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REGULARIZED DISTANCE AND ITS APPLICATIONS

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One of the most powerful tools in studying second order elliptic and parabolic differential equations is the barrier method, i.e. using the comparison principle with a suitable comparison or barrier function to infer some feature of the boundary behavior of a solution to such an equation. For sufficiently smooth domains Ω (e.g. $\partial\Omega \in C^2$), barrier functions can be constructed rather easily in terms of the distance function $d(x) = \operatorname{dist}(x, \partial\Omega)$ because d is a C^2 function near $\partial\Omega$; for less smooth domains it need not be even C^1 (although it is Lipschitz continuous.) These less smooth domains are of interest and several authors have constructed barriers for certain such domains. We consider here a general method for constructing these barriers by introducing a regularized distance, described below.

Our concern here is primarily to develop a general theory of this regularized distance; we merely indicate some applications. We note that many of the ideas are not new and have appeared before in specialized circumstances. For example, essentially a regularized distance has been constructed by Triebel [**24**, §3.2.3] for arbitrary domains and by Nečas [**22**, Theorem 2.1] for Lipschitz domains. Both constructions are slightly different from the one given here. In addition we explore aspects of regularized distance not considered by Triebel or Nečas.

We mention also some authors who have used some of these additional properties of the regularized distance. Výborný used a regularized distance in [**25**] although he took the existence of such a function as the geometric characterization of his domains. Gilbarg and Hörmander constructed and used essentially a regularized distance for $C^{1+\alpha}$ domains in [**3**]. Finally Lieberman used a regularized distance both for $C^{1+\alpha}$ domains and for convex domains in [**18**]. Proofs of the properties asserted there are contained in the present work.

This work is organized as follows. We derive basic properties of the regularized distance, including its existence for arbitrary domains, in §1. A local regularized distance for Lipschitz domains is discussed in §2, and a modified regularized distance especially suited for parabolic equations is constructed in §3. Some simple applications appear in §4.

1. Existence and basic properties of regularized distance. Let Ω be an open subset of $\mathbf{R}^n$ having non-empty boundary $\partial\Omega$. We define the *signed distance* to $\partial\Omega$ by

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & x \in \Omega \\ -\operatorname{dist}(x, \partial\Omega) & x \notin \Omega. \end{cases}$$

We call a function ρ a *regularized distance* for Ω if $\rho \in C^2(\mathbf{R}^2 \setminus \partial\Omega) \cap \mathbf{C}^{0,1}(\mathbf{R}^n)$ and if the ratios $\rho(x)/d(x)$ and $d(x)/\rho(x)$ are positive and uniformly bounded for all $x \in \mathbf{R}^n \setminus \partial\Omega$. To construct a regularized distance, we use a modification of a standard mollification argument.

Lemma 1.1. *Let Ω be an open subset of $\mathbf{R}^n$ having non-empty boundary, and suppose there is a Lipschitz function g for which the ratios d/g and g/d are uniformly bounded and positive in $\mathbf{R}^n \setminus \partial\Omega$. Let L be a positive constant such that $|g(x) - g(y)| \leq \frac{1}{2}L|x - y|$ for all x and y in $\mathbf{R}^n$, let ϕ be a non-negative $C^2(\mathbf{R}^n)$ function with support in the unit ball such that $\int_{\mathbf{R}^n}\phi(z)\,dz = 1$ and define*

$$G(x, \tau)\int_{|z|<1} g(x - (\tau/L)z)\phi(z)\,dz.$$

Then a regularized distance is given by the equation

$$\rho(x) = G(x, \rho(x)). \tag{1.1}$$

Proof. To see that ρ is a regularized distance, we first investigate the properties of G.

For $\tau \neq 0$, we can write

$$G(x, \tau) = \left(-\frac{L}{\tau}\right)^n \int_{\mathbf{R}^n} g(z)\phi(L(x - z)/\tau)\,dz; \tag{1.2}$$

it is clear from this formula that $G \in C^2(\mathbf{R}^{n+1} \setminus \{(x, 0)\})$ since g is continuous and the integration is over a compact set. Moreover

$$G(x, \tau_1) - (x, \tau_2) = \int_{|z|<1} [g(x - (\tau_1/L)z) - g(x - (\tau_2/L)z)]\phi(z)\,dz,$$

so the choice of L implies that

$$\begin{aligned} |G(x, \tau_1) - G(x, \tau_2)| &\leq \int_{|z|<1} \tfrac{1}{2}L|\tau_1 - \tau_2|(|z|/L)\phi(z)\,dz \\ &\leq \tfrac{1}{2}|\tau_1 - \tau_2|. \end{aligned} \tag{1.3}$$

Similarly

$$|G(x_1, \tau) - G(x_2, \tau)| \leq \frac{1}{2}L|x_1 - x_2|. \tag{1.4}$$

We now consider (1.1.). By virtue of (1.3), (1.1) has a unique solution for every $x \in \mathbf{R}^n$, which can be found by iteration. If $\rho_0(x) \equiv 0$ and $\rho_m(x) = G(x, \rho_{m-1}(x))$, then $|\rho_m(x) - \rho_{m-1}(x)| \le \frac{1}{2}|\rho_{m-1}(x) - \rho_{m-2}(x)|$, so the infinite series $\Sigma_{m=1}^{\infty}(\rho_m(x) - \rho_{m-1}(x))$ is uniformly and absolutely convergent, with sum $\rho(x)$ a solution of (1.1). The uniqueness of ρ follows by noting that any two solutions ρ_1 and ρ_2 of (1.1) satisfy the inequality

$$|\rho_1(x) - \rho_2(x)| = |G(x, \rho_1(x)) - G(x, \rho_2(x))| \le \tfrac{1}{2}|\rho_1(x) - \rho_2(x)|.$$

Another application of (1.3) yields

$$G(x,0) - \tfrac{1}{2}|\rho(x)| \le \rho(x) \le G(x,0) + \tfrac{1}{2}|\rho(x)|;$$

since $G(x,0) = g(x)$, we see that $\frac{1}{2} \le \rho(x)/g(x) \le 2$ and hence that ρ/d and d/ρ are positive and uniformly bounded. Combining (1.3), (1.4) and the equation

$$\begin{aligned}\rho(x) - \rho(y) = &(G(x, \rho(x)) - G(x, \rho(y))) \\ &+ (G(x, \rho(y)) - G(y, \rho(y)))\end{aligned}$$

yields

$$|\rho(x) - \rho(y)| \le L|x - y|,$$

so ρ is Lipschitz. That ρ is C^2 follows from the implicit function theorem, and therefore ρ is a regularized distance.

It follows from the proof of Theorem 1.3 below that in fact $\rho \in C^3(\mathbf{R}^n \setminus \partial\Omega)$ and that $|D^k\rho| = O(|\rho|^{1-k})$ for $k = 0, 1, 2, 3$ (cf. [**15**, Lemma 4.13]). In addition it is clear that higher regularity of ρ can be obtained by increasing the regularity of ϕ. In particular if $\phi \in C^\infty(\mathbf{R}^n)$, then $\rho \in C^\infty(\mathbf{R}^n \setminus \partial\Omega)$ and $|D^k\rho| = O(|\rho|^{1-k})$ for all non-negative integers k.

Lemma 1.1 reduces the study of regularized distance to choosing a suitable function g. We illustrate this procedure with various choices for g relevant to the properties of Ω we wish to consider. First we choose $g \equiv d$ to obtain a result similar to that in [**24**, §3.2.3].

COROLLARY 1.2. *Every domain has a regularized distance.*

Proof. We need only verify that d is Lipschitz and use $g = d$ in Lemma 1. We shall show that $|d(x) - d(y)| \le |x - y|$. If x and y are both in Ω or if neither is in Ω, we proceed as in [**4**, §14.6]. Let $z \in \partial\Omega$ be a point such that $|y - z| = |d(y)|$. Then

$$|d(x)| \le |x - z| \le |x - y| + |d(y)|.$$

Reversing the roles of x and y, we obtain

$$|d(x) - d(y)| = \big|\, |d(x)| - |d(y)| \,\big| \le |x - y|.$$

On the other hand, if $x \in \Omega$ and $y \notin \overline{\Omega}$, let $z \in \partial\Omega$ be a point on the line segment joining x and y. Then

$$|x - y| = |x - z| + |z - y| \ge |d(x)| + |d(y)| = |d(x) - d(y)|.$$

Combining these two cases gives the desired result. □

Although this result has some independent interest, there are certain desirable features (cf. [**4**, §14.6]) of the distance function for C^2 domains which do not seem to have an analog for the regularized distance constructed in this corollary. The two such features we shall use in the application are (1) that $|D\rho|$ be bounded away from zero near $\partial\Omega$, and (2) that geometric properties of Ω be represented by analytic properties of ρ. To make (1) precise, we say that a regularized distance is *proper* if there are positive constants c_1 and c_2 such that $|D\rho(x)| \ge c_1$ whenever $0 < |\rho(x)| \le c_2$. With regard to (2), we consider both regularity and convexity. We shall describe regularity of $\partial\Omega$ in terms of the regularity of the function g of Lemma 1. The connection with regularity in terms of a local representation of $\partial\Omega$ will be discussed in the next section.

