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GENERALIZED COMPLETE MAPPINGS, NEOFIELDS, SEQUENCEABLE GROUPS AND BLOCK DESIGNS. II

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In part I we introduced the concepts of generalized complete mapping and generalized near complete mapping of a group and used them to characterize left neofields. In §5 of this second part we describe how (K, λ) complete mappings and near complete mappings may be used to provide new constructions of block designs of Mendelsohn type; and in §6 we describe various methods of obtaining generalized complete mappings.

For the convenience of the reader we again give the definitions of these concepts, but for all other matters we refer to part I. We denote the sections of this paper by 5 and 6 (those of the earlier paper being 1 to 4) and use decimal notation for the numbering of theorems and definitions. A reference to Theorem 2.1 is therefore a reference to the first theorem in §2 of part I. However, bibliographical references for this second part are independent of those in part I.

A number of examples illustrative of the various concepts which we introduce will be found in the Appendix. We shall refer to them by number.

Definitions of a (K, λ) *complete mapping and* (K, λ) *near complete mapping* (Definition 2.4 of part I).

A (K, λ) *complete mapping*, where $K = \{k_1, k_2, \ldots, k_s\}$ and the k_i are integers such that $\sum_{i=1}^{s} k_i = \lambda(|G| - 1)$, is an arrangement of the non-identity elements of a group G (each used λ times) into s cyclic sequences (of distinct elements) of lengths $k_1, k_2, \ldots, k_s$, say

$$(g_{11}\, g_{12} \cdots g_{1k_1})(g_{21}\, g_{22} \cdots g_{2k_2}) \cdots (g_{s1}\, g_{s2} \cdots g_{sk_s}),$$

such that the elements $g_{ij}^{-1} g_{i,j+1}$ (where $i = 1, 2, \ldots, s$; and the second suffix j is added modulo k_i) comprise the non-identity elements of G each counted λ times.

A (k, λ) *near complete mapping*, where $K = \{h_1, h_2, \ldots, h_r; k_1, k_2, \ldots, k_s\}$ and the h_i and k_j are integers such that $\sum_{i=1}^{r} h_i + \sum_{j=1}^{s} k_j = \lambda|G|$, is an arrangement of the elements of G (each used λ times) into r sequences with lengths $h_1, h_2, \ldots, h_r$ and s cyclic sequences with lengths

$k_1, k_2, \ldots, k_s$, say,

$$\left[g'_{11}\, g'_{12} \cdots g'_{1h_1}\right] \cdots \left[g'_{r1}\, g'_{r2} \cdots g'_{rh_r}\right](g_{11}\, g_{12} \cdots g_{1k_1}) \cdots (g_{s1}\, g_{s2} \cdots g_{sk_s})$$

such that the elements $(g'_{ij})^{-1}g'_{i,j+1}$ and $g_{ij}^{-1}g_{i,j+1}$ together with the elements $g_{ik_i}^{-1}g_{i1}$ comprise the non-identity elements of G each counted λ times. (We have $\Sigma(h_i - 1) + \Sigma k_j = \lambda(|G| - 1)$ so it is immediate from the definition itself that $r = \lambda$.)

Definitions of a (k, λ) *complete mapping and a* (k, λ) *near complete mapping* (Definition 2.5 of part I)

A (k, λ) *complete mapping* is a (K, λ) complete mapping such that $K = \{k, k, \ldots, k\}$. For such a generalized complete mapping, $s = \lambda(|G| - 1)/k$.

Similarly, *a* (k, λ) *near complete mapping* is a (K, λ) near complete mapping such that $K = \{h, h, \ldots, h; k, k, \ldots, k\}$ and $k - h = 1$.

5. Block designs of Mendelsohn type. In this section we show how the concepts of (K, λ) complete mapping and (K, λ) near complete mapping can be used to construct block designs of Mendelsohn type whose automorphism group contains a specified group G as subgroup. For this purpose it is convenient to use additive notation for cyclic groups.

DEFINITION 5.1. A block design of Mendelsohn type comprises a set G of v elements and a collection of b cyclically ordered subsets of G called *blocks* of cardinalities $k_1, k_2, \ldots, k_b$, respectively, with the property that every ordered pair of elements of G are consecutive in exactly λ of the blocks. We call such a design a (v, K, λ) *Mendelsohn design*, where K is the set formed by the distinct integers among $k_1, k_2, \ldots, k_b$. More briefly, we shall write it as a (v, K, λ)-MD and denote the set of blocks by B. (See Examples 1–3.)

