PACIFIC JOURNAL OF MATHEMATICS
Vol 117, No 2, 1985

ORTHOGONAL PRIMITIVE IDEMPOTENTS AND BANACH ALGEBRAS ISOMORPHIC WITH l_2

TAQDIR HUSAIN

In this paper, a study of orthogonal primitive idempotents and minimal ideals in topological algebras with orthogonal bases has been made. Among other results, a structure theorem for Banach algebras with orthogonal bases has been proved, similar to Ambrose's structure theorem for H^*-algebras in the separable case. Furthermore, a necessary and sufficient condition is given for Banach algebras with orthogonal bases to be isomorphically homeomorphic with the Hilbert algebra l_2.

1. Introduction. In our papers [5], [6], [7], we introduced the notions of orthogonal and cyclic bases in topological algebras and studied a number of properties of such algebras. For instance, we proved necessary and sufficient conditions under which a topological algebra with an orthogonal basis is isomorphically homeomorphic with the Fréchet algebra s of all complex sequences ([7], [6]). Similar characterization theorems were proved for the Banach algebra l_1 of all complex sequences with absolutely convergent series [3] and the Banach algebra c_0 of all complex null sequences [4].

Here we are concerned with a study of orthogonal primitive idempotents in topological algebras with orthogonal bases. As is well-known, the existence of idempotents in semisimple rings or algebras enables one to represent such rings or algebras as a direct sum of simple rings or algebras. Such a consideration for Hilbert algebras has led us to doubly orthogonal idempotents and to a structure theorem for such algebras. The structure theorem for Banach algebras with orthogonal bases is stronger than the similar result for H^*-algebras proved by Ambrose [1] but only in the separable case.

Specifically, we prove general results regarding orthogonal primitive idempotents in §3. For instance, we identify the maximal family of all orthogonal primitive idempotents. In §4, we prove that there are lots of closed minimal ideals (Theorem 4.4) in any topological algebra with an orthogonal basis. This leads us to a structure theorem for such algebras. Moreover, each Banach algebra, if it has an orthogonal basis, can be expressed as a countable direct sum of simple Banach subalgebras, each of which is isomorphically homeomorphic with the field of complex numbers

(Theorem 4.7). In §5, we consider Hilbert algebras. In this case, the structure theorem can be strengthened by using doubly orthogonal idempotents (Theorem 5.2 and Remark 5.3). Finally, in §6, we prove a necessary and sufficient condition under which a Banach algebra with an orthogonal basis is isomorphically homeomorphic with l_2, the Hilbert algebra of all complex sequences with square absolutely summable series.

2. Preliminaries. Let $\mathscr{A}$ be a Hausdorff topological algebra over the complex field $\mathbf{C}$ (i.e. a complex algebra $\mathscr{A}$ with a Hausdorff topology in which the maps: $(x, y) \to x + y$, $(\lambda, x) \to \lambda x$, $(x, y) \to xy$ are continuous, where $x, y \in \mathscr{A}$ and $\lambda \in \mathbf{C}$.) A countable set $\{x_i\}$ of $\mathscr{A}$ is said to be its *orthogonal basis* if the following two axioms hold:

(i) $x_i x_j = \delta_{ij} x_i$ for all $i, j \geq 1$. In other words, $x_i x_j = 0$ for $i \neq j$ and $x_i^2 = x_i$.

(ii) $\{x_i\}$ is a basis of $\mathscr{A}$ regarded as a topological vector space. In other words, for each $x \in \mathscr{A}$ there exists a unique sequence $\{\lambda_i(x)\}$ of scalars such that

$$x = \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_i(x) x_i = \sum_{i=1}^{\infty} \lambda_i(x) x_i.$$

(In the sequel we denote $\Sigma_{i=1}^{\infty} \lambda_i(x) x_i$ by $\Sigma_i \lambda_i(x) x_i$ unless otherwise stated.) Since λ_i's depend upon x, it is easily seen that each λ_i is a multiplicative linear functional, often called a coordinate functional. In general, each λ_i need not be continuous. Whenever they are, the basis $\{x_i\}$ is called a Schauder orthogonal basis. It is known **[6]** that each orthogonal basis in a LMC (cf. **[6]**) algebra is a Schauder orthogonal basis. Indeed, in particular for Fréchet (i.e. complete metrizable LMC) and Banach algebras, the same is true.

If $\{x_i\}$ is an orthogonal basis in a normed algebra $\mathscr{A}$, without any loss of generality, we may assume that $\|x_i\| = 1$ for all $i \geq 1$, in other words, the basis can be normalized. Thus in this case if $x = \Sigma_i \lambda_i(x) x_i$, then

$$\lim_{i \to \infty} \|\lambda_i(x) x_i\| = \lim_{i \to \infty} |\lambda_i(x)| = 0 \quad \text{for all } x \in \mathscr{A}.$$

Hence for each $x \in \mathscr{A}$, the sequence $\{\lambda_i(x)\}$ is in c_0.

From $x = \Sigma_i \lambda_i(x) x_i$ in a topological algebra, we conclude that $x = 0$ iff $\lambda_i(x) = 0$ for all $i \geq 1$. Furthermore, by the definition of basis, $x_i \neq 0$ for all $i \geq 1$.

We shall freely use the terms and results from **[5]**–**[7]**. For a general reference on bases in Banach space, the reader may consult Day **[2]**, Singer **[9]** and for a general theory of Banach algebras, Zelazko's book **[10]**.

3. Orthogonal and primitive idempotents. In this section, we describe the maximal family of all orthogonal primitive idempotents in a topological algebra with an orthogonal basis.

Since a topological algebra with an orthogonal basis is commutative [**6**], the commonly used terms "left", "right" and "two-sided" in noncommutative cases are redundant. Hence the terms defined by Ambrose [**1**] for noncommutative H^*-algebras, which we are going to use here will not be accompanied by the words "left", "right" and "two-sided".

