in alternating link complements — determining when a given punctured surface in an alternating link complement is incompressible, pairwise incompressible and understanding when a "peripheral tubing" operation (which will produce a closed surface) preserves the incompressibility of a surface.

1. Introduction. Let $L \subset \mathbf{R}^3 \subset S^3 = \mathbf{R}^3 \cup \{\infty\}$ be a non-split prime link which is alternating with respect to the projection $\pi\colon \mathbf{R}^3 \to \mathbf{R}^2$. In [**M**] we introduced the notion of a standard position embedding with respect to $\pi(L)$ for incompressible, pairwise incompressible surfaces in $S^3 - L$. From Theorem 3(c) of [**M**] we can conclude that there are only finitely many such n-punctured surfaces for $n > 0$. However, for a given surface $S \subset S^3 - L$ in standard position, no method was given in [**M**] for determining whether S is incompressible or pairwise incompressible. Furthermore, understanding when a "peripheral tubing" operation preserves the incompressibility of a standard position surface was not discussed in [**M**].

In this paper we give a method for determining when a surface in standard position is incompressible and pairwise incompressible. Using results on branched surfaces in [**F-O**], we adapt our new method to understanding when a "peripheral tubing" operation preserves the incompressibility. Throughout this paper we use the same notation and terminology as in [**M**].

In §2 we amplify on the notion of standard position.

In §3 we develop the notion of a compressing disk or pairwise compressing disk in standard position with respect to a standard positioned surface. From an analysis of a disk in standard position we produce our sought after method. In particular, the following sufficient condition for incompressibility becomes evident.

THEOREM 1. *Let L be a non-split prime alternating link and suppose $S \subset S^3 - L$ is a surface in standard position. For S to be incompressible,* *pairwise incompressible it is sufficient that both of the following situations not occur*:

(a) *There exists a loop* $\alpha \subset S \cap S^2_\pm$ *such that* α *intersects a component of* $(S^2 \cap S^2_\pm) - L$ *more than once*.

(b) *There exist loops* $\alpha \subset S \cap S^2_+$, $\beta \subset S \cap S^2_-$ *and arcs* $a, c \subset \alpha \cap ((S^2 \cap S^2_+) - L)$, $b, c \subset \beta \cap ((S^2 \cap S^2_-) - L)$ *with* $c \subset \alpha \cap \beta$ *such that* a, b, c *are contained in components of* $(S^2 \cap S^2_\pm) - L$ *adjacent to a common bubble*. (*See Figure* 1b.)

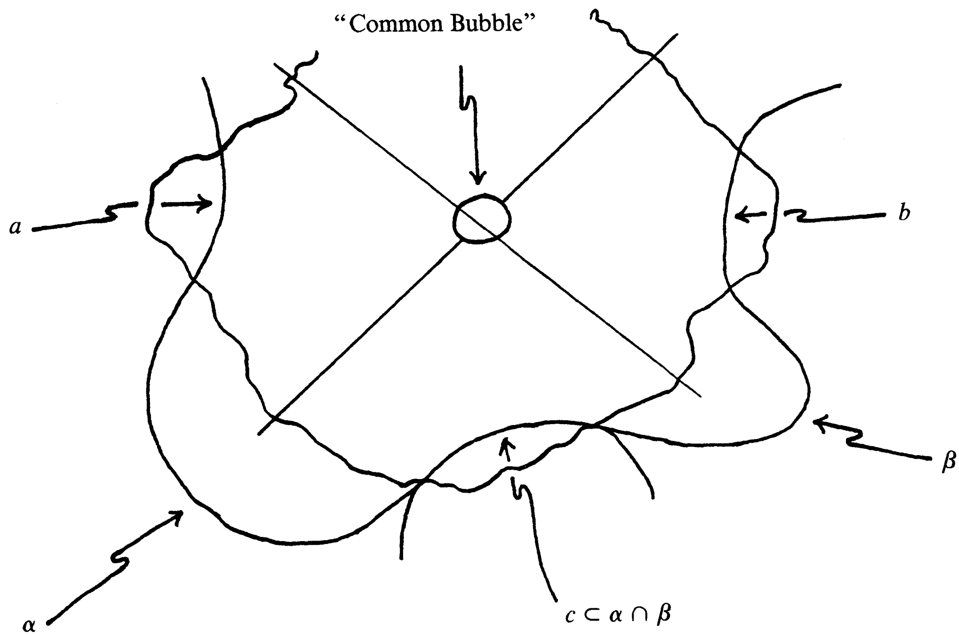


FIGURE 1

This sufficient condition will force some standard position loop patterns of $S \cap S^2_\pm$ to be incompressible, pairwise incompressible independent of $\pi(L)$. In particular, we have the following results in §3. (We use the notation PT. to mean point.)

THEOREM 2. *Let* L *be a non-split prime alternating link and suppose* $S \subset S^3 - L$ *is an* S^2 *in standard position having* 4 *or* 6 *punctures*. *Then*:

(a) *If* $S = S^2 - 4$ *PT.'s is a PSPS loop pattern then* S *is incompressible, pairwise incompressible*.

(b) *If* $S = S^2 - 6$ *PT.'s with* $\{c_1, c_2, c_3\} = S \cap S^2_+$, *where* $\omega_+(c_1) = \omega_+(c_2) = PSPS$ *and* $\omega_+(c_3) = PS^iPS^i$, $i = 1$ *or* 2, *then S is incompressible, pairwise incompressible.*

(c) *If* $S \cap S^2_+$ *is a* P^4 *or* P^6 *loop patterns, or if* $\{c_1, c_2\} = S \cap S^2_+$, $\omega_+(c_1) = P^3SPS$ *and* $\omega_+(c_2) = PSPS$ *then there is one simple isotopy that will determine whether S is incompressible, pairwise incompressible.*

Finally, in §4 we exploit Theorem 2 of **[F-O]** to determine which "peripheral tubings" produce closed incompressible surfaces. A strong familiarity with branched surfaces and their development in **[F-O]** and **[O]** is assumed.

2. Amplifying standard position. Let $S \subset S^3 - L$ be an incompressible and pairwise incompressible surface satisfying the following conditions:

(i) No word $\omega_{\pm}(c)$ associated to S is empty.

$(*)$ (ii) No loop of $S \cap S^2_{\pm}$ meets a bubble in more than one arc.

(iii) Each loop of $S \cap S^2_{\pm}$ bounds a disc in $B^3_{\pm}$.

Conditions (i) and (iii) are the result of the incompressibility of S and are independent of the alternatingness of L. Condition (ii) is the result of the pairwise incompressibility of S and is dependent on the alternating properties of $\pi(L)$. For further details see **[M]**.