DEFINITION 5.2. A (v, k, λ)-MD is a (v, K, λ)-MD such that $k_1 = k_2 = \cdots = k_b = k$. (See Examples 2–5.)

In Example 2 of the Appendix the mapping α: $i \to i + 1$, $\infty \to \infty$ maps B_{ij} onto $B_{i+1,j}$, where $B = \bigcup_{i=0}^{12}\{B_{ij}: j = 1, 2, 3, 4\}$, and defines an automorphism of the Mendelsohn design (G, B), which permutes both the objects and blocks, each in a cycle of length 13. This example and also Example 4 suggest the following concept.

DEFINITION 5.3. Let $D = (G, B)$ be a (v, K, λ)-MD and let L be a group of v permutations $\alpha_1, \alpha_2, \ldots, \alpha_v$ of G such that $G = \bigcup_{i=1}^{v} \alpha_i(g)$, where g is any fixed element of G. Suppose further that there exists a subset $B^* = \{B_1, B_2, \ldots, B_f\}$ of blocks of B such that $B = \bigcup_{i=1}^{v} \{\alpha_i(B_1), \alpha_i(B_2), \ldots, \alpha_i(B_f)\}$. Then it follows that each permutation of L permutes the blocks of B among themselves and is an automorphism of D. We say that D admits L as a *regular group of automorphisms* with the blocks of B^* as *basis blocks*.

We observe that, in Examples 2 and 4 of the Appendix, the elements of G are themselves elements of a group. In each case the left regular repesentation L_G of this group has the properties required by Definition 5.3 and the basis blocks are the sets {(1 4 3 12 9 10), (2 6 4 11 7 9), (5 2 10 8 11 3), (6 1 8 7 12 5)} and $\{a\, a^2\, ba^2\, ba\, ba^3\, a^3\, b)\}$, respectively. Moreover, in each case, the basis blocks define the cyclic sequences of a (K, λ) complete mapping of the group defined on the set G.

We may state the following:

THEOREM 5.1. *Let L_G be the left regular representation of a group $(G, \cdot)$ of order v and suppose that there exists a (v, K, λ)-MD defined on the set G which admits L_G as a regular group of automorphisms with the blocks*

$$(g_{11}\, g_{12} \cdots g_{1k_1}), (g_{21}\, g_{22} \cdots g_{2k_2}) \cdots (g_{s1}\, g_{s2} \cdots g_{sk_s})$$

as basis blocks, where $K = \{k_1, k_2, \ldots, k_s\}$ and where one element x of G does not occur at all among the elements of the basis blocks but every other element of G occurs exactly λ times, then the cyclic sequences

$$\left(x^{-1}g_{11}\, x^{-1}g_{12} \cdots x^{-1}g_{1k_1}\right), \left(x^{-1}g_{21}\, x^{-1}g_{22} \cdots x^{-1}g_{2k_2}\right), \ldots, \left(x^{-1}g_{s1}\, x^{-1}g_{s2} \cdots x^{-1}g_{sk_s}\right)$$

form a (K, λ) complete mapping of the group $(G, \cdot)$.

Conversely, if

$$(g_{11}\, g_{12} \cdots g_{1k_1}), (g_{21}\, g_{22} \cdots g_{2k_2}), \ldots, (g_{s1}\, g_{s2} \cdots g_{sk_s})$$

is a (K, λ) complete mapping of a group $(G, \cdot)$ of order v with $K = \{k_1, k_2, \ldots, k_s\}$ then the blocks of the set

$$B = \bigcup_{g \in G} \left\{(gg_{11}\, gg_{12} \cdots gg_{1k_1}), (gg_{21}\, gg_{22} \cdots gg_{2k_2}), \ldots, (gg_{s1}\, gg_{s2} \cdots gg_{sk_s})\right\}$$

form a (v, K, λ)-MD which admits the left regular representation L_G of

$(G, \cdot)$ as a regular group of automorphisms such that the cyclic sequences which define the (K, λ) complete mapping of G are its basis blocks.