According to Ambrose [**1**], an algebra A is said to be *proper* if for any $x \in A, Ax = xA = \{0\}$ implies $x = 0$.

Throughout this section, $\mathscr{A}$ will stand for a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$.

Now we have some elementary properties:

PROPOSITION 3.1. *If for any $x \in \mathscr{A}$, $xx_i = 0$ for all $i \geq 1$, then $x = 0$. Hence $\mathscr{A}$ is proper.*

Proof. From $x = \Sigma_i \lambda_i(x) x_i$, we obtain $xx_i = \lambda_i(x)x_i = 0$ for all $i \geq 1$. Since $x_i \neq 0$, it follows that $\lambda_i(x) = 0$ for all $i \geq 1$ and so $x = 0$.

PROPOSITION 3.2. *For any $x \in \mathscr{A}$, $x \neq 0$ iff $x^n \neq 0$ for any integer $n \geq 1$.*

Proof. If $x = \Sigma_i \lambda_i(x) x_i$, then $x^n = \Sigma_i \lambda_i(x^n) x_i$. Now $\lambda_i(x^n) = [\lambda_i(x)]^n \neq 0$ for some i iff $\lambda_i(x) \neq 0$ for the same i.

PROPOSITION 3.3. *If $\mathscr{I}$ is an ideal of $\mathscr{A}$ such that $x_i \mathscr{A} \subset \mathscr{I}$ for some i, then $x_i \in \mathscr{I}$.*

Proof. Since $x_i x \in \mathscr{I}$ for all $x \in \mathscr{A}$, in particular when $x = x_i$, we get $x_i^2 = x_i \in \mathscr{I}$.

PROPOSITION 3.4. *$\mathscr{A}$ contains no proper ($\neq \{0\}, \mathscr{A}$) nilpotent ideals. (An ideal $\mathscr{I}$ is called nilpotent if $\mathscr{I}^n = \{0\}$ for some integer $n \geq 1$.)*

Proof. Suppose $\mathscr{I}^n = \{0\}$ for some ideal $\mathscr{I}$ and some integer $n \geq 1$. Then $x^n = 0$ for all $x \in \mathscr{I}$. But then by Proposition 3.2, $x = 0$. Hence $\mathscr{I} = \{0\}$.

DEFINITION 3.5. An element e of an algebra is called *idempotent* if $e^2 = e \neq 0$. A family $\{e_\alpha\}$ of idempotents is said to be *orthogonal* if

$e_\alpha e_\beta = 0$ for $\alpha \neq \beta$. An element is said to be *primitive* [**1**] if it cannot be expressed as a sum of two orthogonal idempotents.

Note that if an algebra has identity e $(ex = xe = x)$ then it is an idempotent, but not conversely.

It is a nontrivial matter to determine the collection of all idempotents in an arbitrary algebra or ring. In our case, however, it turns out to be a simple matter.

We note that by definition each member x_i of an orthogonal basis is an idempotent. Moreover, if σ is any finite subset of the set $\mathbf{N}$ of positive integers, then

$$x_\sigma = \sum_{i \in \sigma} x_i$$

is an idempotent. In general, let x be an idempotent of $\mathscr{A}$. Then from $x = \sum_i \lambda_i(x) x_i$ and $x^2 = x$, we get $[\lambda_i(x)]^2 = \lambda_i(x)$ for all $i \geq 1$, i.e. either $\lambda_i(x) = 0$ or 1 for all $i \geq 1$. Let E be the subset of $\mathbf{N}$ such that $\lambda_i(x) = 1$ for all $i \in E$. Then

$$x = \sum_{i \in E} x_i.$$

Hence the collection of all idempotents in $\mathscr{A}$ is huge. In special cases, the subset E above is always finite.

PROPOSITION 3.6. *Let $\mathscr{A}$ be a normed algebra with an orthogonal basis. If $x = \sum_i \lambda_i(x) x_i$ is an idempotent of $\mathscr{A}$, then $\lambda_i(x) = 0$ for all i but a finite number of them where it is equal to* 1. *In other words, x is an idempotent of $\mathscr{A}$ iff $x = \sum_{i \in \sigma} x_i$, where σ is a finite subset of* $\mathbf{N}$.

Proof. By the above remark, since x is an idempotent, there is a subset E of $\mathbf{N}$ such that $x = \sum_{i \in E} x_i$. Suppose E is infinite. Since the series is convergent, there is i_0 such that for all $i \geq i_0$, $\|x_i\| < 1$. By the orthogonality of basis, $x_i = x_i^k$ for all $k \geq 1$ and so $\|x_i\| = \|x_i^k\| \leq \|x_i\|^k \to 0$ as $k \to \infty$ for all $i \geq i_0$. This shows that $x_i = 0$ for all $i \geq i_0$, which is ruled out by the definition of basis. Hence E is finite.

THEOREM 3.7. *$\{x_i\}$ is the maximal orthogonal family of idempotents of a topological algebra $\mathscr{A}$ with an orthogonal basis $\{x_i\}$.*

Proof. Clearly by the definition of orthogonal basis, $\{x_i\}$ is an orthogonal family of idempotents. To show it is maximal, let $x \in \mathscr{A}$ be such that $x^2 = x \neq 0$ and $xx_i = 0$ for all $i \geq 1$. But then by Proposition

3.1, $x = 0$ which is contrary to the definition of idempotent. Hence there are no other idempotents orthogonal to $\{x_i\}$.

REMARK 3.8. Note that if a topological algebra with an orthogonal basis has identity e, then e cannot belong to the maximal orthogonal family of idempotents.