THEOREM 1.3. *Let Ω and g be as in Lemma* 1.1 *and suppose that $g \in C^1(\mathbf{R}^n)$. Then Ω has a C^1 regularized distance. Let ζ be a continuous, increasing function with $\zeta(0) = 0$. If*

$$|Dg(x) - Dg(y)| \le \zeta(|x - y|) \quad \text{for all } x, y \text{ in } \mathbf{R}^n \tag{1.6}$$

(so Dg is uniformly continuous), if L and ϕ are as in Lemma 1, *and if*

$$K = \int_{|z|<1} |D\phi(z)|\,dx,$$

then ρ is proper and for $i, j = 1, \ldots, n$ we have

$$|D\rho(x) - D\rho(y)| \le 8\zeta(|x - y|) \quad \text{for all } x, y \text{ in } \mathbf{R}^n \tag{1.7a}$$

$$|D_{ij}\rho(x)| \le 2(4K + 3n + 1)\big(L/|\rho(x)|\big)\zeta\big(|\rho(x)|/L\big) \quad \text{for all } x \in \mathbf{R}^n \setminus \partial\Omega. \tag{1.7b}$$

Proof. First we note that G is C^1 since g is C^1 and use the implicit function theorem to see that ρ is C^1.

Before proceeding further, we simply notation as follows. Subscripts on G will denote partial derivatives, τ will be identified with x_{n+1} and the argument $\rho(x)$ will be suppressed from G and its derivatives. Thus

$$G_i(x) = \frac{\partial G}{\partial x_i}(x, \rho(x)), \qquad G_{n+1}(y) = \frac{\partial G}{\partial \tau}(y, \rho(y)), \quad \text{etc.}$$

Also we denote by $G'(x)$ the vector $(G_1(x), \ldots, G_n(x))$.

Differentiating (1.1) yields

$$D\rho(x) - D\rho(y) = (G'(x) - G'(y)) + D\rho(y)(G_{n+1}(x) - G_{n+1}(y)) + G_{n+1}(x)(D\rho(x) - D\rho(y)).$$

Combining this equation with (1.3) and (1.5), we see that

$$|D\rho(x) - D\rho(y)| \leq 2|G'(x) - G'(y)| + 2L|G_{n+1}(x) - G_{n+1}(y)|. \tag{1.8}$$

to obtain (1.7a) we estimate the two terms on the right side of (1.8).

Since $g \in C^1$, we can differentiate under the integral to obtain

$$G'(x) = \int_{|z|<1} Dg(x - (\rho(x)/L)z)\phi(z)\, dz \tag{1.9}$$

$$G_{n+1}(x) = -1/L \int_{|z|<1} z \cdot Dg(x - (\rho(x)/L)z)\phi(z)\, dz.$$

Hence

$$\begin{aligned} &|G'(x) - G'(y)| \\ &\quad \leq \int_{|z|<1} |Dg(x - (\rho(x)/L)z) - Dg(y - (\rho(x)/L)z)|\phi(z)\, dz \\ &\quad\quad + \int_{|z|<1} |Dg(y - (\rho(x)/L)z) - Dg(y - (\rho(y)/L)z)|\phi(z)\, dz \\ &\quad \leq \int_{|z|<1} \big(\zeta(|x-y|) + \zeta(|\rho(x) - \rho(y)|/L)\big)\phi(z)\, dz \\ &\quad \leq 2\zeta(|x-y|) \qquad \text{by (1.5)}. \end{aligned}$$

Similarly

$$|G_{n+1}(x) - G_{n+1}(y)| \leq 2\zeta(|x-y|)/L.$$

Inserting these last two estimates in (1.8) gives (1.7a).

To derive (1.7b), we differentiate (1.1) twice to obtain

$$\begin{aligned} D_{ij}\rho(x) = 1/(1 - G_{n+1}(x)) &\cdot \big[G_{ij}(x) + G_{i,n+1}(x)D_j\rho(x) + G_{j,n+1}(x)D_i\rho(x) \\ &+ G_{n+1,n+1}(x)D_i\rho(x)D_j\rho(x)\big]. \end{aligned} \tag{1.10}$$

So to estimate $D_{ij}\rho$, we must evaluate and estimate the second derivatives of G. The evaluations are routine but tedious. We take the equations (1.9) with $\rho(x)$ replaced by τ, make the substitution $y = x - (\tau/L)z$ (to obtain equations similar to (1.2)), differentiate, and then convert back to the integration variable z. The result is that

$$(1.11)\quad G_{ij}(x) = -(L/\rho(x))\int_{|z|<1} D_i g(x-(\rho(x)/L)z)D_j\phi(z)\,dz$$

$$G_{i,n+1}(x) = -(1/\rho(x))\int_{|z|<1} D_i g(x-(\rho(x)/L)z)d_j\big(z_j\phi(z)\big)\,dz$$

$$G_{n+1,n+1}(x) = -(1/(L\rho(x)))\int_{|z|<1} D_k g(x-(\rho(x)/L)z)D_j\big(z_k z_j\phi(z)\big)\,dz.$$

Since ϕ has compact support,

$$\int_{|z|<1} D_j\phi(z)\,dz = 0$$

and hence

$$G_{ij}(x) = -(L/\rho(x))\int_{|z|<1}\big[D_i g(x-(\rho(x)/L)z) - D_i g(x)\big]D_j\phi(z)\,dz$$

with analogous expressions for $G_{i,n+1}(x)$ and $G_{n+1,n+1}(x)$. The integrals of these expressions are estimated by noting that

$$|D_i g(x-(\rho(x)/L)z) - D_i g(x)| \le \zeta(|\rho(x)|/L)$$

so

$$|G_{ij}(x)| \le KL\zeta(|\rho(x)|/L)/|\rho(x)|$$
$$|G_{i,n+1}(x)| \le (n+K)\zeta(|\rho(x)|/L)/|\rho(x)|$$
$$|G_{n+1,n+1}(x)|l \le (n+1+K)\zeta(\,|\rho(x)/L)/(|\rho(x)|L).$$

Inequality (1.7b) follows readily from these inequalities in conjunction with (1.3), (1.5), and (1.10).

By virtue of the continuity of $D\rho$, we infer that ρ is proper provided $|D\rho|$ is bounded away from zero on $\partial\Omega$. Now if $x \in \partial\Omega$, then

$$|D\rho(x)| = |Dg(x)|\Big/\left(1 + Dg(x)\cdot\int_{|z|<1} z\phi(z)\,dz\right)$$
$$\ge 2|Dg(x)|/(2+L)$$

since $|Dg| \le \frac{1}{2}L$. Thus ρ is proper if $|Dg|$ is bounded away from zero on $\partial\Omega$, so let $x \in \partial\Omega$.

Suppose first that Ω satisfies an interior sphere condition at x with center x_0, and let M be a positive constant such that $g(y) \ge Md(y)$ for all $y \in \mathbf{R}^n$. If ω is the vector from x to x_0, it is clear that

$$d(x+\omega) = t|\omega| \quad \text{for } 0 \le t \le 1,$$

and hence

$$g(x+\omega) - g(x) = g(x+t\omega) \ge Mt|\omega| \quad \text{for } 0 \le t \le 1.$$

Therefore $\omega \cdot Dg(x) \ge M|\omega|$, so $|Dg(x)| \ge M$.