Proof. For the first part, we wish to show that the elements $(x^{-1}g_{ij})^{-1}$, $(x^{-1}g_{i,j+1})$, where $i = 1, 2, \ldots, s$ and the second suffix j is added modulo k_i, comprise the non-identity elements of G each counted λ times. We know that each of the ordered pairs (g, gu), where u is an assigned element of G and g varies through the set G, occurs as a pair of consecutive elements λ times among the blocks of the set

$$B = \bigcup_{y \in G} \{(yg_{11}\, yg_{12} \cdots yg_{1k_1}), (yg_{21}\, yg_{22} \cdots yg_{2k_2}), \ldots, (yg_{s1}\, yg_{s2} \cdots yg_{sk_s})\}.$$

Suppose that the ordered pair $(yg_{ij}, yg_{i,j+1})$ is one of these occurrences. Then $g = yg_{ij}$ and $gu = yg_{i,j+1}$. So

$$u = (yg_{ij})^{-1}(yg_{i,j+1}) = g_{ij}^{-1}g_{i,j+1} = \left(x^{-1}g_{ij}\right)^{-1}\left(x^{-1}g_{i,j+1}\right).$$

No two of the ordered pairs $(yg_{ij}, yg_{i,j+1})$ which are equal to (g, gu) can involve the same g_{ij} since otherwise they would also involve the same y. It follows that the element u of G appears λ times among the elements $(x^{-1}g_{ij})^{-1}(x^{-1}g_{i,j+1})$, as required.

For the converse, we wish to show first that each ordered pair of elements of G occurs consecutively λ times among the blocks of the set B defined in the statement of the theorem. Let (u, v) be an ordered pair of elements of G. In the given (K, λ) complete mapping of the group $(G, \cdot)$, the element $u^{-1}w$ occurs λ times in the form $g_{ij}^{-1}g_{i,j+1}$ for suitable values of i and j. Define t as the solution of the equation $gg_{ij} = u$. Then $w = u(u^{-1}w) = gg_{i,j+1}$ so the ordered pair $(gg_{ij}, gg_{i,j+1})$ for this value of g is the pair (u, w). It follows that the pair (u, w) occurs λ times among the blocks of B and so (G, B) is a (v, K, λ)-MD. The remaining statement of the theorem follows immediately from Definition 5.3.

COROLLARY. *If the group $(G, \cdot)$ of order v is R-sequenceable then there exists a $(v, v-1, 1)$-MD which admits the group L_G of permutations of the left regular representation of $(G, \cdot)$ as a regular group of automorphisms.* (*See Example* 6.)

THEOREM 5.2. *If*

$$\left[g'_{11}\, g'_{12} \cdots g'_{1h_1}\right] \cdots \left[g'_{\lambda 1}\, g'_{\lambda 2} \cdots g'_{\lambda h_\lambda}\right](g_{11}\, g_{12} \cdots g_{1k_1}) \cdots (g_{s1}\, g_{s2} \cdots g_{sk_s})$$

is a (K, λ) *near complete mapping of a group* $(G, \cdot)$ *of order* $v - 1$ *with* $K = \{h_1, h_2, \ldots, h_\lambda; k_1, k_2, \ldots, k_s\}$ *then the blocks of the set*

$$B = \bigcup_{g \in G} \{(gg'_{11}\, gg'_{12} \cdots gg'_{1h_1}\, \infty), (gg'_{21}\, gg'_{22} \cdots gg'_{2h_2}\, \infty), \ldots,$$
$$(gg'_{\lambda 1}\, gg'_{\lambda 2} \cdots gg'_{\lambda h_\lambda}\, \infty), (gg_{11}\, gg_{12} \cdots gg_{1k_1}),$$
$$(gg_{21}\, gg_{22} \cdots gg_{2k_2}), \ldots, (gg_{s1}\, gg_{s2} \cdots gg_{sk_s})\}$$

form a (v, k^*, λ)*-MD on the set* $G^* = G \cup \{\infty\}$, *where* $K^* = \{h_1 + 1, h_2 + 1, \ldots, h_\lambda + 1; k_1, k_2, \ldots, k_s\}$. *Moreover the group of permutations*

$$\bigcup_{g \in G} \begin{pmatrix} g_1 & g_2 & \cdots & g_{v-1} & \infty \\ gg_1 & gg_2 & \cdots & gg_{v-1} & \infty \end{pmatrix} \quad \text{where } G = \{g_1, g_2, \ldots, g_{v-1}\},$$

acts as a regular group of automorphisms on this design with basis blocks $(g'_{11}\ g'_{12} \cdots g'_{1h_1}\ \infty), \ldots, (g'_{\lambda 1}\ g'_{\lambda 2} \cdots g'_{\lambda h_\lambda}\ \infty)$, $(g_{11}\ g_{12} \cdots g_{1k_1}), \ldots,$ $(g_{s1}\ g_{s2} \cdots g_{sk_s})$.