THEOREM 3.9. *Each x_i is a primitive idempotent of $\mathscr{A}$ and $\{x_i\}$ is the maximal family of all orthogonal primitive idempotents of $\mathscr{A}$.*

Proof. Suppose, if possible, $x_i = y + z$ where y, z are orthogonal idempotents, i.e,. $y^2 = y \neq 0$, $z^2 = z \neq 0$ and $yz = 0$. Since y, z are idempotents, there are subsets (possibly infinite) of $\mathbf{N}$ such that $x = \sum_{j \in E} x_j$, $y = \sum_{j \in F} x_j$. Furthermore, $yz = 0$ implies $E \cap F = \varnothing$. Then $x_i = \sum_{j \in E \cup F} x_j$. But this contradicts the uniqueness of the representation of an element in series by the definition of basis, unless $E \cup F = \{i\}$. Hence each x_i is primitive. Since there are no other orthogonal (Theorem 3.7) idempotents than $\{x_i\}$, it follows that $\{x_i\}$ is the maximal family of all orthogonal primitive idempotents.

4. Minimal closed ideals and structure theorems. In this section, we identify all closed minimal ideals with a view to establishing a structure theorem for topological algebras (in particular, Banach algebras) possessing an orthogonal basis.

DEFINITION 4.1. An ideal in a commutative ring (or algebra) is said to be *minimal* if it does not contain any nonzero proper ideal of the ring (or algbebra). An algebra or ring which does not contain any nonzero proper ideals is called *simple*.

Again throughout this section, $\mathscr{A}$ denotes a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$.

PROPOSITION 4.2. *For each i, $x_i\mathscr{A}$ is an ideal of $\mathscr{A}$ and $x_i\mathscr{A} \cap x_j\mathscr{A} = \{0\}$ for $i \neq j$.*

Proof. It is, indeed, easy to verify that $x_i\mathscr{A}$ is an ideal. If $x \in x_i\mathscr{A} \cap x_j\mathscr{A}$ for $i \neq j$, then $x = x_i y = x_j z$ for some $y, z \in \mathscr{A}$. From the expansions of y and z, we derive $x = x_i y = \lambda_i(y) x_i = \lambda_j(z) x_j = x_j z$. Now multiplying this equation by x_i, we get

$$\lambda_i(y) x_i^2 = \lambda_i(y) x_i = \lambda_j(z) x_i x_j = 0.$$

Since $x_i \neq 0$, we conclude that $\lambda_i(y) = 0$ and so $x = 0$.

THEOREM 4.3. *Each* $x_i\mathscr{A}$ $(i \geq 1)$ *is a closed minimal ideal and* $x_i \in x_i\mathscr{A}$ *for all* $i \geq 1$.

Proof. First we show that each $x_i\mathscr{A}$ is a minimal ideal. Let i be fixed. Then by Proposition 4.2, $x_i\mathscr{A}$ is an ideal. For minimality, suppose $\mathscr{I}$ is an ideal of $\mathscr{A}$ such that $0 \neq \mathscr{I} \subset x_i\mathscr{A}$. We show that $\mathscr{I} = x_i\mathscr{A}$. We consider three cases:

Case (a). Suppose $x_j \notin \mathscr{I}$ for all $j \geq 1$. Then for any $x \in \mathscr{I}$, $x \neq 0$ from $x = \Sigma_j \lambda_j(x)x_j$ we have $xx_j = \lambda_j(x)x_j \in \mathscr{I}$ for all $j \geq 1$. Since $x \neq 0$ implies there is some j such that $\lambda_j(x) \neq 0$ and so $(\lambda_j(x))^{-1}(\lambda_j(x)x_j) = x_j \in \mathscr{I}$, a contradiction.

Case (b). Suppose there is x_j $(j \neq i)$ such that $x_j \in \mathscr{I}$. But then for all $x \in \mathscr{A}$, $xx_j \in \mathscr{I}$ and so $\mathscr{A}x_j \subset \mathscr{I} \subset \mathscr{A}x_i$. By Proposition 4.2, $\mathscr{A}x_i \cap \mathscr{A}x_j = \{0\}$ for $i \neq j$ and we conclude that $\mathscr{A}x_j = \{0\}$ for $j \neq i$. But this means that $x_j^2 = x_j = 0$, which is ruled out. Hence $\mathscr{I}$ contains no x_j, $j \neq i$.

Case (c). Thus $\mathscr{I}$ must contain x_i. Since $\mathscr{I}$ is an ideal, for all $x \in \mathscr{A}$, $xx_i \in \mathscr{I}$ and we have $\mathscr{A}x_i \subset \mathscr{I}$. Since $\mathscr{I} \subset x_i\mathscr{A}$, we have proved that $x_i\mathscr{A} = \mathscr{I}$ and $x_i \in x_i\mathscr{A}$.

To complete the proof, we must show that $x_i\mathscr{A}$ is closed. Let $y \in \overline{x_i\mathscr{A}}$ (closure). There is a net $\{x_\alpha x_i\} \subset x_i\mathscr{A}$ such that $(x_\alpha x_i) \overset{(\alpha)}{\to} y$. Since the multiplication in $\mathscr{A}$ is continuous, we have

$$(x_\alpha x_i)x_i = (x_\alpha x_i) \overset{(\alpha)}{\to} yx_i \in \mathscr{A}x_i.$$

Since $\mathscr{A}$ is Hausdorff, $y = yx_i \in \mathscr{A}x_i$ and so $\mathscr{A}x_i$ is closed for each i.

Now we prove a structure theorem.

THEOREM 4.4. *Each Hausdorff topological algebra* $\mathscr{A}$ *with an orthogonal basis* $\{x_i\}$ *can be written as a countable direct sum of closed minimal ideals* $\{x_i\mathscr{A}\}$.