To complete the proof, we need only show that the set of points of $\partial\Omega$ at which Ω satisfies an interior sphere condition is dense in $\partial\Omega$, so let $x \in \partial\Omega$ and $\delta > 0$ be arbitrary, and choose $x_1 \in \Omega$ and $x_2 \in \partial\Omega$ such that

$$|x - x_1| < \delta/2, \quad |x_1 - x_2| = d(x_1).$$

Since $d(x_1) < \delta/2$, it follows that $|x - x_2| < \delta$; also Ω satisfies an interior sphere condition (*with center* x_1 *and radius* $d(x_1)$) at x_2. Thus the set of points of $\partial\Omega$ at which Ω satisfies an interior sphere condition is dense in $\partial\Omega$. From the continuity of Dg it follows that $|Dg| \ge M$ on $\partial\Omega$ and hence ρ is proper. □

We remark that a regularized distance constructed with smoother g or ϕ will obey an estimate on its higher derivatives analogous to (1.7). Moreover modulus of continuity estimates for these higher derivatives can be obtained in terms of the function ζ (cf. [**3**, Lemma 2.8] and Theorem 4.1 below) under the hypotheses of the preceding theorem. We remark also that the condition that g/d and d/g be uniformly bounded is equivalent, in this case, to Dg being bounded away from zero on $\partial\Omega$.

We now consider convex domains. For technical reasons (which will become clear later), we only consider bounded convex domains.

THEOREM 1.4. *Let Ω be an open, bounded, convex subset of $\mathbf{R}^n$. Then Ω has a concave proper regularized distance ρ, i.e. the matrix $(D_{ij}\rho(x))$ is negative semi-definite for $x \in \mathbf{R}^n \setminus \partial\Omega$.*

Proof. Without loss of generality, we may assume that $0 \in \Omega$. Define the function $h(x)$ by

$$h(x) = \inf\{\lambda > 0 \colon x \in \lambda\Omega\}.$$

(This is the well-known Minkowski distance function. Although all of our assertions concerning h can be derived from known results in the theory of convexity, e.g. from results on [17], we shall give elementary direct proofs.) We shall show that

h is convex, i.e., $h(tx + (1-t)y) \le th(x) + (1-t)h(y)$ if $0 < t < 1$

h is uniformly Lipschitz, $|x|/|h(x)|$ is uniformly bounded for $x \ne 0$.

To prove the convexity, suppose $x \in \lambda\Omega$ and $y \in \mu\Omega$, say $x = \lambda x_1$, $y = \mu y_1$. Then

$$tx + (1-t)y = t\lambda x_1 + (1-t)\mu y_1.$$

Setting $\alpha = t\lambda/(t\lambda + (1-t)\mu)$ and noting that $0 < \alpha < 1$, we obtain

$$tx + (1-t)y = (t\lambda + (1-t)\mu)[\alpha x_1 + (1-\alpha)y_1].$$

Since Ω is convex, this equation implies that

$$tx + (1-t)y \in (t\lambda + (1-t)\mu)\Omega$$

so $h(tx + (1-t)y) \le t\lambda + (1-t)\mu$, and hence h is convex.

To prove the uniform Lipschitz continuity, set $r = 2/d(0)$, and suppose $x \in \lambda\Omega$, say $x = \lambda x_1$. Then

$$y = \lambda x_1 + y - x = \lambda x_1 + [(1/r)(y-x)/|y-x|]r|y-x|.$$

Clearly $z = (1/r)(y-x)/|y-x| \in \Omega$ since $|z| = 1/r \le d(0)/2$, i.e. z is closer to 0 than $\partial\Omega$ is. Thus, setting $\alpha = \lambda/(\lambda + r|y-x|)$,

$$y = \lambda x_1 + r|y-x|z = (\lambda + r|y-x|)(\alpha x_1 + (1-\alpha)z),$$

so $h(y) \le h(x) + r|y-x|$. Upon reversing the roles of x and y, we obtain

$$|h(x) - h(y)| \le r|x-y|.$$

To obtain a bound on $|x|/|h(x)|$, suppose $|y| \le K$ for all $y \in \Omega$. Then for $x \ne 0$, we have

$$|(1/2K)|x|y| \le |x|/2 + |x|$$

for all $y \in \Omega$ and hence $x \notin (|x|/2K)\Omega$. Since Ω is convex and $0 \in \Omega$, it follows that $x \notin \lambda\Omega$ for all $\lambda \le |x|/2K$. Hence $|h(x)| \ge |x|/2K$ for $x \ne 0$, so $|x|/|h(x)| \le 2K$.

Let us now define

$$g(x) = 1 - h(x),$$

and note that g is uniformly Lipschitz since h is. Since $g = 0$ on $\partial\Omega$, it follows that if $|x - z| = d(x)$ and $z \in \partial\Omega$, then

$$|g(x)| = |g(x) - g(z)| \le r|x-z| = r|d(x)|.$$

Also it is readily seen that $(1/h(x))x \in \partial\Omega$, so

$$|d(x)| \leq |x \in (1/h(x))x| = (|x|/|h(x)|)|g(x)| \leq 2K|g(x)|.$$

Clearly $g > 0$ in Ω and $g < 0$ outside $\overline{\Omega}$, so the ratios g/d and d/g are positive and uniformly bounded. Therefore g satisfies the hypotheses of Lemma 1. Let G and ρ be as in that lemma. Since g is concave, ϕ is non-negative, and

$$\begin{aligned} &G(tx + (1-t)y, t\tau + (1-t)\sigma) \\ &\quad = \int_{|z|<1} g(t[x - (\tau/L)z] + (1-t)[y - (\sigma/L)z])\phi(x)\,dz, \end{aligned}$$

it follows that G is jointly concave in the variables x and τ. We now define ρ_m inductively by $\rho_0 \equiv 0$ and

$$\rho_m(x) = \tfrac{2}{3}G(x, \rho_{m-1}(x)) + \tfrac{1}{3}\rho_{m-1}(x),$$

and observe that $\rho(x) = \lim_{m\to\infty} \rho_m(x)$. Now if ρ_{m-1} is concave, then

$$\begin{aligned} \rho_m(tx + (1-t)y) &= \tfrac{2}{3}G(tx + (1-t)y, \rho_{m-1}(tx + (1-t)y)) \\ &\quad + \tfrac{1}{3}\rho_{m-1}(tx + (1-t)y) \\ &\geq \tfrac{2}{3}G(tx + (1-t)y, t\rho_{m-1}(x) + (1-t)\rho_{m-1}(y)) \\ &\quad + \tfrac{1}{3}|\rho_{m-1}(tx + (1-t)y) - t\rho_{m-1}(x) - (1-t)\rho_{m-1}(y)| \\ &\quad + \tfrac{1}{3}\rho_{m-1}(tx + (1-t)y) \\ &= \tfrac{2}{3}G(tx + (1-t)y, t\rho_{m-1}(x) + (1-t)\rho_{m-1}(y)) \\ &\quad + \tfrac{1}{3}(t\rho_{m-1}(x) + (1-t)\rho_{m-1}(y)). \end{aligned}$$

Since ρ_0 and G are concave, sending $m \to \infty$ shows that ρ is concave. The semi-definiteness of $(D_{ij}\rho)$ is proved by standard calculus arguments (see [**1**], Thm. 3.6).

To show that ρ is proper, we observe that

$$h(tx) = th(x) \quad \text{for all } t > 0 \text{ and all } x \in \mathbf{R}^n,$$

and hence that

$$G(tx, t\tau) = (1-t) + tG(x, \tau) \quad \text{for all } t > 0 \text{ and all } (x, \tau) \in \mathbf{R}^{n+1}.$$

Differentiating this equation with respect to t, setting $t = 1$ and $\tau = \rho(x)$ yields

$$x^i G_i(x) + \rho(x)G_{n+1}(x) = -1 + \rho(x).$$

This equation and the equation for $D\rho$ imply that

$$x \cdot D\rho(x) = \rho(x) + (1/(G_{n+1}(x) - 1)).$$

If $0 < |\rho(x)| \le 1/3$, it follows from this equation and (1.3) that

$$|D\rho(x)| \ge 1/(3|x|).$$

Since $|d(x)| \ge |\rho(x)|/2r$ and $|x| \le K$, we conclude that ρ is proper. □

We remark that the regularized distance constructed in this theorem is as regular as the domain (in the sense used in Theorem 1.3). To see this, let g_1 be a function with the properties described in Theorem 1.3 on a bounded convex domain Ω. Then $h(x)$ is defined implicitly by the equation $g_1(x/h(x)) = 0$. But

$$\frac{d}{dt}(g_1(x/t)) = -x \cdot Dg_1(x/t)/t^2,$$

and it is easy to check that this is non-zero in a neighborhood of $\partial\Omega$. Hence h has the same regularity as g_1 near $\partial\Omega$, and therefore so does ρ. Of course away from $\partial\Omega$, ρ is C^2. This observation will not be used in the applications.