LEMMA 3. *Let S be as above. S can be isotoped so that, in addition to the preceding conditions, S satisfies the following*:

(a) *No loop* $S \cap S^2_{\pm}$ *meets both a bubble and an arc of* $L \cap S^2$ *having an endpoint on that bubble.*

(b) *No loop of* $S \cap S^2_{\pm}$ *meets a component of* $L \cap S^2_{\pm}$ *more than once.*

(c) *There do not exist two loops* $\alpha \subset S \cap S^2_+$ *and* $\beta \subset S \cap S^2_-$, *with arcs* $a, b \subset \alpha \cap \beta$ *such that* $\mathring{a}$ *and* $\mathring{b}$ *are contained in adjacent components of* $(S^2 \cap S^2_{\pm}) - L$, *and* $\partial a \cap \partial b = \varnothing$.

Proof. To show (a) we consider a loop, C, of $S \cap S^2_{\pm}$ which meets both a bubble and an arc of $L \cap S^2$ having an endpoint on the bubble. Figure 2 shows the four possible configurations that can occur. The reader should convince himself that if Figure 2(a) occurs, then Figure 2(b) must occur, and if Figure 2(c) occurs, then Figure 2(d) must occur. An isotopy that will remove the configurations in Figures 2(b) and (d) will thus remove the configurations in Figures 2(a) and (c).

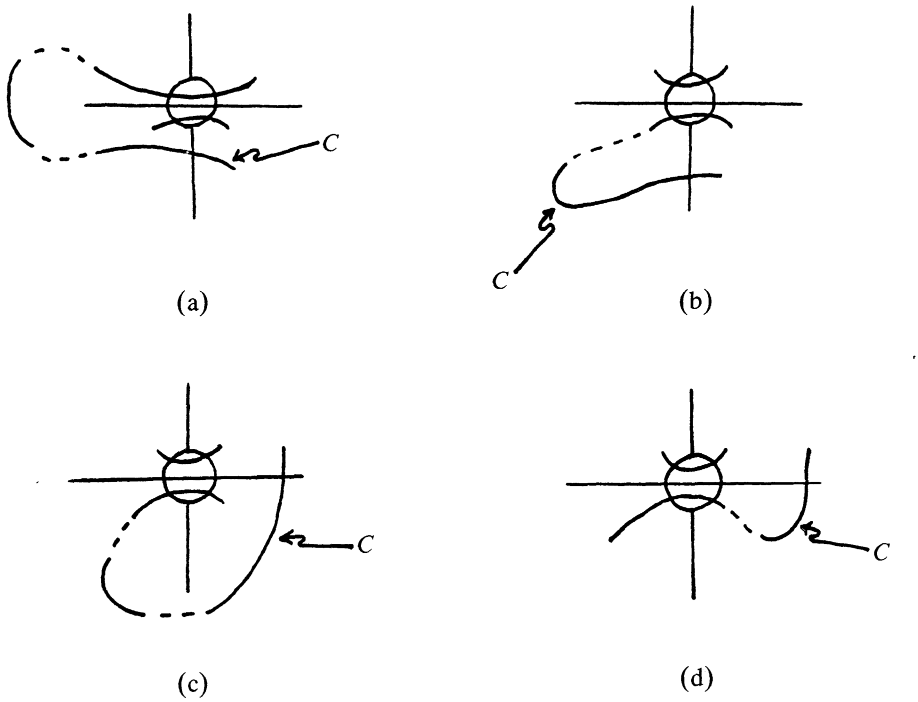


FIGURE 2

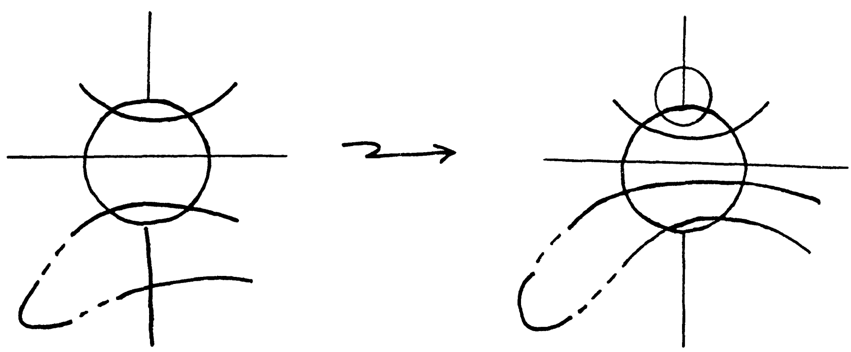


FIGURE 3

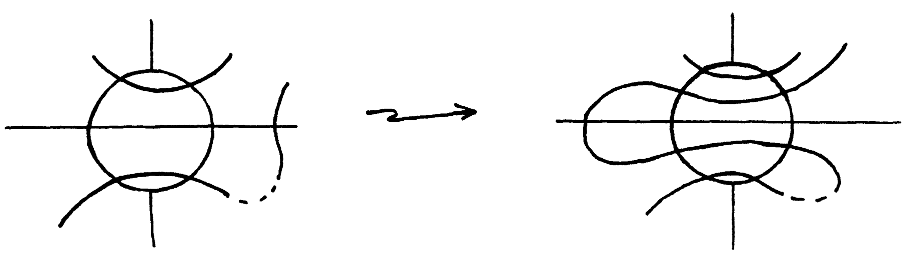


FIGURE 4

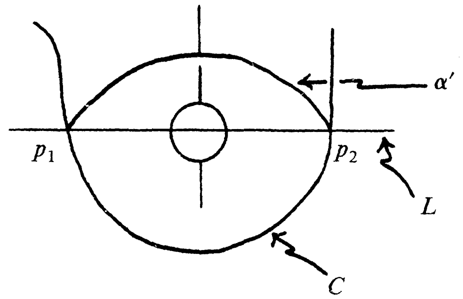


FIGURE 5

Suppose C corresponds to Figure 2(b) and assume C is the innermost such loop. By sliding the point of C that is transverse to L across and under the bubble, and dragging S along, we produce a new loop that has two arcs on the same bubble. (See Figure 3.) This adds a saddle to S, but we are now in the situation of Figure 5(b) of [**M**], where two saddles can be eliminated.

Suppose C corresponds to Figure 2(d) and assume C is the innermost such loop. By sliding the point of C that is transverse to L across and over the bubble, we produce a new loop that has two arcs on the same bubble. (See Figure 4.) In a similar fashion to the isotopy depicted in Figure 3, this adds a saddle to S, but we are again in the situation of Figure 5(b) or [**M**], where two saddles can be eliminated.