Proof. It is easy to check directly that each ordered pair (g, ∞) and also each ordered pair (∞, g) occur λ times as a pair of consecutive elements among the blocks of the set B. The remainder of the proof is exactly similar to that of the second part of Theorem 5.1.

COROLLARY. *If the group* $(G, \cdot)$ *of order* $v - 1$ *is sequenceable then there exists a* $(v, v, 1)$*-MD which admits the group of permutations*

$$\bigcup_{g \in G} \begin{pmatrix} g_1 & g_2 & \cdots & g_{v-1} & \infty \\ gg_1 & gg_2 & \cdots & gg_{v-1} & \infty \end{pmatrix}$$

as a regular group of automorphisms.

Examples 7 and 8 illustrate Theorem 5.2.

As special cases of Theorems 5.1 and 5.2, we see that a (k, λ) complete mapping of a group of order v and a (k, λ) near complete mapping of a group of order $v - 1$ each give rise to a (v, k, λ)-MD. We consider next the question of when such a design is resolvable.

Since the number of blocks in a (v, k, λ)-MD is $\lambda v(v - 1)/k$, which must be an integer, we have $v(v - 1) \equiv 0 \bmod k$ in the case when $\lambda = 1$.

DEFINITION 5.4. If the blocks of $(v, k, 1)$-MD for which $v \equiv 1 \bmod k$ can be partitioned into v sets each containing $(v - 1)/k$ blocks which are pairwise disjoint (as sets), we say that the $(v, k, 1)$-MD is *resolvable* and any such parititon is called a *resolution* of the design.

Each set of $(v-1)/k$ pairwise disjoint blocks together with the singleton which is the only element not in any of its blocks is called a *parallel class* of the resolution. Any resolution of this kind has v parallel classes.

These definitions were first given in [**2**].

DEFINITION 5.5. If the blocks of a $(v, k, 1)$-MD for which $v \equiv 0 \bmod k$ can be partitioned into $v-1$ sets each containing v/k blocks which are pairwise disjoint (as sets), we shall again say that the $(v, k, 1)$-MD is *resolvable*.

Each set of v/k pairwise disjoint blocks will be called a *parallel class*.

A Mendelsohn design which is resolvable in either of the above senses will be denoted as a $(v, k, 1)$-RMD.

By Theorem 5.1, Theorem 5.2 and the above two definitions, we have

THEOREM 5.3. *Every* $(v, k, 1)$-MD *obtained from a* $(k, 1)$ *complete mapping in the manner of Theorem* 5.1 *is resolvable. Likewise, every* $(v, k, 1)$-MD *derived from a* $(k, 1)$ *near complete mapping in the manner of Theorem* 5.2 *is resolvable.*

Note that by Theorems 5.1 and 5.3, a $(v, k, 1)$-MD derived from a $(k, 1)$ complete mapping of a cyclic group must be cyclic[1] and resolvable. It is proved in [**2**] and [**3**] that a $(v, 3, 1)$-RMD exists if $v \equiv 1 \pmod 3$ and in [**4**] that a $(v, 3, 1)$-RMD exists if $v \equiv 0 \pmod 3$. On the other hand, it is proved in [**5**] that a cyclic $(v, 3, 1)$-MD exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$. It is conjectured (see [**7**] and [**8**]) that $(3, 1)$ complete mappings of the cyclic group C_v exist for all $v \equiv 1 \pmod 6$. If this were proved, we should then have that cyclic $(v, 3, 1)$-RMD's exist if and only if $v \equiv 1 \pmod 6$ (the other direction of the implication is easy to see). We shall give a general account of the construction of complete mappings and near complete mappings in the next section.

DEFINITION 5.6. Let S be a given set and let $B_i = (a_{i1}\ a_{i2} \cdots a_{ik})$ be a cyclically ordered subset of k elements of S. Then the elements a_{ir} and $a_{i,r+t}$, where addition of the second suffix is modulo k, are said to be *t-apart* in the cyclic k-tuple B_i.

A (v, k, λ) Mendelsohn design (G, B) is said to be *l-fold perfect* if each ordered pair (x, y) of elements of G appears t-apart in exactly λ of

[1] That is, it admits a regular group of automorphisms which is cyclic.

the blocks of B for all $t = 1, 2, \ldots, l$. If $l = k - 1$, the design (G, B) is said to be *perfect*. We call such a design a (v, k, λ)-PMD.