Proof. Since for each $x \in \mathscr{A}$ there is a unique sequence $\{\lambda_i(x)\}$ of scalars such that $x = \Sigma_i \lambda_i(x)x_i$, from this we obtain: $xx_i = \lambda_i(x)x_i$ for all $i \geq 1$. Hence $x = \Sigma_i(xx_i)$. Since $xx_i \in \mathscr{A}x_i$ for all $i \geq 1$ and $x_i\mathscr{A} \cap x_j\mathscr{A} = \{0\}$ for $i \neq j$ (Proposition 4.2), we conclude that $\mathscr{A} = \oplus \Sigma_i(\mathscr{A}x_i)$, a direct sum in which each $\mathscr{A}x_i$ is a closed minimal ideal of $\mathscr{A}$ by Theorem 4.3.

We know ([**6**], Corollary 1.5) that if the basis of a topological algebra is a Schauder basis, then the algebra is semisimple. Since every basis in a Fréchet and, in particular, Banach algebra is a Schauder basis, it follows that every Fréchet as well as Banach algebra with an orthogonal basis is semisimple. We can improve Theorem 4.4 for Banach algebras.

First assume that $\mathscr{A}$ is a Hausdorff topological algebra with a Schauder orthogonal basis $\{x_i\}$. Then each associated coordinate functional $\lambda_i(x) = \lambda_i$ is a continuous multiplicative linear functional on $\mathscr{A}$. Hence its kernel

$$\mathscr{M}_i = \{x \in \mathscr{A} : \lambda_i(x) = 0\}$$

is a closed maximal ideal of $\mathscr{A}$ for each $i \geq 1$. Moreover, these are all the closed maximal ideals of $\mathscr{A}$ (cf. [**6**], Theorem 2.1). Thus the quotient $\mathscr{A}/\mathscr{M}_i$ is a Hausdorff topological algebra in the quotient topology.

Now we have:

PROPOSITION 4.5. *Let $\mathscr{A}$ be a Hausdorff topological algebra with an orthogonal Schauder basis $\{x_i\}$. Then there exists a continuous isomorphism of $\mathscr{A}/\mathscr{M}_i$ onto $\mathscr{A}x_i$ for each fixed $i \geq 1$.*

Proof. For each $\dot{x} \in \mathscr{A}/\mathscr{M}_i$, set $\psi(\dot{x}) = xx_i$, where $\dot{x} = x + \mathscr{M}_i$ is a coset in $\mathscr{A}/\mathscr{M}_i$. First we check that ψ is a well-defined map. For this, let $\dot{x} = x + \mathscr{M}_i = y + \mathscr{M}_i$, then $x - y \in \mathscr{M}_i$ and so $\lambda_i(x - y) = 0$. Whence $\lambda_i(x)x_i = \lambda_i(y)x_i$ and so $xx_i = \lambda_i(x)x_i = \lambda_i(y)x_i = yx_i$. It is easy to verify that ψ is linear. Further

$$\psi(\dot{x}\dot{y}) = \psi((xy)^{\cdot}) = (xy)x_i = (xx_i)(yx_i) = \psi(\dot{x})\psi(\dot{y}),$$

because $x_i^2 = x_i$. This proves that ψ is an algebra homomorphism of $\mathscr{A}/\mathscr{M}_i$ into $\mathscr{A}x_i$. ψ is one-one: suppose $xx_i = 0$, then $\lambda_i(x)x_i = 0$ implies $\lambda_i(x) = 0$ because $x_i \neq 0$. Thus $x \in \mathscr{M}_i$, which proves that ψ is injective. Now if xx_i is an arbitrary element of $\mathscr{A}x_i$ for some $x \in \mathscr{A}$, then $xx_i = \lambda_i(x)x_i = \lambda_i(x + y)x_i$ for all $y \in \mathscr{M}_i$ and so $\psi(\dot{x}) = xx_i$ shows that ψ is an isomorphism of $\mathscr{A}/\mathscr{M}_i$ onto $\mathscr{A}x_i$.

We endow $\mathscr{A}/\mathscr{M}_i$ with the quotient topology and $\mathscr{A}x_i$ with the induced Hausdorff topology from $\mathscr{A}$. To prove the continuity of ψ, first we denote the quotient map: $\mathscr{A} \to \mathscr{A}/\mathscr{M}_i$ by φ and the map: $x \to xx_i$ by g. Then $\psi(\varphi(x)) = g(x)$ for all $x \in \mathscr{A}$. Clearly φ, being the quotient map, is continuous and open, whereas g, being the multiplication map in the topological algebra, is also continuous. Thus for any neighbourhood U of 0 in $\mathscr{A}x_i$ (note $\mathscr{A}x_i$ being an ideal of $\mathscr{A}$ is an algebra), $g^{-1}(U)$ is a neighbourhood of 0 in $\mathscr{A}$ by the continuity of g and hence $\varphi(g^{-1}(U))$ is a neighbourhood of 0 in $\mathscr{A}/\mathscr{M}_i$ by the openness of φ. Since

$$\psi^{-1}(U) = \varphi(g^{-1}(U)),$$

we have proved the continuity of ψ.

Now in the case where $\mathscr{A}$ is a Banach algebra with an orthogonal basis, we can describe the nature of closed minimal ideals $\mathscr{A}x_i$.