Condition (b) follows from the incompressibility of S. Suppose C is a loop of $S \cap S^2_\pm$ and α is an arc of $L \cap S^2_\pm$ such that $p_1, p_2 \subseteq C \cap \alpha$, where p_1 and p_2 are two successive points on α. Let $D \subset S$ be the disc in $B^3_\pm$ whose boundary is C. Then there is an arc $\alpha' \subset D$ with $\partial\alpha' = \{p_1, p_2\}$ and α' isotopic to α, as illustrated in Figure 5. This means S is ∂-incompressible and since S is not an annulus, S is compressible.

Finally, Condition (c) follows from the pairwise incompressibility of S. Suppose $\alpha \subset S \cap S^2_+$ and $\beta \subset S \cap S^2_-$ with $a, b \subset \alpha \cap \beta$ arcs contained in adjacent components of $(S^2 \cap S^2_\pm) - L$, $\partial a \cap \partial b = \varnothing$, as shown in Figure 6. We can take points $p_1 \in a$ and $p_1 \in b$, and arcs, R_+ and R_-, such that $\{p_1, p_2\} = \partial R_\pm$ and $R_\pm \subset S \cap B^2_\pm$. Since p_1 and p_2 are in adjacent components of $(S^2 \cap S^2_\pm) - L$, $R_+ \cup R_-$ is isotopic to a meridian of L. Assume α and β are innermost such loops $R_+ \cup R_-$ bounds a punctured disc along which S can be pairwise compressed. □

We refine the notion of *standard position* to mean that a surface S satisfies the conditions of $(*)$ and Lemma 3. Notice that the alternating hypothesis for L has not been used in the proof that surfaces can be isotoped into standard position.

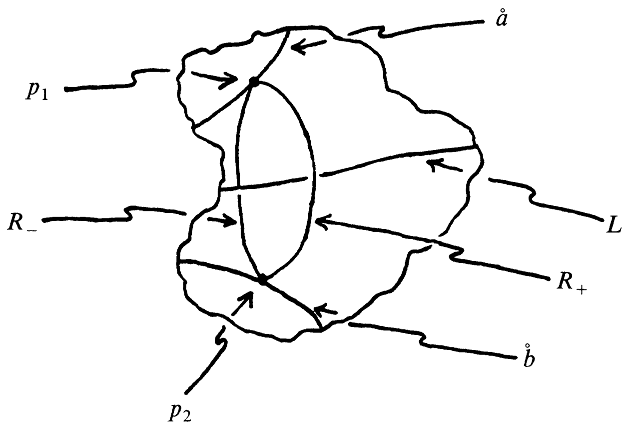


FIGURE 6

3. Compressing discs in standard position. Let $S \subset S^3 - L$ be a surface in standard position. From §2 we know that every incompressible, pairwise incompressible surface can be isotoped into standard position. We do not know that every surface in standard position is incompressible. In fact, there are many easily constructed examples to the contrary.

Suppose $D \subset S^3 - L$ is an embedding of either a disc (D^2) or a disc minus a point $(D^2 - \text{PT.})$ with $\partial D \subset S$. (If $D = D^2 - \text{PT.}$ then D is a disc that is punctured once by a component of L.)

As in [M] we can isotope D to meet each ball bounded by a bubble in saddle-shaped discs. (See Figure 4 of [M].) We may suppose that the meridian boundary of $D = D^2 - \text{PT.}$ does not intersect the bubbles and that D meets S^2_+, S^2_- and S transversely.

Each component of $D \cap S^2_\pm$ is either a loop or an arc whose endpoints are on $\partial D \subseteq S$.

For each arc $\alpha_1 \subset D \cap S^2_\pm$ there are additional arcs $\alpha_2, \ldots, \alpha_k \subset D \cap S^2_\pm$ and arcs $\beta_1, \ldots, \beta_k \subset \partial D \cap B^3_\pm$ such that $\alpha_i \cap \partial D = (\alpha_i \cap \beta_i) \cup (\alpha_i \cap \beta_{i-1}) = \partial\alpha_i$ (where the index is mod k). We refer to the loop $\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2 \cup \cdots \cup \alpha_k \cup \beta_k \subset B^3_\pm$ as a *loop of arcs.*

LEMMA 4. *Let $S \subset S^3 - L$ be a surface in standard position, and suppose $D \subset S^3 - L$ is either a compressing disc or a pairwise compressing* *disc. Then D can be replaced by a similar disc, D', such that:*

(a) *No component of $D' \cap S^2_\pm$ that is contained in a single component of $(S^2 \cap S^2_\pm) - L$ is either a loop or an arc which together with an arc on $S \cap S^2_\pm$ bounds a disc in $(S^2 \cap S^2_\pm) - L$.*

(b) *No loop of $D' \cap S^2_\pm$ or loop of arcs of D' intersects a bubble in more than one arc.*

(c) *Each loop of $D' \cap S^2_\pm$ or loop of arcs in $B^3_\pm$ bounds a disc, $D'' \subset D' \cap B^3_\pm$.*

(d) *No loops of arcs of D' intersects a bubble and a component of $S \cap S^2_\pm$ which intersect each other.*

Proof. Let C be an innermost loop of $D \cap S^2_+$ in S^2_+, or a loop of arcs of D in B^3_+. Let C' be a nearby isotopic circle on $D \cap \mathring{B}^3_+$. C' bounds a disc $d \subset \mathring{B}^3_+ - S$ with $d \cap D = \partial d_1 = C'$, since $S \cap S^2_\pm$ satisfies (iii) of $(*)$. Surgering D along d we produce a new compressing disc that we still call D, and we then perform the same procedure on a loop of $(D \cap S^2_+) - C$ which is innermost among loops of $(D \cap S^2_+) - C$ or another loop of arcs of D; and so on. When all loops of $S \cap S^2_+$ have been considered, we operate in the same way on loops of $S \cap S^2_-$ and loops of arcs of D in B^3_-. After this has been done the conditions on loops in (a) and condition (c) automatically hold.

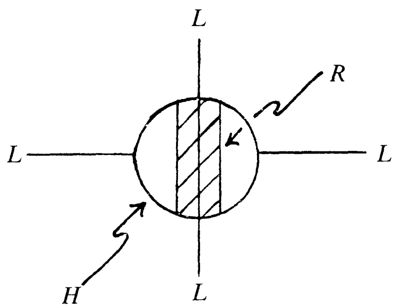
For the condition on arcs in (a), let $\alpha \subset D \cap S^2_+$ be an arc, $\alpha' \subset D \cap \mathring{B}^3_+$ be an isotopic arc, $\beta \subset S \cap \mathring{B}^3_+$ be an arc, and $d \subset \mathring{B}^3_+$ be a disc such that $\alpha' \cap \beta = \partial d$ and $\mathring{d} \cap S = \mathring{d} \cap D = \varnothing$. Surgering D along the half-disc d produces two new discs, one of which must be a compressing or pairwise compressing disc; once again, call this new disc D. D still satisfies (a) for loops and (c). Performing this operation on all such α in $D \cap S^2_+$ and then in $D \cap S^2_-$ will finish off condition (a).