We show next that Mendelsohn designs which are l-fold perfect can be constructed with the aid of generalized complete and near complete mappings which have appropriate properties and we give a number of examples of such constructions.

DEFINITION 5.7. Let

$$(g_{11}\, g_{12} \cdots g_{1k})(g_{21}\, g_{22} \cdots g_{2k}) \cdots (g_{s1}\, g_{s2} \cdots g_{sk})$$

be a (k, λ) complete mapping of a group $(G, \cdot)$ of order v such that for each value of t, $t = 1, 2, \ldots, l$, the elements $g_{ij}^{-1} g_{i,j+t}$ (where $i = 1, 2, \ldots, s$; and the second suffix j is added modulo k) comprise the non-identity elements of G each counted λ times. Then the complete mapping is said to be an *l-fold perfect (k, λ) complete mapping*. If $l = k - 1$, the mapping is said to be a *perfect (k, λ) complete mapping*.

We note that a 1-fold perfect (k, λ) complete mapping is a (k, λ) complete mapping as given in Definition 2.5 of [**10**].

DEFINITION 5.8. Let

$$[g'_{11}\, g'_{12} \cdots g'_{1h}] \cdots [g'_{\lambda 1}\, g'_{\lambda 2} \cdots g'_{\lambda h}](g_{11}\, g_{12} \cdots g_{1k}) \cdots (g_{s1}\, g_{s2} \cdots g_{sk}), \quad \text{where } k - h = 1,$$

be a (k, λ) near complete mapping of a group $(G, \cdot)$ of order v and let B^* denote the set of $\lambda + s$ k-tuples

$$(g'_{11}\, g'_{12} \cdots g'_{1k})(g'_{21}\, g'_{22} \cdots g'_{2k}) \cdots (g'_{\lambda 1}\, g'_{\lambda 2} \cdots g'_{\lambda k})(g_{11}\, g_{12} \cdots g_{1k}) \cdots (g_{s1}\, g_{s2} \cdots g_{sk}), \quad \text{where } g'_{ik} = \infty \text{ for } i = 1, 2, \ldots, \lambda.$$

If, for each value of t, $t = 1, 2, \ldots, l$, the elements of the set

$$\left(\bigcup_{i=1}^{\lambda} \{ g'^{-1}_{ij} g'_{i,j+t}\ j = 1, 2, \ldots, k-1 \} \cup \left(\bigcup_{i=1}^{s} \{ g_{ij}^{-1} g_{i,j+t};\ j = 1, 2, \ldots, k \} \right) \right.$$

comprise all the non-identity elements of the set $G \cup \{\infty\}$ each counted λ times where we define $g^{-1}\infty = \infty$, then the near complete mapping is called *l-fold perfect*. If $l = k - 1$, the mapping is called *perfect*.

A 1-fold perfect (k, λ) near complete mapping is a (k, λ) near complete mapping as given in Definition 2.5 of [**10**].

THEOREM 5.4. *If there exists an l-fold perfect (k, λ) complete mapping of a group $(G, \cdot)$ of order v then, by the construction of Theorem* 5.1, *there exists a (v, k, λ)-MD which is l-fold perfect and on which the left regular representation L_G of G acts as a regular group of automorphisms. Likewise,*

if there exists an l-fold perfect (k, λ) *near complete mapping of the group then, by the construction of Theorem* 5.2, *there exists a* $(v + 1, k, \lambda)$*-MD with these same properties.*

Proof. The proof is closely similar to that of Theorems 5.1 and 5.2. We shall prove the second statement of the theorem.

We wish to show that each ordered pair of elements of the set $G \cup \{\infty\}$ occur t-apart (for $t = 1, 2, \ldots, l$) λ times among the blocks of the set

$$B = \bigcup_{g \in G} \{(gg'_{11}\, gg'_{12} \cdots gg'_{1,k-1}\, \infty), (gg'_{21}\, gg'_{22} \cdots gg'_{2,k-1}\, \infty), .,$$
$$(gg'_{\lambda 1}\, gg'_{\lambda 2} \cdots gg'_{\lambda,k-1}\, \infty), (gg_{11}\, gg_{12} \cdots gg_{1k}),$$
$$(gg_{21}\, gg_{22} \cdots gg_{2k}), \ldots, (gg_{s1}\, gg_{s2} \cdots gg_{sk})\}.$$