THEOREM 4.6. *Let $\mathscr{A}$ be a Banach algebra with an orthogonal basis $\{x_i\}$. Then each closed minimal ideal $\mathscr{A}x_i$ $(i \geq 1)$ is isomorphically homeomorphic with the field* **C** *of all complex numbers.*

Proof. Since $\mathscr{A}x_i$ is a closed ideal of the Banach algebra $\mathscr{A}$, it is itself a Banach algebra under the induced norm topology. Furthermore, $\mathscr{A}x_i$ has identity, viz. x_i. Hence the algebra $\mathscr{A}/\mathscr{M}_i$, being isomorphic to $\mathscr{A}x_i$ (Proposition 4.5) also has identity. Since $\mathscr{M}_i$ is a closed maximal ideal of $\mathscr{A}$, $\mathscr{A}/\mathscr{M}_i$ is a Banach division algebra and hence by the Gelfand-Mazur theorem, $\mathscr{A}/\mathscr{M}_i$ is isomorphic and homeomorphic with the field of complex numbers. But the continuous isomorphism ψ (Proposition 4.5) of the Banach algebra $\mathscr{A}/\mathscr{M}_i$ onto the Banach algebra $\mathscr{A}x_i$ is open by the open mapping theorem and so ψ is an isomorphism and homeomorphism. Since $\mathscr{A}/\mathscr{M}_i$ has been shown to be isomorphically homeomorphic with the field of complex numbers, the same holds for each $\mathscr{A}x_i$. This completes the proof.

Now we prove a structure theorem for Banach algebras with an orthogonal basis which is similar to, but stronger than, Ambrose's structure theorem [**1**] for H^*-algebras in the separable case.

THEOREM 4.7. *Every Banach algebra $\mathscr{A}$ with an orthogonal basis can be expressed as a countable direct sum of simple Banach subalgebras $\mathscr{A}x_i$ $(i \geq 1)$, each of which is isomorphically homeomorphic with the field of complex numbers.*

Proof. It follows from Theorems 4.4 and 4.6.

5. Doubly orthogonal idempotents and Hilbert algebras. In this section we consider those topological algebras with orthogonal bases whose topology is given by a norm which in turn is induced by an inner product functional.

Let H denote a Banach algebra such that its norm is given by: $\|x\| = +\sqrt{\langle x, x\rangle}$, where $\langle\ ,\ \rangle$ is the inner product functional. In other words, H is a Hilbert space as well as a Banach algebra in the norm induced by $\langle\ ,\ \rangle$. Such algebras are called *Hilbert algebras.*

It is well-known that each separable Hilbert space H contains a sequence $\{u_i\}$ of orthonormal vectors (i.e. $\langle u_i, u_j\rangle = \delta_{ij}$) which forms a basis of H. More explicitly, one has:

(i) $x = \Sigma_i\langle x, u_i\rangle u_i$ for each $x \in H$,

(ii) $\langle x, y\rangle = \Sigma_i\langle x, u_i\rangle\overline{\langle y, u_i\rangle}$,

(iii) $\|x\|^2 = \Sigma_i|\langle x, x_i\rangle|^2$ (Parsevals' equality).

Thus from (iii) and other known results it follows that each separable Hilbert space is isometric with l_2, the Hilbert space of all complex sequences $\{a_i\}$ with $\Sigma_i|a_i|^2 < \infty$.

DEFINITION 5.1. A collection $\{e_\alpha\}$ of elements in a Hilbert algebra is said to be *doubly orthogonal* if $\langle e_\alpha, e_\beta\rangle = 0$ and $e_\alpha e_\beta = 0$ for $\alpha \neq \beta$.

Since each separable Hilbert space H has a sequence $\{u_i\}$ of orthonormal vectors as pointed out above, it is interesting to ask if there exists a Hilbert algebra structure on H such that the sequence $\{u_i\}$ is doubly orthogonal. We answer this question below, in the affirmative.

It is clear that the canonical basis $\{e_i\}$, where $e_i = \{\delta_{ij}\}_{j\geq 1}$, $i = 1, 2, \ldots$, of l_2 is unconditional i.e. the series $x = \Sigma_i\langle x, e_i\rangle e_i$ for each $x \in l_2$ is unconditionally convergent [2]. Hence an orthonormal basis of any separable Hilbert space, being isometric with l_2, is also unconditional.

With these remarks, we have:

THEOREM 5.2. *Let $\mathscr{H}$ be a separable Hilbert space with an orthonormal basis $\{x_i\}$. Then there exists a multiplication on $\mathscr{H}$ and an equivalent norm making $\mathscr{H}$ a Hilbert algebra with $\{x_i\}$ as its doubly orthogonal basis.*

Proof. First we define a multiplication on $\mathscr{H}$ as follows: for $x = \Sigma_i\langle x, x_i\rangle x_i$, $y = \Sigma_i\langle y, x_i\rangle x_i$, we define

$$x * y = \sum_i \langle x, x_i\rangle\langle y, x_i\rangle x_i.$$

The series is convergent in $\mathscr{H}$ because the basis $\{x_i\}$ is unconditional as remarked above. Thus for all $x, y \in \mathscr{H}$, $x * y \in \mathscr{H}$. It is easy to check that $\mathscr{H}$ with this multiplication is an algebra. Further, we define another norm on $\mathscr{H}$ by: $\|x\|' = \sqrt{\Sigma_i|\langle x, x_i\rangle|^2}$. Clearly this is the transported norm from l_2 and so $\|\cdot\|'$ is equivalent to the initial norm on $\mathscr{H}$ by isometry. We see that

$$\begin{aligned}\|x * y\|' &= \sqrt{\Sigma_i |\langle x, x_i\rangle \langle y, x_i\rangle|^2}\\ &\le \sqrt{\Sigma_i |\langle x, x_i\rangle|^2}\sqrt{\Sigma_i |\langle y, x_i\rangle|^2}\\ &= \|x\|'\|y\|'.\end{aligned}$$

Hence $\mathscr{H}$ is a Banach algebra with the equivalent norm and the product $*$. Since the new norm is induced from an inner product functional

$$\langle x, y\rangle' = \sum_i \langle x, x_i\rangle \langle \overline{y, x_i}\rangle$$

(the series is convergent by the Cauchy-Schwarz inequality), it follows that $\mathscr{H}$ is a Hilbert algebra. Clearly

$$\langle x_j, x_k\rangle' = \sum_i \langle x_j, x_i\rangle \langle \overline{x_k, x_i}\rangle = 0$$

and $x_j * x_k = 0$ if $j \neq k$. Thus $\{x_i\}$ is doubly orthogonal.