We fruther remark that it can be shown that the regularized distance from the proof of Corollary 1.2 is concave if Ω is convex; however it is not apparent that this regularized distance is proper.

2. Local regularized distance. From the construction of regularized distance in §1, it is clear that local properties of ρ are determined by local properties of g. In this section we explore some of these local properties and their relationship with properties of Ω. Everything we wish to consider can be described via a local representation of $\partial\Omega$, defined below. We remark that our basic construction parallels that of [**22**, Theorem 2.1] (see also [**15**, §4]) although the idea of a proper regularized distance does not appear there.

To be specific, let $x_0 \in \partial\Omega$, let $r > 0$, and define

$$B_r = \{x \in \mathbf{R}^n : |x - x_0| < r\}, \qquad \Omega_r = B_r \cap \Omega.$$

If there is a constant $r > 0$ and a function $\rho \in C^2(B_r \setminus \partial\Omega)$ which is Lipschitz continuous on $\overline{B}_r$ and such that the ratios d/ρ and ρ/d are bounded and positive on $B_r \setminus \partial\Omega$, we call ρ a *local regularized distance* at x_0. If there are positive constants δ and A, an orthonormal coordinate system $Y = (y', y^n) = (y^1, \ldots, y^{n-1}, y^n)$ with origin at x_0, and a function f such that

$$\Omega_{4\delta} = \{y \in B_{4\delta} : y^n > f(y')\}$$

and

$$|f(y_1') - f(y_2')| \le A|y_1' - y_2'| \quad \text{for } y_1', y_2' \text{ in } B_{4\delta},$$

we call f a *local representation* for $\partial\Omega$ at x_0. We note that not every domain has a local representation at every point of its boundary.

For a fixed $x_0 \in \partial\Omega$ at which a local representation f exists, it is easy to verify that $g(y) = y^n - f(y')$ obeys the hypotheses of Lemma 1.1 with $\mathbf{R}^n$ replaced by $B_{2\delta}$. Clearly g is Lipschitz in $B_{2\delta}$ with

$$|g(x) - g(y)| \leq \max\{1, A\}|x - y| \quad \text{for all } x,\, y \text{ in } B_{2\delta},$$

and the ratios d/g and g/d are positive in $B_{2\delta}$ with

$$|d(x)| \leq |g(x)| \leq (A^2 + 1)^{1/2}|d(x)| \quad \text{for } x \in B_{2\delta}.$$

If we choose $L = 2\max\{1, A\}$, the construction of ρ in Lemma 1.1 can be carried out for $x \in B_\delta$. To see this, we observe first that $G(x, \tau)$ is defined for $x \in B_\delta$ and $|\tau|/L \leq \delta$. Since $f(0) = 0$, it follows from (1.4) and the definition of g that $|G(x, \tau)| \leq \frac{1}{2}L|x| + \frac{1}{2}\tau$. Hence $G(x, \cdot)$ is a self-map of the interval $[-L\delta, L\delta]$ if $x \in B_\delta$, so the construction of ρ in Lemma 1.1 can be carried out in this case. Moreover $|\rho|/L < \delta$. Finally ρ is proper because $g(x + \varepsilon\omega) - g(x) = \varepsilon$ for ω a unit vector in the y^n-direction, so $G_n(x) \equiv 1$ in B_δ. In fact it is clear that $|D\rho| \geq 1/2$ in B_δ.

When additional hypotheses are placed on f, they imply additional properties for ρ.

THEOREM 2.1. *Let $x_0 \in \partial\Omega$. If $\partial\Omega$ has a C^1 local representation f at x_0, then Ω has a C^1 local regularized distance ρ at x_0. Specifically if ζ is a increasing function such that $\zeta(0) = 0$ and if*

$$|Df(x') - Df(y')| \leq \zeta(|x - y|) \quad \textit{for } |x'|,\, |y'| < 4\delta,$$

then there is a positive constant C_1, determined only by A, δ, and n, such that

$$|D\rho(x) - D\rho(y)| \leq 10\zeta(|x - y|) \quad \textit{for } x,\, y \textit{ in } B_\delta$$
$$|D^2\rho(x)| \leq C_1\zeta(|\rho(x)|/2\max\{1, A\})/|\rho(x)| \quad \textit{for } x \in B_\delta. \qquad \square$$

We say that Ω is convex at $x_0 \in \partial\Omega$ if Ω_r is convex for some $r > 0$. When Ω is convex at x_0, a elementary calculation (*cf.* [**1**, *Proposition* 3.4] shows that $\partial\Omega$ has a convex local representation at x_0. From this fact, we can proceed as in the last half of the proof of Theorem 1.4 to infer the next theorem.

THEOREM 2.2. *Let $x_0 \in \partial\Omega$ be a point at which Ω is convex. Then Ω has a concave local regularized distance at x_0.* $\square$

Finally we show that the two senses of regularity (that of Theorem 2.1 and that of Theorem 1.3) are essentially equivalent.

THEOREM 2.3. *Let Ω be an open set in $\mathbf{R}^n$. If there is a function $g \in C^1(\mathbf{R}^n)$ such that the hypotheses of Lemma* 1.1 *are satisfied, then there is a local represesntation for $\partial\Omega$ at each point of $\partial\Omega$. If there is a local representation for $\partial\Omega$ at each point of $\partial\Omega$ and if the constants A and δ in the definition of local representation can be chosen independent of the particular point x_0, then there is a function $g \in C^1(\mathbf{R}^n)$ which satisfies the hypotheses of Lemma* 1.1.

Proof. The first implication is a simple consequence of the implicit function theorem. The second implication follows by a construction of the function g.

Let $\{B_i\}$ be a sequence of open balls of radius δ such that $\bigcup_{i=1}^{\infty} B_i = \mathbf{R}^n$ and such that there is a positive integer N such that each point in $\mathbf{R}^n$ lies in at most N of these balls. (For example, the B_i's could be centered at all points whose coordinates are integer multiplies of δ/n). If $B_i \cap \partial\Omega$ is non-empty, then let ρ_i be a C^1 regularized distance defined in B_i and if $B_i \cap \partial\Omega$ is empty let ρ_i be the regularized distance for Ω constructed in Corollary 1.2. Let (η_i) be a C^2 partition of unit subordinate to (B_i) with $\sup_i \sup_{B_i} |D\eta_i|$ finite and define

$$g(x) = \sum_{i=1}^{\infty} \eta_i(x)\rho_i(x).$$

It is clear that $g \in C^1(\mathbf{R}^N)$ and that g is uniformly Lipschitz on $\mathbf{R}^n$. Since there are positive constants c_1 and c_2 such that

$$c_1|d(x)| \le |\rho_i(x)| \le c_2|d(x)| \quad \text{for } i = 1,2,\ldots \quad \text{and} \quad x \in \mathbf{R}^n,$$

it follows that g is the required function. □

Of course detailed information about the modulus of continuity of the function g can be obtained from a knowledge of the moduli of continuity of the local representations and vice versa. A closer examination of certain aspects of the regularity of domains, especially comparison of different definitions, can be found in [**2**].

3. Anisotropic regularized distance. In certain circumstances, e.g., for parabolic equations, the independent variables are not all treated equally. Rather than discuss this situation in its fullest generality, we shall work here only with an anisotropic regularized distance suitable for parabolic equations.

To conform with normal usage, we change our notation slightly. We consider open sets Q in $\mathbf{R}^{n+1}$ and label points in $\mathbf{R}^{n+1}$ by $(x, t) = (x^1,\ldots,x^n, t)$. When g is defined on a subset of $\mathbf{R}^n$, we set

$$Dg = (D_1g,\ldots,D_ng), \quad g_t = D_{n+1}g, \text{ etc.}$$

provided the derivatives exist. For any open set $Q \subset \mathbf{R}^{n+1}$ and $t \in \mathbf{R}$, we define

$$Q(t) = \{x \in \mathbf{R}^n\colon (x, t) \in Q\}, \qquad I = \{t \in \mathbf{R}\colon Q(t) \neq \varnothing\}.$$

For $(x, t) \in \mathbf{R}^n$ let $d_p(x, t)$ be the signed distance from (x, t) to $\partial Q(t)$, and for $(x_0, t_0) \in \mathbf{R}^{n+1}$ and $r > 0$, we define

$$C_r = \left\{(x, t) \in \mathbf{R}^{n+1}\colon |x - x_n| < r, |t - t_0| < r^2\right\}, \qquad Q_r = C_r \cap Q.$$

By a *regularized parabolic distance*, we mean a function ρ such that

(i) $D\rho$, $D^2\rho$, and ρ_t exist and are continuous on $\mathbf{R}^n \times I \setminus \partial Q$,

(ii) there is a constant C such that

$$|\rho(x_1, t_2) - \rho(x_2, t_2)| \leq C\left(|x_1 - x_2| + |t_1 - t_2|^{1/2}\right),$$

(iii) the ratios ρ/d_p and d_p/ρ are bounded and continuous on $\mathbf{R}^n \times I \setminus \partial Q$. Let $(x_0, t_0) \in \partial Q$. If there is a positive constant r with $[t_0 - r, t_0 + r] \subset I$ and a function ρ satisfying the definition of regularized parabolic distance with $\mathbf{R}^n \times I$ replaced by C_r, we call ρ a *local regularized parabolic distance* at (x_0, t_0).