For (b), suppose C is some loop of $D \cap S^2_+$ or loop of arcs of D in B^3_+ ($D \cap S^2_-$ and D in B^3_- are treated similarly) which meets the upper hemisphere H of a bubble in two or more arcs. Let $d_1 \subset S^2_+ \cup (S \cap B^3_+)$ be a disc bounded by C, chosen so that $d_1 \cap H$ contains a rectangle R whose boundary consists of two arcs of C and two arcs of ∂H. Replacing C if necessary by another loop of $D \cap S^2_+$ or loop of arcs of D in B^3_+, we may assume $D \cap \operatorname{int}(R) = \varnothing$. We now have two possibilities, according to whether R meets L or not; see Figure 7. C bounds a disc $d_2 \subset D \cap \mathring{B}^3_+$. Let $B \subset d_2$ be a band joining the two arcs of $C \cap R$; if $(R \cap L) \neq \varnothing$ (Figure 7a), the two arcs belong to the same saddle σ, so $B \cup \sigma$ contains a circle isotopic in $S^3 - L$ to a meridian of L. For $D = D^2$, this is impossible. When $D = D^2 - \text{PT.}$, we meridionally surger D along the core circle of $B \cup \sigma$ into an annulus and a new $D^2 - \text{PT.}$, which must be pairwise compressing with fewer saddles.

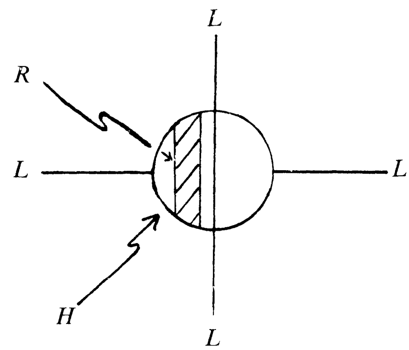
If $R \cap L = \varnothing$, let $d_3 \subset B^3_+$ be a disc with ∂d_3 consisting of an arc of B and an arc of R; we may assume $d_3 \cap D \subset \partial d_3$. Then we may use d_3 to isotope D so as to eliminate the two saddles of D containing the two arcs of $R \cap B$.

For (d), let C_1 be a component of $S \cap S^2_+$ which meets the upper hemisphere H of a bubble. Suppose C_2 is a loop of arcs of D in B^3_+ that meets both C_1 and H. Let d_1 be the disc C_1 bounds in $S \cap B^3_+$. Let $d_2 \subset (S \cap B^3_+) \cup S^2_+$ be the disc that C_2 bounds such that $(d_1 \cup H) \cap d_2$ contains a hexagon R whose boundary consists of two arcs in C_1, two arcs in ∂H, and two arcs in C_2.

We have two possibilities, either $R \cap L = \varnothing$ (Figure 8a) or $R \cap L \neq \varnothing$ (Figure 8b). In Figure 8b the loop of arcs is labeled as C_3. Since another loop of arcs must be attached to C_3 via the saddle in the bubble we must also see C_2 of Figure 8a occurring. So we can assume $D \cap R = \varnothing$.



(a)



(b)

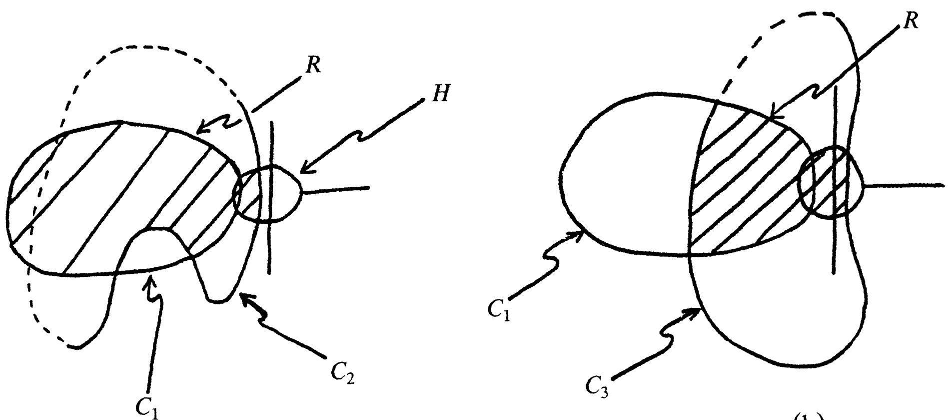
FIGURE 7

Suppose C_1 and C_2 correspond to Figure 8a. By sliding the arc $d_1 \cap c_2$ that is on ∂R across d_1 and through the bubble, and dragging D along, we produce a new loop of arcs (or possibly a loop of $D \cap S^2_+$) that has two arcs on the same bubble. (See Figure 9). This adds a saddle to D, but we are now in the situation of Figure 7b, where two saddles can be eliminated. □

If $D \subset S^3 - L$ is a compressing or pairwise compressing disc of a standard position surface such that D satisfies the conditions of Lemma 4 then we say D is in standard position with respect to S. Notice that we have not used the hypothesis that L is alternating.

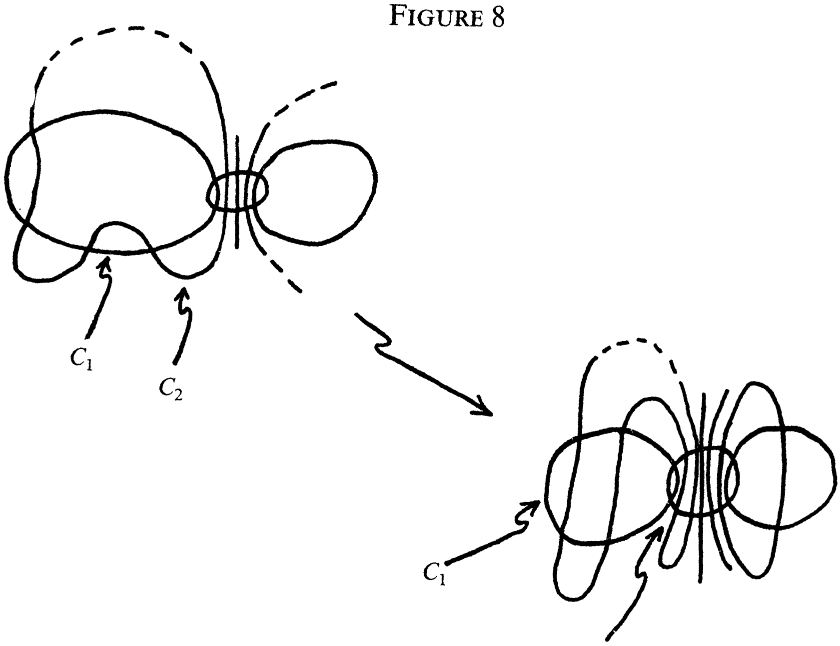
LEMMA 5. *If $\pi(L)$ is alternting and $S \subset S^3 - L$ is a surface in standard position and $D \subset S^3 - L$ is a compressing or pairwise compressing disc of S in standard position w.r.t. S then each component of $D \cap S^2_\pm$ is an arc.*