REMARK 5.3. Now in the case of a separable Hilbert space $\mathscr{H}$, one can always regard it as a Hilbert algebra and therefore it can be expressed as a countable direct sum of closed minimal ideals $\{\mathscr{H}x_i\}$ in which each x_i is doubly orthogonal and each $\mathscr{H}x_i$ is isomorphically homeomorphic with the field of complex numbers, in view of Theorems 4.7 and 5.2.

6. Banach algebras isomorphic with l_2. In this section, among other results, a necessary and sufficient condition for a Banach algebra with an orthogonal basis to be isomorphically homeomorphic with l_2, is given.

We have already shown when such Banach algebras are isomorphically homeomorphic with l_1 and c_0 (cf. [3], [4]).

First of all, we note that if $a = \{a_i\} \in l_2$ then $a^* = \{\overline{a_i}\} \in l_2$, where $\bar{a}_i$ is the complex conjugate of the complex number a_i. Furthermore,

$$\|a\|_2 = \sqrt{\Sigma_i |a_i|^2} = \sqrt{\Sigma_i |\bar{a}_i|^2} = \|a^*\|_2.$$

In other words, the involution: $x \to x^*$ in l_2 is norm preserving.

We recall that l_2 has a Schauder basis $\{e_i\}$ satisfying:

(i) The basis is unconditional.

(ii) $\{e_i\}$ is doubly orthogonal, i.e., $\langle e_i, e_j\rangle = e_i e_j = 0$ for $i \neq j$.

(iii) $\{e_i\}$ is an orthonormal family, i.e., $\langle e_i, e_j\rangle = \delta_{ij}$ as well as orthogonal $e_i e_j = \delta_{ij} e_j$ for all i, j.

(iv) The basis $\{e_i\}$ is boundedly complete and shrinking (cf. for instance [2] for definition). (For, l_2 is reflexive and so the James Theorem [2] applies.)

With these remarks, first we show that each topological algebra with an orthogonal unconditional basis carries a natural involution.

PROPOSITION 6.1. *Let $\mathscr{A}$ be a topological algebra with an unconditional orthogonal basis $\{x_i\}$. Then for each $x = \sum_i \lambda_i(x) x_i \in \mathscr{A}$,*

$$x^* = \sum_i \overline{\lambda_i(x)} x_i$$

defines an involution (henceforth called natural involution) on $\mathscr{A}$. Furthermore, for each $x \in \mathscr{A}$, $x \neq 0$ iff $x^* \neq 0$ and also $xx^* \neq 0$ iff $x \neq 0$.

Proof. Since the basis $\{x_i\}$ is unconditional, the series defining x^* is convergent in $\mathscr{A}$. Moreover, it is easy to check that $x^{**} = x$, $(x + y)^* = x^* + y^*$, $(\mu x)^* = \bar{\mu} x^*$ and $(xy)^* = x^* y^* = y^* x^*$. For the rest, we observe that $\lambda_i(x^*) = \overline{\lambda_i(x)}$ and $\lambda_i(xx^*) = |\lambda_i(x)|^2$. Hence $\lambda_i(x^*) = 0$ or $\lambda_i(xx^*) = 0$ iff $\lambda_i(x) = 0$.

PROPOSITION 6.2. *If $\mathscr{A}$ is a normed algebra with an unconditional orthogonal basis $\{x_i\}$, then for all $x = \sum_i \lambda_i(x) x_i \in \mathscr{A}$,*

$$\|xx_i\| = \|x^* x_i\|$$

for all $i \geq 1$.

Proof. Since $xx_i = \lambda_i(x) x_i$, for all $i \geq 1$ we have

$$\|xx_i\| = \|\lambda_i(x) x_i\| = \|\overline{\lambda_i(x)} x_i\| = \|x^* x_i\|.$$

PROPOSITION 6.3. *Let $\mathscr{A}$ be a Banach algebra with an orthogonal basis $\{x_i\}$. Then for any sequence $\{a_i\} \in l_2$, the sequences $\{y_n y_n^*\}$ and $\{|y_n|^2\}$ converge in $\mathscr{A}$, where $y_n = \sum_{i=1}^n a_i x_i$, $y_n^* = \sum_{i=1}^n \bar{a}_i x_i$ and $|y_n| = \sum_{i=1}^n |a_i| x_i$.*

Proof. As before, without any loss of generality, we may normalize the basis $\{x_i\}$ and so assume $\|x_i\| = 1$ for all $i \geq 1$. Since $\{x_i\}$ is orthogonal, we have

$$y_n y_n^* = |y_n|^2 = \sum_{i=1}^n |a_i|^2 x_i^2.$$

For $n > m$,

$$\begin{aligned}\|y_n y_n^* - y_m y_m^*\| &= \left\| |y_n|^2 - |y_m|^2 \right\| \\ &= \left\| \sum_{i=m+1}^n |a_i|^2 x_i^2 \right\| \leq \sum_{i=m+1}^n |a_i|^2 \to 0\end{aligned}$$

as $m, n \to \infty$. So the sequences $\{y_n y_n^*\}$, $\{|y_n|^2\}$ being Cauchy sequences in the Banach algebra $\mathscr{A}$, are convergent.

Incidently, the existence of a (an) (orthogonal) basis $\{x_i\}$ in a Banach space (algebra) $\mathscr{A}$ implies that there exists an inner product topology which is coarser than the initial one. Specifically, if $x = \Sigma_i \lambda_i(x) x_i$, $y = \Sigma_i \lambda_i(y) x_i$, then

$$\langle x, y\rangle = \sum_i \frac{\lambda_i(x)\overline{\lambda_i(y)}}{i^2}$$

defines an inner product. However, this inner product topology need not be equivalent to the initial one.