For simplicity we consider only local regularized parabolic distances, which we call regularized distances in C_r for brevity.

If there are constants δ and A, an orthonormal coordinate system $Y = (y', y^n)$ with origin at x_0 and a function f such that

$$Q_{4\delta} = \{(y, t) \in C_{4\delta}\colon y^n > f(y', t)\}$$

and

$$|f(y_1', t_1) - f(y_2', t_2)| \leq A\left(|y_1' - y_2'| + |t_1 - t_2|^{1/2}\right)$$

$$\text{for } |y_1'|, |y_2'|, |t_1|^{1/2}, |t_2|^{1/2} \leq 4\delta,$$

we call f a local representation for ∂Q at (x_0, t_0). We note that if f is a local representation for ∂Q at (x_0, t_0), then (x_0, t_0) must lie on the lateral surface of ∂Q (see **[11]** for the definition).

From this f we can construct a regularized distance in C_δ. We let ϕ be a non-negative $C^2(\mathbf{R}^{n-1})$ function with support in $\{|z| < 1\}$ such that $\int_{|z|<1}\phi(z)\,dz = 1$, we let η be a non-negative $C^2(\mathbf{R})$ function with support in $(-1, 0)$ with $\int_{-1}^0\eta(s)\,ds = 1$, and we set

$$K = \int_{-1}^0 |\eta'(s)|ds + \int_{|z|<1} |D\phi(z)|dz + \int_{-1}^0 |\eta''(s)|ds;$$

note that $K \geq 2$. Setting $L = 4(A^2 + 1)^{1/2}$ and

$$F(y, t, \tau, \sigma)$$
$$= y^n - \int_{-1}^0 \int_{|z|<1} f\left(y' - (\tau/L)z, t - (\sigma/K^2L^2)s\right)\phi(z)\,dz\,\eta(s)\,ds,$$

for $(y, t) \in C_\delta$, $|\tau| < L\delta$, $|\sigma| < L^2\delta^2$, it is readily verified that

$$|D_i F(y, t, \tau, \sigma)| \le L/4, \quad |D_\sigma F(y, t, \tau, \sigma)| \le 1/(4\sigma^{1/2}),$$

$$|D_\tau F(y, t, \tau, \sigma)| \le 1/4.$$

If we set $G(y, t, \tau) = F(y, t, \tau, \tau^2/2)$, it follows that a regularized distance in C_δ is given implicitly by the equation

$$\rho(y, t) = G(y, t, \rho(y, t)). \tag{3.1}$$

Although convexity is not of particular concern for such equations, we desire a parabolic analog of the condition $f \in C^1(B_{4\delta})$. To describe the analog, consider an increasing, continuous function ζ such that $\zeta(\cdot)/\cdot$ is decreasing. We then assume of f that

$$|Df(y_1', t_1) - Df(y_2', t_2)| \le \zeta(|y_1' - y_2'|) + \zeta\left(|t_1 - t_2|^{1/2}\right) \tag{3.2a}$$

$$|f(y, t_1) - f(y, t_2)| \le |t_1 - t_2|^{1/2}\zeta\left(|t_1 - t_2|^{1/2}\right). \tag{3.2b}$$

(When $\zeta(s) = Cs^\alpha$ for constants $\alpha < 1$ and C, this definition says that $f \in H^{1+\alpha, 1/2+\alpha/2}$ in the sense of **[16]** and that $\partial Q \cap C_{4\delta}$ is a surface of class $\Lambda^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$ in the sense of [7].) from these inequalities we can infer estimates on $D\rho$, $D^2\rho$, and ρ_t analogous to (1.7).

THEOREM 3.1. *Let Q be an open set on $\mathbf{R}^{n+1}$, let $(x_0, t_0) \in \partial Q$, and suppose ∂Q has a local representation f at (x_0, t_0) for which* (3.2) *holds. Then there is a positive constant, $C = C(K, L, n)$ such that the regularized distance ρ constructed above obeys the estimates*

$$|D\rho(x_1, t_1) - D\rho(x_2, t_2)| \le C\left[\zeta(|x_1 - x_2|) + \zeta\left(|t_1 - t_2|^{1/2}\right)\right] \tag{3.3a}$$

$$|D^2\rho(x, t)| \le C\zeta(|\rho(x, t)|/L)/|\rho(x, t)| \tag{3.3b}$$

$$|\rho_t(x, t)| \le C\zeta(|\rho(x, t)|/L)/|\rho(x, t)|. \tag{3.3c}$$

Proof. To verify (3.3a), we proceed as in Theorem 1.3 making judicious use of (3.2a, b). For brevity, we write $\rho_1 = \rho(x_1, t_1)$, $\rho_2 = \rho(x_2, t_2)$, and $f(j, k, m, r) = f(x_j' - (\rho_k/L)z,\ t_m - (\rho_r^2/2K^2L^2)s)$ and similarly for derivatives of f. Also we assume, by relabelling points if necessary, that $|\rho_1| \ge |\rho_2|$.

An examination of the proof of Theorem 1.3 and the fact that $G_\tau = F_\tau + \tau F_\sigma$ show that we need only estimate

$$A_1 = DF(x_1, t_1, \rho_1, \rho_1^2/2) - DF(x_1, t_1, \rho_1, \rho_2^2/2)$$

and

$$B_1 = \rho_1 F_\sigma(x_1, t_1, \rho_1, \rho_1^2/2) - \rho_2 F_\sigma(x_2, t_2, \rho_2, \rho_2^2/2).$$

To estimate A_1, we examine the two cases $(|\rho_1^2 - \rho_2^2|/2K^2L^2)^{1/2} \le |\rho_1 - \rho_2|/2L$ and $(|\rho_1^2 - \rho_2^2|/2K^2L^2)^{1/2} > |\rho_1 - \rho_2|/2L$. In the first case, we write

$$A_1 = \int_{-1}^{0}\int_{|z|<1} [Df(1,1,1,1) - Df(1,1,1,2)]\,\phi(z)\,dz\,\eta(s)\,ds$$

and use (3.2a) and the monontonicity of ζ to infer that

$$|A_1| \le \zeta(|\rho_1 - \rho_2|/2L).$$

In the second case, we write

$$A_1 = (-L/\rho_1)\int_{-1}^{0} [f(1,1,1,1) - f(1,1,1,2)]\,D\phi(z)\,dz\,\eta(s)\,ds$$

and use (3.2b) and the monotonicity of $\zeta(\cdot)/\cdot$ to infer that

$$|A_1| \le |\rho_1 + \rho_2|\zeta(|\rho_1 - \rho_2|/2L)/K|\rho_1|.$$

Since $K \ge 2$, $|\rho_1| \ge |\rho_2|$, and $|\rho_1 - \rho_2| \le 2L\max\{|x_1 - x_2|, |t_1 - t_2|^{1/2}\}$, it follows that in either case

$$|A_1| \le \zeta(|x_1 - x_2|) + \zeta(|t_1 - t_2|^{1/2}).$$

To estimate B_1, we observe that

$$\begin{aligned} B_1 = {} & \rho_1[F_\sigma(x_1, t_1, \rho_1, \rho_1^2/2) - F_\sigma(x_1, t_1, \rho_1, \rho_1^2/2)] \\ & + \rho_1[F_\sigma(x_2, t_1, \rho_1, \rho_1^2/2) - F_\sigma(x_2, t_2, \rho_1, \rho_1^2/2)] \\ & + [\rho_1 F_\sigma(x_2, t_2, \rho_1, \rho_1^2/2) - \rho_2 F_\sigma(x_2, t_2, \rho_2, \rho_1^2/2)] \\ & + \rho_2[F_\sigma(x_2, t_2, \rho_2, \rho_1^2/2) - F_\sigma(x_2, t_2, \rho_2, \rho_2^2/2)], \end{aligned}$$