Proof. The reader should convince himself that if S is surgered along D to produce a new surface, S', (possible not connected) the new surface will satisfy ($*$). In particular, S' will be in "standard position" in the sense of [**M**]. If $C \subset D \cap S^2_\pm$ is a loop then there is a loop $C' \subset S' \cap S^2_\pm$ parallel to C. Using notation from [**M**], $\omega_\pm(C') = P^iS^j$, where $0 \leq i \leq 1$ and $j \geq 0$. By Lemma 2 of [**M**] this is impossible. □



(a) (b)

FIGURE 8



TWO SADDLE TO ELIMINATE

FIGURE 9

From Lemma 5, it is easy to see that $D \cap S^2_\pm \subset D$ has at least one of the following two alternating properties:

(i) There exists a component of $D - (D \cap S^2_\pm)$ whose boundary is an arc in $D \cap S^2_\pm$ and an arc in ∂D (Figure 10a).

(**) (ii) There exist two adjacent components of $D - (D \cap S^2_\pm)$, R_1 and R_2, such that $\partial R_\iota = \alpha_i \cup \beta_i$, where $\alpha_i \subset D \cap S^2_\pm$ and $B_i \subset \partial D$ are arcs. Furthermore, α_1 and α_2 each intersect one single bubble and $\alpha_1 \cap \alpha_2 \neq \varnothing$. (See Figure 10b.)

Proof of Theorem 1. It should be clear that if (i) of (**) occurs then either situation (a) of the theorem occurs or condition (a) of Lemma 4 does not hold. Since we can assume any compressing disc satisfies Lemma 4 then only situation (a) of the theorem occurs.

If (ii) of (**) occur then Figure 11b (prior to the isotopy) shows how region R_1 and R_2 would be situated. This forces a portion of S to look like Figure 1. Thus, situation (b) of the theorem occurs. □