Now we give a necessary and sufficient condition under which a Banach algebra with an orthogonal basis is isomorphically homeomorphic with l_2.

THEOREM 6.4. *Let $\mathscr{A}$ be a Banach algebra with an unconditional orthogonal basis $\{x_i\}$. Then $\mathscr{A}$ is isomorphically homeomorphic with l_2 iff*

(i) *For all $x, y \in \mathscr{A}$, $\Sigma_i |\lambda_i(xy)| < \infty$.*

(ii) *For all $x \in \mathscr{A}$ and some $\alpha > 0$,*

$$\sum_i |\lambda_i(x)|^2 \geq \alpha \|x\|^2.$$

Proof. If ψ is an isomorphic homeomorphism of $\mathscr{A}$ onto l_2, then $\{\psi(x_i)\}$ becomes an orthogonal basis of l_2 and so in view of Theorem 1.8 **[6]**, we may identify $\{\psi(x_i)\}$ with the canonical basis $\{e_i\}$ of l_2. Hence by Bessel's inequality, which is valid in l_2, (i) and (ii) follow from the fact that ψ is a homeomorphism.

For the converse, assume (i) and (ii) hold. For each $x \in \mathscr{A}$ we have a unique representation of x by $x = \Sigma_i \lambda_i(x) x$. Now by replacing y by x^* in (i), we obtain

$$\sum_i |\lambda_i(xx^*)| = \sum_i |\lambda_i(x)|^2 < \infty,$$

for all $x = \Sigma_i \lambda_i(x) x_i \in \mathscr{A}$. If we set

$$\phi(x) = \{\lambda_i(x)\},$$

then the last equation says that $\phi(x) \in l_2$ for all $x \in \mathscr{A}$. Since the operations in l_2 are pointwise, it is easy to see that ϕ is an algebra homomorphism of $\mathscr{A}$ into l_2. By the definition of basis, ϕ is injective. To prove that ϕ maps $\mathscr{A}$ onto l_2, let $a = \{a_i\} \in l_2$. Consider the sequence $\{y_n\}$ of partial sums:

$$y_n = \sum_{i=1}^{n} a_i x_i.$$

First, for $x = \Sigma_i \lambda_i(x) x_i$, $y = \Sigma_i \lambda(y) x_i$, we define

$$\langle x, y\rangle' = \sum_i \lambda_i(x)\overline{\lambda_i(y)}.$$

Clearly by (i), the series in the last equation being absolutely convergent is convergent. Hence $\langle x, y\rangle'$ is defined for all $x, y \in A$ and defines an inner product and hence a norm:

$$\|x\|' = +\sqrt{\langle x, x\rangle'} = \sqrt{\Sigma_i |\lambda_i(x)|^2}.$$

By (ii), we have:

$$(*) \qquad \|x\| \leq \alpha^{-1/2}\|x\|' = \alpha^{-1/2}\|\phi(x)\|_2$$

for all $x \in \mathscr{A}$. Now clearly for $m < n$,

$$\|y_n - y_m\|'^2 = \langle y_n - y_m, y_n - y_m\rangle' = \sum_{i=m+1}^{n} |a_i|^2 \to 0$$

as $m, n \to \infty$ because $\{a_i\} \in l_2$. Hence from $(*)$ it follows that $\{y_n\}$ is a Cauchy sequence in $\mathscr{A}$. Since $\mathscr{A}$ is complete, there is $x \in \mathscr{A}$ with

$$x = \lim_n y_n = \sum_i a_i x_i.$$

In other words, $\lambda_i(x) = a_i$ for all $i \geq 1$, i.e. $\phi(x) = a$. This shows that ϕ is an isomorphism of $\mathscr{A}$ onto l_2. Hence ϕ^{-1}: $l_2 \to \mathscr{A}$ is also an isomorphism. By $(*)$, we also have

$$\|\phi^{-1}(a)\| \leq \alpha^{-1/2}\|a\|$$

for all $a \in l_2$. This shows that ϕ^{-1} is continuous. By an application of the Banach open mapping theorem, ϕ^{-1} is open and therefore ϕ is an isomorphic homeomorphism.

REMARK 6.5. (a) *Theorem* 6.4 *establishes only an isomorphic homeomorphism of* $\mathscr{A}$ *onto* l_2 *but not an isometry. The latter can be obtained if we modify conditions* (i) *and* (ii) *as follows*:

(i)′ *For all* $x, y \in \mathscr{A}$, $\Sigma_i |\lambda_i(xy)| \leq \|xy\|$.

(ii)′ *For all* $x \in \mathscr{A}$, $\Sigma |\lambda_i(x)|^2 \geq \|x\|^2$.

For, (i)′ *and* (ii)′ *imply*

$$\|x\|^2 = \sum_i |\lambda_i(x)|^2 = \|\phi(x)\|_2^2.$$

(b) *Since the basis* $\{e_i\}$ *of* l_2 *is boundedly complete and shrinking*, *Theorem* 6.4 *implies that an unconditional orthogonal basis of a Banach* *algebra $\mathscr{A}$ satisfying* (i) *and* (ii) *of Theorem* 6.4, *is also boundedly complete and shrinking. Hence such a Banach algebra must be reflexive by James' Theorem* ([**2**]).