and denote the terms in square brackets by B_2, B_3, B_4, B_5. For B_2 the two cases are $|x_1 - x_2| \le |\rho_1|/2^{1/2}KL$ and $|x_1 - x_2| > |\rho_1|/2^{1/2}KL$. In the first case we write

$$\begin{aligned} B_2 = (1/\rho_1^2)\int_{-1}^{0}\int_{|z|<1} & [f(1,1,1,1) - f(2,1,1,1)]\,\phi(z)\,dz \\ & \times(\eta(s) + s\eta'(s))\,ds \end{aligned}$$

and we observe that

$$\int_{-1}^{0} (\eta(s) + s\eta'(s))\,ds = 0 \tag{3.4}$$

and that

$$\begin{aligned} & f(1,1,1,1) - f(2,1,1,1) \\ & \quad = (x_1 - x_2)\cdot\int_0^1 Df(\lambda x_1 + (1-\lambda)x_2 - (\rho_1/L)z, t_1 \\ & \qquad\qquad - (\rho_1^2/2K^2L^2)s)\,d\lambda. \end{aligned}$$

Thus, writing $\bar{x} = \lambda x_1 + (1-\lambda)x_2 - (\rho_1/L)z$, $\bar{\eta}(s) = \eta(s) + s\eta'(s)$, we have

$$B_2 = \int_{-1}^{0}\int_{|z|<1}\int_{0}^{1}\left\{Df\left(\bar{x}, t_1 - \left(\rho_1^2/2K^2L^2\right)s\right) - Df\left(\bar{x}, t_1\right)\right\} d\lambda \times \phi(z)\,dz\,\bar{\eta}(s)\,ds$$

and hence, using (3.2a) and the monotonicity of $\zeta(\cdot)/\cdot$,

$$\begin{aligned} B_2 &\le (1+K)|x_1 - x_2|\zeta\left(|\rho_1|/2^{1/2}KL\right)/|\rho_1|^2 \\ &\le (1+K)\zeta\left(|x_1 - x_2|\right)/2^{1/2}KL|\rho_1|. \end{aligned}$$

In the second case we observe that

$$f(j,1,1,1) = (-\rho_1/L)\int_0^1 z\cdot Df\left(x_j - (\lambda\rho_1/L)z, t_1 - \left(\rho_1^2/2K^2L^2\right)s\right) d\lambda + f\left(x_j, t_1 - \left(\rho_1^2/2K^2L^2\right)s\right).$$

If we write $\bar{x}_j = x_j - (\lambda\rho_1/L)s$ and $\bar{t} = t_1 - (\rho_1^2/2K^2L^2)s$, define $\bar{\eta}$ as before, and use (3.4), we see that

$$\begin{aligned} B_2 = \left(1/\rho_1^2\right)\Bigg[&-(\rho_1/L)\int_{-1}^{0}\int_{|z|<1}\int_0^1 z\cdot\left\{Df\left(\bar{x}_1, \bar{t}\right) - Df\left(\bar{x}_2, \bar{t}\right)\right\} d\lambda \times\phi(z)\bar{\eta}(s)\,ds \\ &+ \sum_{j=1}^{2}(-1)^j\int_{-1}^{0}\int_{|z|<1}\left\{f\left(x_j, \bar{t}\right) - f\left(x_j, t_1\right)\right\}\phi(z)\,dz\,\bar{\eta}(s)\,ds\Bigg] \end{aligned}$$

and therefore, using (3.2) and the monotonicity of ζ

$$\begin{aligned} |B_2| &\le (1+K)\left[\zeta\left(|x_1 - x_2|\right)/L|\rho_1| + 2\zeta\left(|\rho_1|/2^{1/2}KL\right)/2^{1/2}KL|\rho_1|\right] \\ &\le 2(1+K)\zeta\left(|x_1 - x_2|\right)/L|\rho_1|. \end{aligned}$$

In either case this inequality is valid for B_2.

To estimate

$$B_3 = \left(1/\rho_1^2\right)\int_{-1}^{0}\int_{|z|<1}\left[f(2,1,1,1) - f(2,1,2,1)\right]\phi(z)\,dz \times\left(\eta(s) + s\eta'(s)\right) ds,$$

we consider the cases $|t_1 - t_2| \le \rho_1^2$ and $|t_1 - t_2| > \rho_1^2$. In the first case, we use the inequality

$$|f(2,1,1,1) - f(2,1,2,1)| \le |t_1 - t_2|^{1/2}\zeta\left(|t_1 - t_2|^{1/2}\right) \le |\rho_1|\zeta\left(|t_1 - t_2|^{1/2}\right).$$

In the second case, we use (3.4) to infer that

$$B_3 = \left(1/\rho_1^2\right)\int_{-1}^{0}\int_{|z|<1}\left([f(2,1,1,1) - f(x_2 - (\rho_1/L)z, t_1)]\right.$$
$$\left. + \left[f(x_2 - (\rho_1/L)z, t_2) - f(2,1,2,1)\right]\right)\phi(z)\,dz\,(\eta(s) + s\eta'(s))\,ds,$$

in which case

$$|B_3| \le \left(2/\rho_1^2\right)\left(|\rho_1|/2^{1/2}KL\right)\zeta\left(|\rho_1|/2^{1/2}KL\right)$$
$$\le \zeta\left(|\rho_1|\right)/|\rho_1| \le \zeta\left(|t_1 - t_2|^{1/2}\right)/|\rho_1|.$$

In either case, $|B_3| \le (1 + K)\zeta(|t_1 - t_2|^{1/2})|\rho_1|$.

To estimate B_4, we see that

$$B_4 = \int_{-1}^{0}\int_{|z|<1}\left[f(2,2,1,1)/\rho_1 - f(2,2,2,1)/\rho_2\right]\phi(z)\,dz$$
$$\times(\eta(s) + s\eta'(s))\,ds$$

by observing that

$$f(2,j,2,1) = -(\rho_1/L)\int_0^1 z\cdot Df\left(x_2 - \lambda(\rho_j/L)z, t_2 - \left(\rho_1^2/2K^2L^2\right)s\right)d\lambda$$
$$+ f\left(x_2, t_2 - \left(\rho_1^2/2K^2L^2\right)s\right)$$

and using (3.2a). Finally B_5 is estimated as B_3 was. It follows that there is a constant $C' = C'(K, L)$ such that

$$|B_1| \le C'\left[\zeta\left(|x_1 - x_2|\right) + \zeta\left(|t_1 - t_2|^{1/2}\right)\right],$$

which in conjunction with the estimate for A_1, implies (3.3a).

To prove (3.3b) and (3.3c), we note that, as in Theorem 1.3, we can obtain the following estimates:

$$\left|D_{ij}F(x,t,\tau,\sigma)\right| \le KL\zeta\left(|\tau|/L\right)/|\tau|$$
$$\left|D_{i\sigma}F(x,t,\tau,\sigma)\right| \le (1 + K)\zeta\left(\sigma^{1/2}/KL\right)/|\sigma|$$
$$\left|D_{i\tau}F(x,t,\tau,\sigma)\right| \le (n + K)\zeta\left(|\tau|/L\right)/|\tau|$$
$$\left|D_{\sigma\tau}F(x,t,\tau,\sigma)\right| \le (1 + K)\zeta\left(\sigma^{1/2}/KL\right)/(\sigma L)$$
$$\left|D_{\sigma\sigma}F(x,t,\tau,\sigma)\right| \le (1 + K)\zeta\left(\sigma^{1/2}/KL\right)/\left(KL\sigma^{3/2}\right)$$
$$\left|F_t(x,t,\tau,\sigma)\right| \le LK\zeta\left(\sigma^{1/2}/KL\right)/\sigma.$$

The desired estimates follow from these inequalities and the method of Theorem 1.3. □

4. Applications. We now describe briefly some applications of the regularized distance.

The first application is an extension of the Hopf boundary point lemma [**6**]; this extension is similar to results of Kamynin and Khimchenko [**12**]. Before stating the result, we give some definitions.