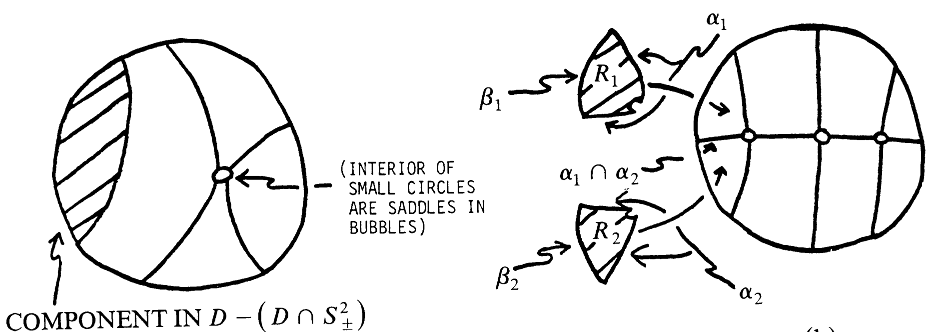


FIGURE 10

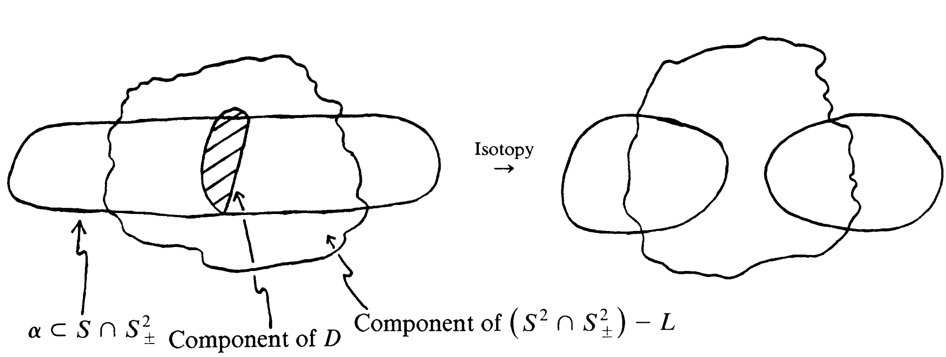


FIGURE 11a

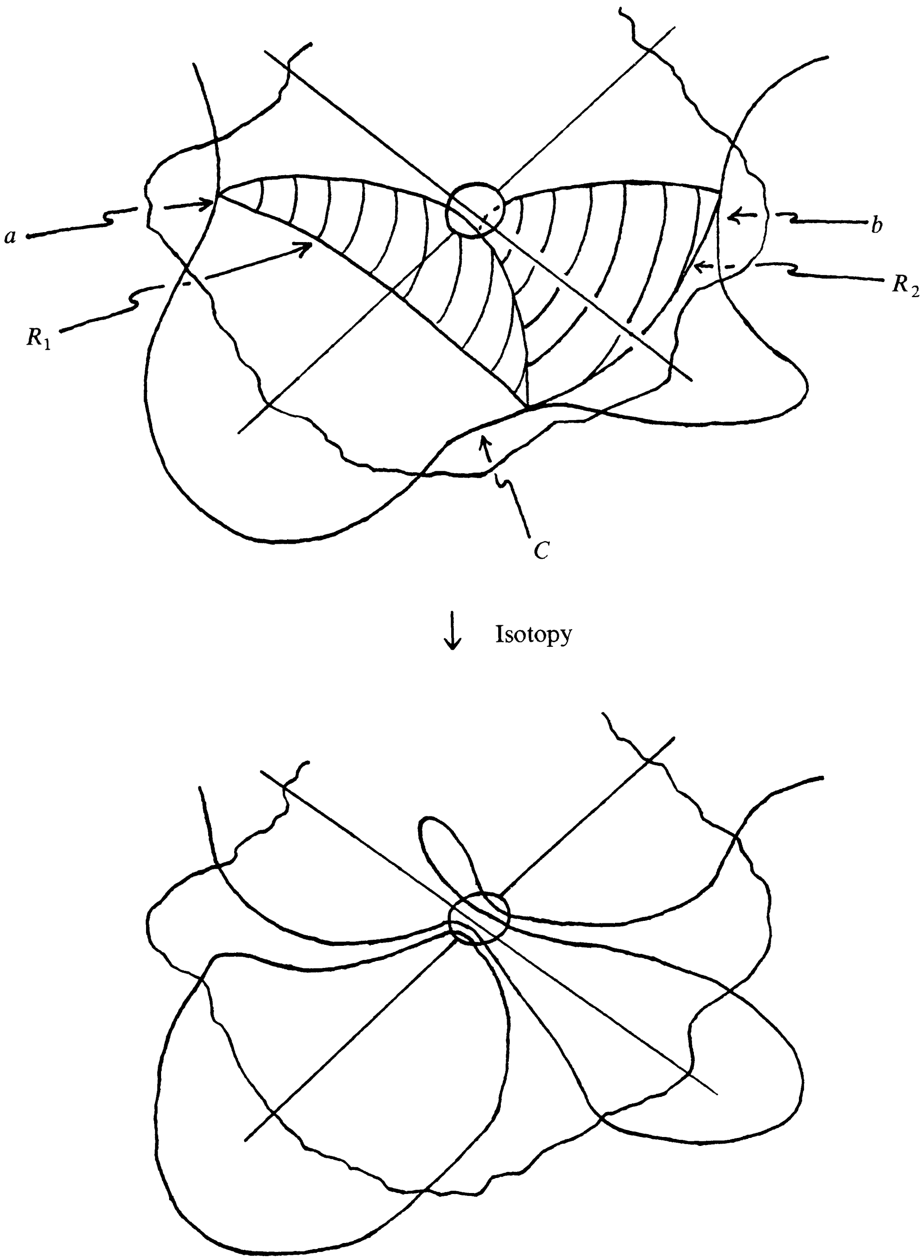


FIGURE 11b

Our method for determining incompressibility, pairwise incompressibility is quite evident now. Suppose for some standard position surface $S \subset S^3 - L$, $S \cap S^2_{\pm}$ satisfies situation (a) or (b) of Theorem 1. We then isotope S along the corresponding components of a possibly existing D.

The resulting $S \cap S^2_\pm$ will not necessarily be standard, but $D \cap S^2_\pm$ will still satisfy $(**)$. In particular, Figure 11a shows the isotopy when (a) of Theorem 1 occurs and Figure 11b shows the isotopy when (b) of Theorem 1 occurs. Since $D \cap S^2_\pm$ still satisfies $(**)$ then $S \cap S^2_\pm$ satisfies situation (a) or (b) of Theorem 1. We continue in this fashion until either (a) and (b) no longer hold, or $S \cap S^2_\pm$ violates (iii) of $(*)$ or Lemma 3(b), (c). In the former case S is incompressible, pairwise incompressible. In the latter case S is not.

Proof of Theorem 2. We start by establishing the following claim.

Claim. If $\alpha \subset S \cap S^2_\pm$ and $\omega_\pm(\alpha) = PSPS$ then α satisfies the sufficient conditions in Theorem 1.

To prove the claim we assume first that α intersects a component of $(S^2 \cap S^2_\pm) - L$ more than once. This would mean that α did not satisfy Lemma 3a and, thus, S was not in standard position. (See Figure 12.) Second, we assume there are arcs $a, b, c \subset \alpha \cap ((S^2 \cap S^2_\pm) - L)$ as in situation (b) of Theorem 1. (We must have $b \subset \alpha \cap ((S^2 \cap S^2_\pm) - L)$ since a $PSPS$ loop in always innermost.) But as Figure 13 illustrates $S \cap S^2_\mp$ will then violate (ii) of $(*)$. Thus the claim is established.

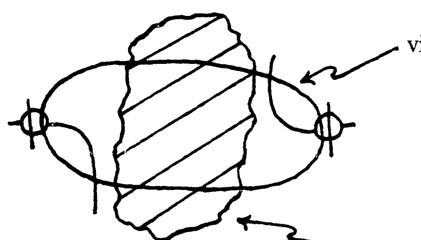


FIGURE 12

The proof of the claim essentially proves part (a). For part (b), the reader can easily construct a similar argument for PS^2PS^2 and $PSPSPS$ loops.

For part (c), we notice if a P^4, P^6, or P^3SPS loop satisfies (a) of the theorem then it must violate (b) of Lemma 3. (See figure 14.)

If a P^4, P^6, or P^3SPS loop satisfies (b) of Theorem 1, then there is a component of the loop intersecting $(S^2 \cap S^2_\pm) - L$ that can be isotopied rel ∂ into a bubble (in fact the "common bubble"). Figure 15 illustrates