(c) *Banach algebras with unconditional orthogonal bases satisfying* (i) *and* (ii) *of Theorem* 6.4 *are isomorphically homeomorphic with their duals because l_2 is.*

In view of Remark 6.5 (c), *it is possible to recover the Riesz representation theorem as follows*:

THEOREM 6.6. *Let $\mathscr{A}$ be a Banach algebra with an unconditional orthogonal basis $\{x_i\}$ satisfying conditions* (i) *and* (ii) *of Theorem* 6.4. *Then for $y \in \mathscr{A}$,*

$$f_y(x) = \sum_i \lambda_i(xy) \qquad (x \in \mathscr{A})$$

defines a bounded linear functional on $\mathscr{A}$ and the map: $y \to f_y$ of $\mathscr{A}$ onto $\mathscr{A}'$ is an isometry.

Proof. By Theorem 6.4, $\mathscr{A}$ is isomorphically homeomorphic with l_2. Since the absolute convergence of the series implies its convergence, for each $y \in \mathscr{A}$,

$$f_y(x) = \sum_i \lambda_i(xy) \qquad (x \in \mathscr{A})$$

defines a linear functional on $\mathscr{A}$. Moreover,

$$|f_y(x)| \leq \sum_i |\lambda_i(xy)| \leq \sqrt{\Sigma_i |\lambda_i(x)|^2}\sqrt{\Sigma_i |\lambda_i(y)|^2} < \infty$$

shows that f_y is bounded. Hence $f_y \in \mathscr{A}'$ for each $y \in \mathscr{A}$. To show that the map: $y \to f_y$ is one-to-one, suppose $f_{y_1} = f_{y_2}$ for some $y_1, y_2 \in \mathscr{A}$. Then $f_{y_1}(x) = f_{y_2}(x)$ for all $x \in \mathscr{A}$. If $y_1 \neq y_2$, then there is λ_i such that $\lambda_i(y_1) \neq \lambda_i(y_2)$ in the expansions $y_1 = \Sigma_i \lambda_i(y_1)x_i$, $y_2 = \Sigma_i \lambda_i(y_2)x_i$. But then $f_{y_1}(x_i) = \lambda_i(y_1) \neq \lambda_i(y_2) = f_{y_2}(x_i)$ contradicts the supposition. To see that the map: $y \to f_y$ is also onto and isometric, we denote by φ: $x \to \{\lambda_i(x)\}$ the map of $\mathscr{A}$ onto l_2 as considered in Theorem 6.4. Then the conjugate map φ': $l_2' \to \mathscr{A}'$ as well as its inverse φ'^{-1}: $\mathscr{A} \to l_2'$ are isomorphisms and homeomorphisms. Now let $f \in \mathscr{A}'$. Then $\phi'^{-1}(f) \in l_2'$ and so there exists $a = \{a_i\} \in l_2$ by the Riesz representation theorem for l_2 such that

$$\begin{aligned}\varphi'^{-1}(f)(\alpha) &= \langle \alpha, a \rangle \quad \text{for all } \alpha = \{\alpha_i\} \in l_2 \\ &= \sum_i \bar{a}_i \alpha_i.\end{aligned}$$

By the isomorphism of φ, for each $\alpha = \{\alpha_i\} \in l_2$ there is $x \in \mathscr{A}$ with $\lambda_i(x) = \alpha_i$ i.e. $\varphi(x) = \alpha$ and a $y_0 \in \mathscr{A}$ with $\lambda_i(y_0) = \bar{a}_i$ for $i \geq 1$. Thus for all $x \in \mathscr{A}$, we have

$$f(x) = \varphi'^{-1}(f(\varphi(x))) = \sum_i \bar{a}_i \alpha_i = \sum_i \lambda_i(xy_0)$$
$$= f_{y_0}(x) \quad \text{for all } x \in \mathscr{A}.$$

Hence $f = f_{y_0}$. Furthermore,

$$\|y_0\|' = \sqrt{\Sigma_i |\lambda_i(y_0)|^2} = \sup_{\|x\| \leq 1} |\langle x, y_0 \rangle'|$$
$$= \sup_{\|x\| \leq 1} \left\| \sum_i \lambda_i(xy_0) \right\| = \sup_{\|x\| \leq 1} |f_{y_0}(x)| = \|f_{y_0}\|,$$

where as usual $\|x\|' = \sqrt{\Sigma_i |\lambda_i(x)|^2}$ and $\langle x, y \rangle' = \Sigma_i \lambda_i(x)\overline{\lambda_i(y)}$. This completes the proof.

REFERENCES

[1] W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc., **44** (1945), 364–386.

[2] M. M. Day, *Normed Linear Spaces*, Springer Verlag, NY, Berlin, 1958.

[3] T. Husain, *Topological algebras isomorphic with l_1 or $H(D)$*, Punjab University Math. J., Special Issue, Lahore, Pakistan, **XIV-XV** (1981–82), 1–14.

[4] ______, *Banach algebras isomorphic with c_0*, Ibid. (to appear).

[5] T. Husain and J. Liang, *Multiplicative functionals on Fréchet algebras with bases*, Canadian J. Math., **29** (1977), 270–276.

[6] T. Husain and S. Watson, *Topological algebras with orthogonal Schauder bases*, Pacific J. Math., **91** (1980), 339–347.

[7] ______, *Algebras with unconditional orthogonal bases*, Proc. Amer. Math. Soc., **79** (1980), 539–545.

[8] S. Watson, *$\mathscr{F}$-algebras with cyclic bases*, Comm. Math., **23** (1982), 141–146.

[9] I. Singer, *Bases in Banach Spaces I*, Springer Verlag, Berlin, 1970.

[10] W. Zelazko, *Banach Algebras*, Elsevier, Amsterdam, 1973.

Received August 4, 1983 and in revised form September 29, 1983. This paper was written when the author was visiting the Department of Mathematics, Institute of Advanced Studies, Australian National University, Canberra. This work was partially supported by a NSERC grant.

MCMASTER UNIVERSITY
HAMILTON, ONTARIO, CANADA (L85 4K1)

Current address: Institute of Advanced Studies
Australian National University
Canberra, Australia