Let (u, w) be an ordered pair of elements of G. In the given l-fold perfect (k, λ) near complete mapping of the group $(G, \cdot)$, the element $u^{-1}w$ occurs λ times in the form $g'^{-1}_{ij} g'_{i,j+t}$ or $g^{-1}_{ij} g_{i,j+t}$ for suitable values of i and j, where the second suffix is added modulo k. Define g as the solution of the equation $gg^*_{ij} = u$ where g^*_{ij} is the first member of one of the products which is equal to $u^{-1}w$. Then $w = u(u^{-1}w) = gg^*_{i,j+t}$ where $g^*_{i,j+t}$ is the second member of the product. It follows that, for this value of g, the ordered pair $(gg^*_{ij}, gg^*_{i,j+t})$ is the pair (u, w). So, the pair (u, w), for each choice of u, $w \in G$, occurs λ times among the blocks of B. We have also to show that each ordered pair (u, ∞) and each ordered pair (∞, u) for $u \in G$ occur t-apart λ times. To see this, define g_t as the solution of the equation $g_t g'_{t,k-t} = u$ $(i = 1, 2, \ldots, \lambda)$. Then $(g_t g'_{i,k-t}, \infty)$ is the ordered pair (u, ∞). Since i can take λ values, we get λ occurrences of this ordered pair in the set of blocks B. Finally, define $\bar{g}_i$ as the solution of the equation $\bar{g}_i g'_{it} = u$ $(i = 1, 2, \ldots, \lambda)$. Then $(\infty, \bar{g}_i g'_{it})$ is the ordered pair (∞, u). Again i can take λ values so we get λ occurrences of this ordered pair in the set of blocks B.

The general problem of constructing l-fold perfect generalized complete mappings turns out to be a difficult one. Even for cyclic groups, the problem is not trivial. However, we have obtained a number of examples; in particular, Examples 9, 10, 11 and 12 of the Appendix.

We remark that any $(3, \lambda)$ complete mapping must be perfect. See Theorem 5.6 below.

In [**12**], an l-fold perfect $(v - 1, 1)$ complete mapping of a group $(G, \cdot)$ of order v has been called an R_l*-sequencing* of the group (see also Definition 6.2) and it has been shown in [**6**], page 237, Theorem 7.4.1, and [**11**] that if a group G is R_l-sequenceable then it is possible to construct at

least $l + 1$ mutually orthogonal latin squares based on the Cayley table of G. Unfortunately, the problem of deciding which groups are R_l-sequenceable for a particular value of l seems not to be a simple one. However, we have

THEOREM 5.5. *The elementary abelian group of order p^n has a $(p^n - 1, 1)$ complete mapping which is perfect.*

Proof. This follows from the fact that such groups are R_{p^n-2}-sequenceable. (See [**12**].)

Computer searches reported in [**12**] have shown that no 2-fold perfect $(8, 1)$ complete mapping of the cyclic group C_9 exists, that no 3-fold perfect $(14, 1)$ complete mapping of the cyclic group C_{15} exists but that this latter group has 32 isomorphically distinct 2-fold perfect $(14, 1)$ complete mappings. (Three of these latter were used to construct our Example 9.)

From Definition 2.5 of [**10**], it follows that, for an l-fold perfect (k, λ) complete mapping of a group $(G, \cdot)$ of order v, k divides $\lambda(v - 1)$. Consequently, for the corresponding l-fold perfect (v, k, λ)-MD constructed as in Theorem 5.4 we must have $\lambda(v - 1) \equiv 0 \pmod{k}$. The analogous result for an l-fold perfect (k, λ) near complete mapping of a group of order $v - 1$ is $\lambda v \equiv 0 \bmod k$ since in this case we have $\lambda(k - 1) + sk = \lambda(v - 1)$, where s is as in Definition 2.4 of [**10**]. So, for the corresponding l-fold perfect (v, k, λ)-MD constructed as in Theorem 5.4, we have $\lambda v \equiv 0 \bmod k$ in this case.

In [**2**], a $(v, k, 1)$ resolvable perfect Mendelsohn design is constructed for all sufficiently large v with $v \equiv 1 \pmod{k}$. Also, in [**13**], Mendelsohn has obtained $(v, k, 1)$ perfect Mendelsohn designs with $v(v - 1) \equiv 0 \pmod{k}$ for the case when k is an odd prime and v is sufficiently large. Theorem 5.4 enlarges the spectrum of (v, k, λ) perfect Mendelsohn designs in both these cases. Moreover, the designs obtained admit a regular group of automorphisms and, when $\lambda = 1$, they are resolvable.