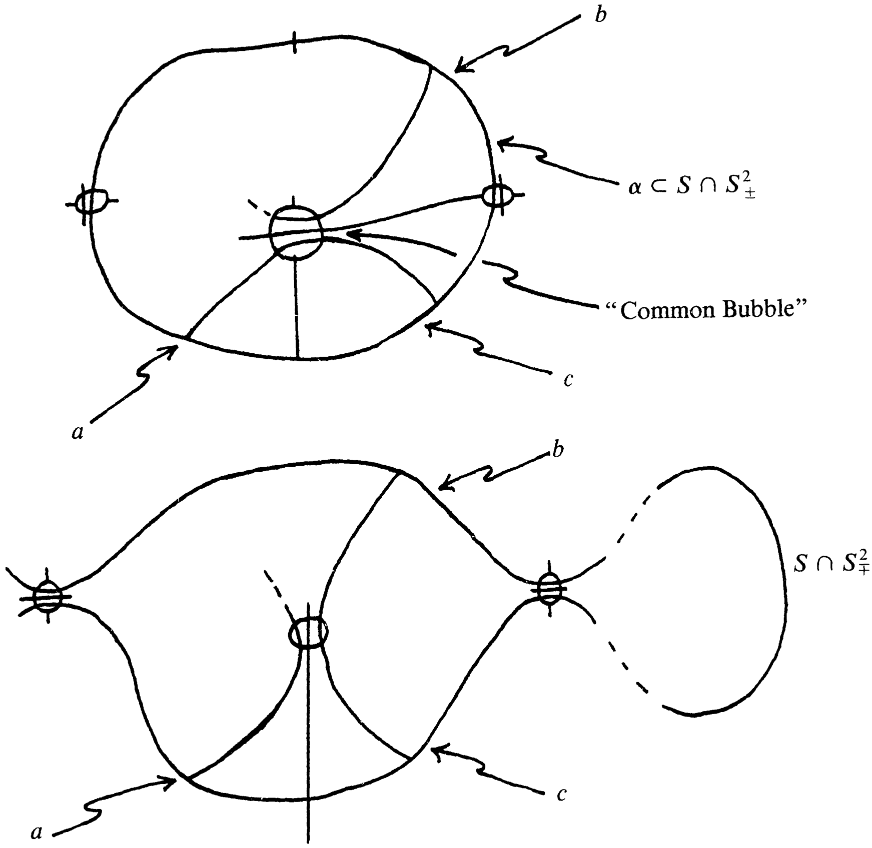


FIGURE 13

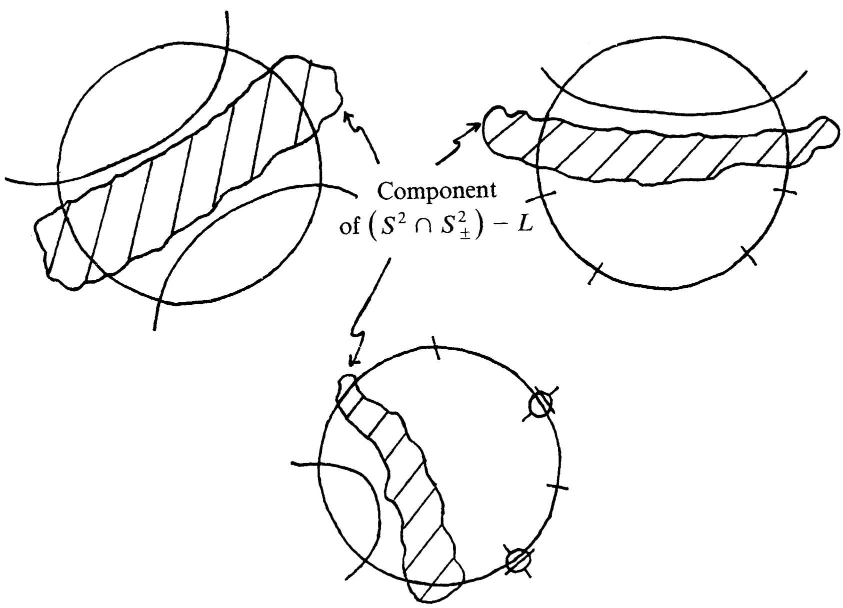


FIGURE 14

this configuration and the possible isotopy of sliding the loop past the "common bubble". This isotopy is the result of first applying the isotopy in Figure 11b once and then applying isotopies that were used for part (a) of Lemma 3. □

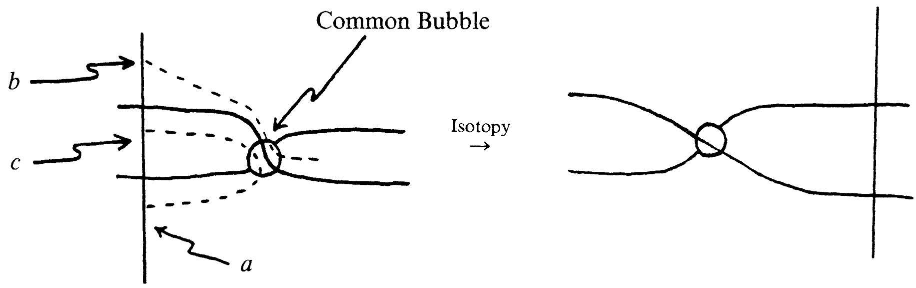


FIGURE 15

4. Peripheral tubing. From [**M**] we know that all closed imcompressible surfaces in alternating link complements are the result of peripheral tubing together meridian boundary components of incompressible, pairwise incompressible surfaces. However, not all peripheral tubing preserve incompressibility. In determining when incompressibility is preserved we use some recent results on branched surfaces, [**F-O**]. For the purposes of this paper we need only concern ourselves with the branched surfaces that are described in §3 of [**10**] and the result pertaining to them, namely Theorem 3.2 of [**10**]. (We use the same terminology as used in [**F-O**] and [**O**].)

From [**F-O**] we know if S is a closed incompressible surface it is carried by a branched surface, B. In particular, if S is obtained by peripheral tubing on a pairwise incompressible surface S' then $B -$ (branching set) must be a union of meridial annuli and a component that is isotopic to int(S'). (See [**O**].) Near the branching set B resembles one of the configurations in Figure 16. To determine which types of branching set can occur on B such that B carries only incompressible surfaces, we apply Theorem 3.2 of [**O**], i.e. B must satisfy the following conditions:

(1) There are no monogons in $(S^3 - L) - B$.

(2) $\partial_h N$ is incompressible.

Here, $\partial_h N$ is the portion of the boundary of a fibered neighborhood, N, of B that is transverse to fibers. If B satisfies the above we then must check that no negative weights can be assigned to B to produce a surface.

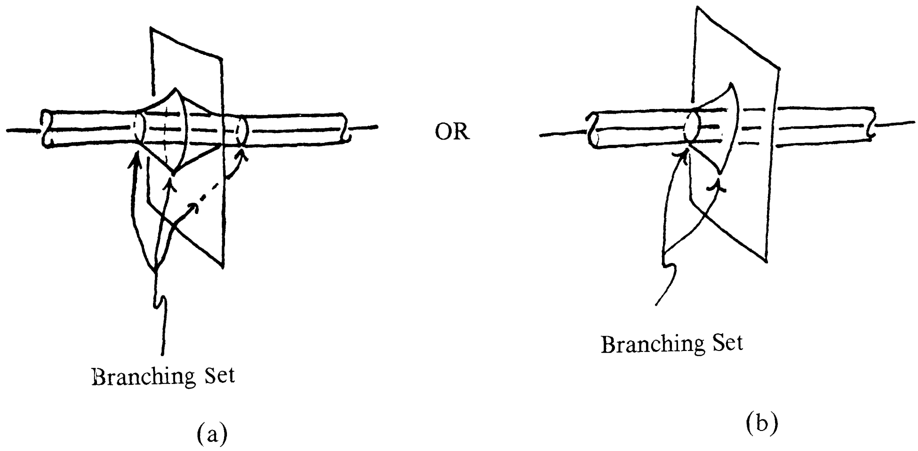


FIGURE 16

To determine when $\partial_h N$ is incompressible we first need to develop a notion of a closed surface in standard position and a compressing disc in standard position.

We say a closed surface, S, is in standard position if $S \cap S^2_{\pm}$ satisfies the following:

(i) No word $\omega_{\pm}(c)$ associated to S is empty.

(ii) No loop of $S \cap S^2_{\pm}$ meets a bubble in a non-meridian fashion

(***) (Figure 7b). (Thus no loop meets a bubble more than twice, as illustrated in Figure 7a.)

(iii) Each loop of $S \cap S^2_{\pm}$ bounds a disc in $B^3_{\pm}$.

The argument that a closed incompressible surface is isotopic to a surface satisfying (i) and (iii) of (***) is the same as for a surface with boundary since it is dependent only on incompressibility. The argument that any surface can be isotoped such that no loop of $S \cap S^2_{\pm}$ meets a bubble in a non-meridian fashion is the same as previous arguments which are independent of incompressibility and alternatingness. Thus every closed incompressible surface is isotopic to a surface in standard position.