Suppose Ω has a C^1 local representation f at $x_0 \in \partial\Omega$. We say that Ω is *Dini* at x_0 if $Df(0) = 0$ and if there is a function ω such that

(4.1a) ω is increasing and continuous on $[0, 8\delta]$

(4.1b) $\omega(0) = 0$

(4.1c) $\int_0^{8\delta} (\omega(s)/s)\, ds$ is finite

(4.2) $|Df(x') - Df(y')| \leq \omega(|x' - y'|)$ for $|x'|, |y'| < 4\delta$.

We say that Ω satisfies an *interior Dini condition* at $x_0 \in \partial\Omega$ if there is an open set $\Omega' \subset \Omega$ such that $x_0 \in \partial\Omega'$ and Ω' is Dini at x_0. We use the constants A and δ, the function ω, and the coordinate system Y from the various definitions without further comment. We denote by ν the unit vector in the y^n direction and by d the distance to $\partial\Omega'$, and we follows the summation convention for repeated indices.

THEOREM 4.1. *Let $\mathscr{L}$ be a linear second-order differential operator*

$$\mathscr{L}u = a^{ij}D_{ij}u + b^iD_iu + cu$$

defined on an open set $\Omega \subset \mathbf{R}^n$ with the matrix $(a^{ij}(x))$ positive semi-definite and symmetric for all $x \in \Omega$. Suppose that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, that $\mathscr{L}u \leq 0$ in Ω, that there is a point $x_0 \in \partial\Omega$ for which $u(x_0) \geq 0$ and $u(x_0) \geq u(x)$ for all $x \in \Omega$ and that Ω satisfies an interior Dini condition at x_0. Let $\overline{\omega}$ be a function satisfying (4.1), *let A_2 be a positive constant, set $\omega_1(x) = \overline{\omega}(d(x))/\omega(2d(x))$, $\omega_2(x) = \overline{\omega}(d(x))/d(x)$, and suppose that,*

(4.3a) $0 < \sum_i a^{ii}(x) \leq a^{ij}(x)\nu_i\nu_j/(20\omega(|x - x_0|))^2$

(4.3b) $\sum_i a^{ii}(x) \leq A_2\omega_1(x)a^{ij}(x)\nu_i\nu_j$

(4.3c) $b(x)\nu_i \geq -A_2\omega_2(x)a^{ij}(x)\nu_i\nu_j$

(4.3d) $\sum_i |b^i(x)| \leq A_2[\omega_2(x)/\omega(|x - x_0|)]a^{ij}(x)\nu_i\nu_j$

(4.3e) $0 \geq c(x) \geq -A_2[\omega_2(x)/d(x)]a^{ij}(x)\nu_u\nu_j$

for all $x \in \Omega'_\delta$. Then for any vector μ such that $\mu \cdot \nu > 0$, we have

$$\limsup_{t \to 0}(u(x_0 + t\mu) - u(x_0))/t < 0.$$

Proof. By standard arguments, it suffices to find a function $h \in C^2(\Omega'_\delta) \cap C^1(\overline{\Omega}'_\delta)$ such that $\mathscr{L}h \geq 0$ in Ω'_δ, $h = 0$ on $\partial\Omega'_\delta \setminus \partial B_\delta$, $h > 0$ on Ω'_δ, $\partial h/\partial y^n > 0$ at x_0. We determine h as a function of local regularized distance on Ω', say $h(x) = k(\rho(x))$, and we suppose that $k(0) = 0$ and that $k' > 0$ and $k'' \geq 0$ on $(0, \delta)$; these properties will be verified from the explicit formula for k. By direct calculation,

$$\mathscr{L}h = k'' a^{ij} D_i\rho D_j\rho + k'\left(b^i D_i\rho + a^{ij} D_{ij}\rho\right) + kc$$

and

$$\begin{aligned} a^{ij} D_i\rho D_i\rho &= a^{ij}\nu_i\nu_j + a^{ij}(D_i\rho - \nu_i)\nu_j + a^{ij} D_i\rho(D_j\rho - \nu_j) \\ &\geq \tfrac{1}{2}a^{ij}\nu_i\nu_j - \tfrac{1}{2}a^{ij} D_\rho D_j\rho - a^{ij}(D_i\rho - \nu_i)(D_j\rho - \nu_j). \end{aligned}$$

Since $D\rho(x_0) = \nu$, we infer from (4.3a) and Theorem 2.1 that

$$a^{ij} D_i\rho D_j\rho \geq \tfrac{1}{6}a^{ij}\nu_i\nu_j.$$

The other terms in the expression for $\mathscr{L}h$ are estimated similarly via Theorem 2.1, (4.1a), (4.3b–e), and the inequality $k(\rho) \leq \rho k'(\rho)$ which follows from $k'' \geq 0$. Hence for some constant $H = H(A, A_2, n)$ we have

$$\mathscr{L}h \geq \tfrac{1}{6}a^{ij}\nu_i\nu_j\left(k'' - Hk'\overline{\omega}(2\rho)/\rho\right).$$

Taking (4.1c) into account, we see that the desired function is

$$k(\rho) = \int_0^\rho \exp\left(H\int_0^r \overline{\omega}(2s)/s\,ds\right) dr. \qquad \square$$

It is instructive to compare this theorem with [**12**, Theorem 1] and similar results of Kamynin and Khimchenko. The proof of our Theorem 3.1 is simple and our method is well-suited to conditions stated purely in terms of the distance function d. On the other hand, Kamynin and Khimchenko consider a relaxation of our (4.3b). In addition they are able to prove more easily the sharpness of the Dini condition (4.1c) for this result and those discussed below.

Of course an analogous boundary part lemma for parabolic equations can be proved via Theorem 3.1 (cf. [**9**], [**13**]). We mention also [**21**] in which a boundary-point-type lemma is proved for Lipschitz domains. All of these boundary point results lead to uniqueness theorems for suitable boundary value problems.

Another application of the regularized distance occurs in the study of regularity at the boundary for solutions of boundary vlaue problems. For linear equations we refer to [**10**] and [**11**], and we only remark that the results there can be obtained via regularized distance. For nonlinear equations, we refer to [**18**] and [**19**] although we remark that some of the results thee concerning convex domains can be improved by using Theorem 1.4. For example [**18**, Theorem A.1] can be proved directly by the methods of [**18**, Chapter II] since a global regularized distance is available. Similarly the hypothesis that $\partial\Omega \in C^2$ can be removed from [**10**, Theorems 3.3 and 3.9]. In particular we have the following result: if Ω is bounded and convex, if $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of the minimal surface equation

$$\mathscr{M}u = \left(1 + |Du|^2\right)\Delta u - D_i u D_j u D_{ij} u = 0$$

in Ω, and if $u|_{\partial\Omega}$ is Lipschitz, then u is Hölder continuous on $\overline{\Omega}$. In connection with these results on nonlinear equations, we raise an important question concerning the regularized distance: Is there a natural analog of Serrin's curvature conditions [**23**] for domains which are less smooth than C^2? Since in the theory of elliptic equations, these conditions are used to infer estimates on the Hessian matrix $(D_{ij}d)$ of the distance function, it seems reasonable to assume that the analog exists. So far, however, we have not been able to find a simple geometric condition which implies the appropriate behavior for the Hessian $(D_{ij}\rho)$ of the regularized distance.

As a final application, we discuss extensions of functions on $\partial\Omega$ to globally defined functions. In what follows, we assume Ω to be bounded although an unbounded Ω can also be handled by our methods. Our first step is to define appropriate regularity classes of functions in terms of certain norms. We denote by ζ a continuous increasing function with $\zeta(0) = 0$ and $\zeta(t) > 0$ for $t > 0$ such that $\zeta(t)/t$ is a decreasing function of t. (From [**20**, §3.5] it follows that assuming this last property involves no loss of generality for our purposes), and we get $Z(t) = \log_t \zeta(t)$ (Z = capital ζ). For u defined on Ω and $k \geq 0$ an integer, we define

$$|u|_0 = \sup_{\Omega} u, \quad \|u\|_k = \sum_{j=0}^{k} |D^j u|_0$$

$$[u]_Z = \sup\{|u(x) - u(y)|/\zeta(|x - y|) : x \neq y \text{ in } \Omega\}$$

$$|u|_{k+Z} = \|u\|_k + [D^k u]_Z.$$

We note that $|u|_{k+Z}$ is the usual Hölder norm $|\ |_{k+\alpha}$ when $\zeta(t) = t^\alpha$ for some $0 < \alpha \leq 1$. We also note that $|\ |_0 = \|\ \|_0$ and that $|\ |_k \geq \|\ \|_k$ for any integer k. We denote by $H_{k+Z}(\Omega)$ the set of all u for which $|u|_{k+Z}$ is finite. Let γ be a positive function defined on $\{t > 0\}$ with the property that there are positive constants γ_1 and γ_2 such that

$$\gamma_1\gamma(2t) \le \gamma(t) \le \gamma_2\gamma(2t) \quad \text{for all } t > 0.$$