THEOREM 5.6. *Every $(3, \lambda)$ complete or near complete mapping is perfect.*

Proof. Let $(g_{11}\ g_{12}\ g_{13})(g_{21}\ g_{22}\ g_{23}) \cdots (g_{s1}\ g_{s2}\ g_{s3})$ be a $(3, \lambda)$ complete mapping of a group $(G, \cdot)$ of order v. Each pair of elements are either 1-apart or 2-apart.

Since $g_{ij}^{-1}g_{i,j+2} = (g_{i,j+2}^{-1}g_{ij})^{-1}$, where the second suffix is added modulo 3, and since the elements $g_{1,j+2}^{-1}g_{1j}$ comprise the non-identity elements of G each counted λ times by definition of a complete mapping, the complete mapping is 2-fold perfect and so perfect.

Next let $[g'_{11}\ g'_{12}]\cdots[g'_{\lambda 1}\ g'_{\lambda 2}](g_{11}\ g_{12}\ g_{13})\cdots(g_{s1}\ g_{s2}\ g_{s3})$ be a $(3, \lambda)$ near complete mapping. The elements $g_{i1}'^{-1}g'_{i2}$ and $g_{ij}^{-1}g_{i,j+1}$ (for $j = 1, 2$) together with the elements $g_{i3}^{-1}g_{i1}$ comprise the non-identity elements of G each counted λ times. Consequently, so also do the elements $g_{i2}'^{-1}g'_{i1}$ and $g_{i,j+1}^{-1}g_{ij}$ for $(j = 1, 2)$ together with the elements $g_{i1}^{-1}g_{i3}$. From this, we see at once that the set B^* of $\lambda + s$ triples $(g'_{11}\ g'_{12}\ \infty)\cdots(g'_{\lambda 1}\ g'_{\lambda 2}\ \infty)(g_{11}\ g_{12}\ g_{13})\cdots(g_{s1}\ g_{s2}\ g_{s3})$ satisfies the condition given in Definition 5.8 for the near complete mapping to be 2-fold perfect and so perfect.

From Theorem 5.6, it follows that sufficient conditions for the existence of a $(v, 3, \lambda)$-PMD are the existence of a $(3, \lambda)$ complete mapping of a group of order v or of a $(3, \lambda)$ near complete mapping of a group of order $v - 1$. In either case the perfect Mendelsohn design admits the group G as a regular group of automorphisms. (See Example 13.)

6. Constructions for generalized complete mappings. In this final section, we provide a number of different constructions for generalized complete and near complete mappings.

We begin with the following simple observations:

(i) If we repeat the cycles and sequences of a (K, λ) generalized complete or near complete mapping h times, we get an $(hK, h\lambda)$ generalized complete mapping. Such a construction is called *trivial* and the generalized complete mapping so obtained is called *trivially decomposable*.

(ii) If we reverse the order of the elements in all the cycles and sequences of a generalized complete or near complete mapping, we get another such mapping. Moreover, if

$$\left[g'_{11}\ g'_{12}\cdots g'_{1h_1}\right]\cdots\left[g'_{\lambda 1}\ g'_{\lambda 2}\cdots g'_{\lambda h_\lambda}\right](g_{11}\ g_{12}\cdots g_{1k_1})\cdots(g_{s1}\ g_{s2}\cdots g_{sk_s})$$

is a (K, λ) near complete mapping which satisfies the condition

$$\left\{g_{ij}'^{-1}g'_{i,j+1}: i = 1,2,\ldots,\lambda;\ j = 1,2,\ldots,h_i\right\} = \left\{g_{i,j+1}'^{-1}g'_{ij}: i = 1,2,\ldots,\lambda;\ j = 1,2,\ldots,h_i\right\},$$

then the mapping

$$\left[g'_{11}\ g'_{12}\cdots g'_{1h_1}\right]\cdots\left[g'_{\lambda 1}\ g'_{\lambda 2}\cdots g'_{\lambda h_\lambda}\right](g_{1k_1}\ g_{1,k_1-1}\cdots g_{11})\cdots(g_{sk_s}\ g_{s,k_s-1}\cdots g_{s1})$$

is another near complete mapping. (See Example 14.)

(iii) If we adjoin a (K, λ) generalized complete mapping to its reverse, we get a $(2K, 2\lambda)$ generalized mapping. A mapping which can be so constructed is called *patterned*.