We use the same definition for a compressing disc in standard position with respect to a standard position surface as was developed in §3. Lemma 4d is a vacuous condition for closed surfaces. The rest of the proof of Lemma 4 was independent of boundary. Thus a compressing disc can always be replaced by a standardized disc.

When $\pi(L)$ is alternating Lemma 5 still applies to a compressing disc in standard position. Thus Theorem 1 and the isotopies of Figure 11 still hold. So the same method of determining incompressibilities can be used on closed surfaces.

Let S' be an incompressible, pairwise incompressible surface in standard position and let S be a surface in standard position that is the result of peripheral tubing of S' with annuli that lie on peripheral tori in standard position. (If T is a peripheral torus in standard position then $T \cap S^2_\pm$ is made up of loops that encircle components of $L \cap S^2_\pm$ as shown in Figure 17.)

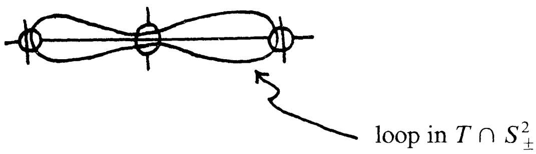


FIGURE 17

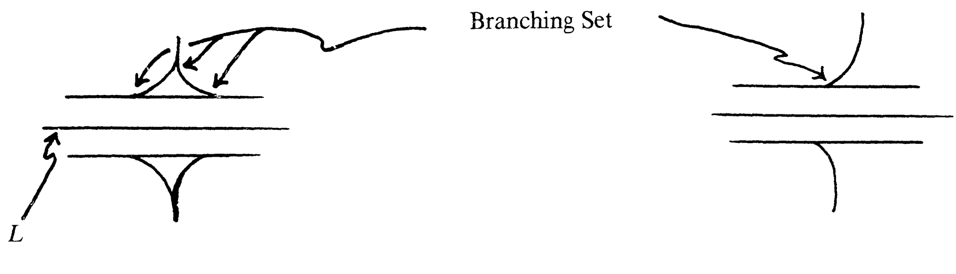
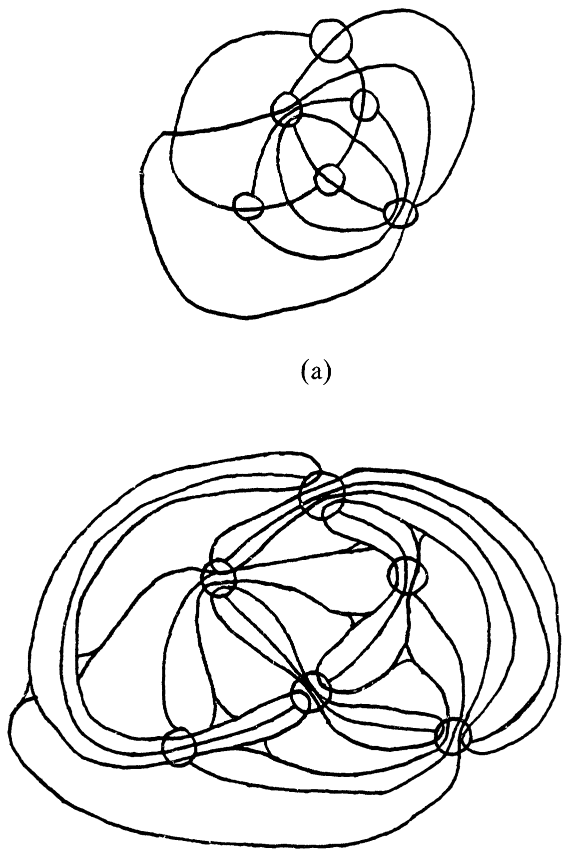


FIGURE 18

If B is a branched surface that carries S then away from the branching set $B \cap S^2_\pm$ looks like either $T \cap S^2_\pm$ or $S' \cap S^2_\pm$. Near the branching set (which is near where $S' \cap S^2_\pm$ intersects L) $B \cap S^2_\pm$ resembles Figure 18. Figure 18a(b) corresponds to a branching set of the type shown in Figure 16a(b).

To determine whether there exists a B associated with S' satisfying the conditions of Theorem 3.2 of [**O**] we first assume that all branching sets are of the type shown in Figure 16a. The only resulting components of $\partial_h N$ that must be checked for incompressibility are components that correspond to interior and exterior peripheral tubing of S'. If both of these components are incompressible then the corresponding B carries only incompressible surfaces. If either component is compressible we locate all standard discs and replace appropriate branching sets of type Figure 16a with the type in Figure 16b, ensuring along the way that no monogons are added. Finally we check to see if the resulting branched surface carries a surface.



(a)

(b)

FIGURE 19

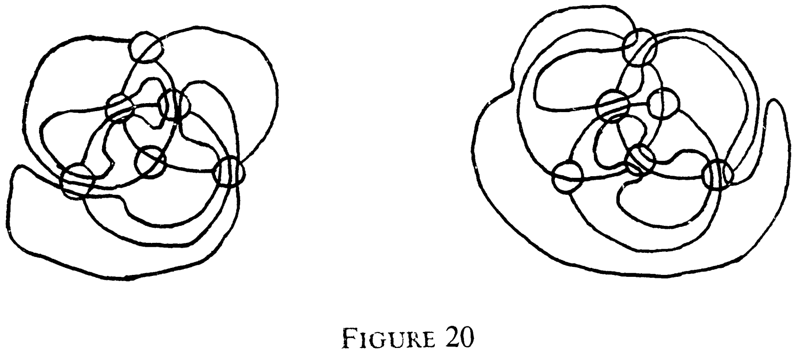


FIGURE 20

At this point an example may be useful. Figure 19a shows the borromean rings with an incompressible, pairwise incompressible 4-punctured sphere in a *PSPS*, *PSPS* pattern. The associated branched surface, where all of the branching set are of the type in Figure 16a, is shown in Figure 19b.

Figure 20 shows the two components of $\partial_h N$ that correspond to interior and exterior peripheral tubing. The reader should notice that these components violate (iii) of (***) i.e. the two sets of curves in Figure 20 bound annuli in $B^3_{\pm}$. This eliminates any possible choice for changing branching sets.

References

[F-O] W. Floyd and U. Oertel, *Incompressible surfaces via branched surfaces*, to appear in Pacific J. Math.

[M] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, to appear in Topology.

[O] U. Oertel, *Closed incompressible surfaces in complements of star links*, to appear in Pacific J. Math.

Received February 9, 1983 and in revised form September 15, 1983.

RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08903