For $\delta > 0$, we write $\Omega_\delta = \{x \in \Omega\colon d(x) > \delta\}$, and for $a \ge 0$ and k a non-negative integer such that $t^a\gamma(t)$ and $t^k\gamma(t)$ are increasing functions of t, we define the norms

$$|u|_a^{(\Gamma)} = \sup_{\delta>0}\{\delta^a\gamma(\delta)|u|_{a;\Omega_\delta}\}$$

$$\|u\|_k^{(\Gamma)} = \sup_{\delta>0}\{\delta^a\gamma(\delta)\|u\|_{k;\Omega_\delta}\}.$$

In analogy with the weighted Hölder spaces of **[3]** we denote by $H_a^{(\Gamma)}$ and $C_k^{(\Gamma)}$ the set of all functions u on Ω with finite norm $|u|_a^{(\Gamma)}$ and $\|u\|_k^{(\Gamma)}$ respectively. Thus (1.7) can be reformulated as saying $\rho \in C_2^{(-1-Z)}$. In fact a stronger result is valid.

THEOREM 4.1. *Under the hypotheses of Theorem* 1.3 (*including* (1.6)) Ω *has a regularized distance* $\rho \in H_a^{(-1-Z)}$ *for any* $a \ge 2$.

Proof. As in the proof of Theorem 1.3, we see that $\rho \in C_k^{(-1-Z)}$ for all integers $k > 2$. We first show that $H_{2+\alpha}^{(-1-Z)} \supset C_2^{(-1-Z)} \cap C_3^{(-1-Z)}$ for all $\alpha \in (0,1]$, and then that $H_2^{(-1-Z)} \supset H_{1+Z} \cap C_2^{(-1-Z)}$, thus establishing the theorem for $2 \le a \le 3$. The general case $a > 3$ is handled similarly.

Let $u \in C_2^{(-1-Z)} \cap C_3^{(-1-Z)}$. Setting

$$H = \sup_{\delta>0}\{(\delta^{1+\alpha}/\zeta(\delta))[D^2u]_{\alpha;\Omega_\delta}\},$$

$$d^* = \max\{1, \operatorname{diam}\Omega\},$$

we see by some simple algebra that

$$|u|_{2+\alpha}^{(-1-Z)} \le H + (d^*)^\alpha\|u\|_2^{(-1-Z)}.$$

Hence we need only estimate H in terms of $c_1 = |D^3u|_0^{(2-Z)}$ and $c_2 = |D^2u|_0^{(1-Z)}$ since $c_1 \le \|u\|_3^{(-1-Z)}$ and $c_2 \le \|u\|_2^{(-1-Z)}$. Now fix $\delta > 0$, positive integers $i \le n$ and $j \le n$, and x and y in Ω_δ. If $|x - y| < \delta/2$, then $x + t(y - x) \in \Omega_{\delta/2}$ for $0 \le t \le 1$, so

$$\begin{aligned}|D_{ij}u(x) - D_{ij}u(y)| &= \left|\int_0^1 D_{ijk}u(x + t(y - x))\,dt(x_k - y_k)\right| \\ &\le 4c_1|x - y|\zeta(\delta/2)\delta^{-2} \le 4c_1|x - y|^\alpha\zeta(\delta)\delta^{-1-\alpha}.\end{aligned}$$

On the other hand if $|x - y| \ge \delta/2$, then

$$\begin{aligned}|D_{ij}u(x) - D_{ij}u(y)| &\le |D_{ij}u(x)| + |D_{ij}u(y)| \\ &\le 2c_2\zeta(\delta)\delta^{-1} \le 4c_2|x - y|^\alpha\zeta(\delta)\delta^{-1-\alpha}.\end{aligned}$$

Hence $H \leq 4n(c_1 + c_2)$, yielding $H_{2+\alpha}^{(-1-Z)} \supset C_2^{(-1-Z)} \cap C_3^{(-1-Z)}$.

To show that $H_2^{(-1-Z)} \supset C_2^{(-1-Z)} \cap H_{1+Z}$, we proceed as before except that in the case $|x - y| \geq \delta/2$, we use the estimate

$$|Du(x) - Du(y)| \leq \zeta(|x - y|) = \big(\zeta(|x - y|)/|x - y|\big)|x - y|$$

$$\leq 2|x - y|\zeta(\delta)/\delta. \qquad \square$$

It should be noted that the inclusions $H_{2+\alpha}^{(-1-Z)} \supset C_2^{(-1-Z)} \cap C_3^{(-1-Z)}$ and $H_2^{(-1-Z)} \supset H_{1+Z} \cap C_2^{(-1-Z)}$ were established without any restrictions on the smoothness of $\partial\Omega$. Under slightly stronger hypotheses (e.g., $\partial\Omega$ is Lipschitz and $Z = \beta$, a constant, with $0 < \beta < 1$), better results are valid; see [**3**, Lemma 2.1].

We are now ready to discuss extensions of functions on $\partial\Omega$. For k and ζ as above, we write H'_{k+Z} for the set of all $h \in H_{k+Z}(\mathbf{R}^n)$ with compact support and we write $H_{k+Z}(\partial\Omega)$ for the set of all h with finite norm:

$$|h|_{k+Z;\partial\Omega} = \inf\big\{|\tilde{h}|_{k+Z}: \tilde{h} \in H'_{k+Z} \text{ and } \tilde{h} = h \text{ on } \partial\Omega\big\}.$$

We note that if h is continuous on $\partial\Omega$, then $h \in H_Z(\partial\Omega)$ for some ζ. Moreover if $\partial\Omega$ has a local representation at each $x_0 \in \partial\Omega$ and if $|h(x) - h(y)| \leq \zeta(|x - y|)$ for all x and y in $\partial\Omega$ and some ζ, then $h \in H_Z(\partial\Omega)$ and $|h|_{Z;\partial\Omega} = 1$. If, in addition, all the local representations f are in C^k for some $k \geq 1$ then $H_{k+Z}(\partial\Omega)$ (for suitable ζ) and $H_{j+Z}(\partial\Omega)$ (for all $j < k$ and all ζ) can be defined via the function g given by $g(y') = h(y', f(y'))$ being in the appropriate function space. For brevity we write $\partial\Omega \in H_{k+Z}$ if $\partial\Omega$ has a local H_{k+Z} represenentation at each $x_0 \in \partial\Omega$. In this case we can extend functions on $\partial\Omega$ in a convenient fashion (cf. [**5**, Lemma 1]).

THEOREM 4.2. (a) *If $h_0 \in H_Z(\partial\Omega)$, then there is $\tilde{h} \in H'_Z$ such that $\tilde{h} = h_0$ on $\partial\Omega$ and $|\tilde{h}|_a^{(-Z)} \leq c(a, \Omega)|h|_{Z;\partial\Omega}$ for all $a \geq 1$.* (b) *Suppose $\partial\Omega \in H_{1+Z}$ and let ν be the inner normal to $\partial\Omega$. If $h_0 \in H_{1+Z}(\partial\Omega)$ and if $h_1 \in H_Z(\partial\Omega)$ then there is $\tilde{h} \in H'_{1+Z}$ such that*

$$\tilde{h} = h_0, \qquad D\tilde{h} \cdot \nu = h_1 \quad \text{on } \partial\Omega$$

and

$$|h|_a^{(-1-Z)} \leq c(a, \Omega)\big(|h|_{1+Z;\partial\Omega} + |h_1|_{Z;\partial\Omega}\big)$$

for all $a \geq 2$.

Proof. Set $H(x, \tau) = \int_{|z|<1} h_0(x - (\tau/L)z)\phi(z)\, dz$ with ϕ as in §1.

(a) $\tilde{h}(x) = H(x, \rho(x))$ gives the desired function.

(b) Note that according to (a), $1/D\rho \cdot \nu$ can be extended to a function $F \in H'_Z \cap H_a^{(-Z)}$. Using the obvious definition for $H_1(x, \tau)$, we set $h_2(x) = H(x, \rho(x))$ and $h_3(x) = H_1(x, \rho(x))$. Then

$$\tilde{h}(x) = h_2(x) + (h_3(x) - Dh_2(x) \cdot D\rho(x))F(x)\rho(x)$$

gives the desired function. □

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