A generalized complete mapping obtained from a given generalized complete mapping by the methods of (ii) above has the parameters K and λ unchanged. However, this invariant property is combinatorial rather than algebraic because this type of transformation does change the properties of the corresponding neofield. (See Examples 15 and 16.)

A construction very similar to that of (ii) which yields a $(v + 1, 2)$ generalized near complete mapping of a group G of order v if G is quasisequenceable is the following. Suppose that $b_0, b_1, b_2, \ldots, b_{v-1}$ is the terrace associated with the quasisequencing (see [1] for the definitions). It has the property that the $v - 1$ elements $b_0^{-1}b_1, b_1^{-1}b_2, \ldots, b_{v-2}^{-1}b_{v-1}$ contain each element of order 2 of G once and, for each other non-identity element of G, either g twice or g^{-1} twice or both g and g^{-1}. Consequently, $[b_0\, b_1\, b_2 \cdots b_{v-1}][b_{v-1}\, b_{v-2} \cdots b_0]$ is a $(v + 1, 2)$ generalized near complete mapping of G. It can be expressed in canonical form by multiplying the elements of the first sequence by b_0^{-1} and those of the second by b_{v-1}^{-1}. Moreover, if $b_{v-1}^{-1}b_0 = b_i^{-1}b_{i+1}$ then

$$[b_0\, b_1\, b_2 \cdots b_{v-1}][b_i\, b_{i-1} \cdots b_0\, b_{v-1}\, b_{v-2} \cdots b_{i+2}\, b_{i+1}]$$

also is a $(v + 1, 2)$ generalized complete mapping. (Compare page 329 of [1].)

A slightly less obvious way of constructing new generalized complete mappings comes from the observation that, if a group G can be expressed as a direct product $G = G_1 \times G_2$ then a complete mapping of G can be constructed from complete mappings of its direct factors G_1 and G_2. Precisely, we have

THEOREM 6.2. *If there exists a (k_1, λ_1) complete mapping of the group G_1 and a (k_2, λ_2) complete mapping of the group G_2, then there exists a (K, λ) complete mapping of the group $G_1 \times G_2$, where $\lambda = \lambda_1 \cdot \lambda_2$, $K = \{k, k, \ldots, k, k_1, k_1, \ldots, k_1, k_2, k_2, \ldots, k_2\}$ and k is the least common multiple of k_1 and k_2.*

Proof. Suppose that

$$\sigma_i = \left(g_{11}^{(i)}\, g_{12}^{(i)} \cdots g_{1k_i}^{(i)}\right)\left(g_{21}^{(i)}\, g_{22}^{(i)} \cdots g_{2k_i}^{(i)}\right) \cdots \left(g_{s_i,1}^{(i)}\, g_{s_i,2}^{(i)} \cdots g_{s_i,k_i}^{(i)}\right)$$

is a (k_i, λ_i) complete mapping of G_i, $i = 1, 2$. Let G_i^* denote the set of non-identity elements of G_i. Then we may regard σ_i as a permutation from $\lambda_i G_i^*$ to $\lambda_i G_i^*$ with cyclic structure $\{k_i, k_i, \ldots, k_i\}$ and such that $\sigma_i(g_{j,l}^{(i)}) = g_{j,l+1}^{(i)}$. Here, the second suffix is added modulo k_i and $\lambda_i G_i^*$ is a set

which consists of the non-identity elements of G_i each counted λ times. The permutation σ_i satisfies the extra condition that the set $\{g^{-1}\sigma_i(g): g \in \lambda_i G_i^*\}$ coincides with the set $\lambda_i G_i^*$.

We shall obtain the desired (K, λ) complete mapping of $G_1 \times G_2$ by constructing its associated permutation σ.

We define $\sigma_i(e_i) = e_i$, where e_i is the identity element of G_i. Let σ be the permutation from $\lambda(G_1 \times G_2)^*$ to $\lambda(G_1 \times G_2)^*$ defined by $\sigma(g_1, g_2) = (\sigma_1(g_1), \sigma_2(g_2))$ where $(g_1, g_2) \in \lambda(G_1 \times G_2)^*$. Thus,

$$\sigma\left(g_{j_1 l_1}^{(1)}, g_{j_2 l_2}^{(2)}\right) = \left(g_{j_1, l_1+1}^{(1)}, g_{j_2, l_2+1}^{(2)}\right), \quad \sigma\left(g_{j_1 l_1}^{(1)}, e_2\right) = \left(g_{j_1, l_1+1}^{(2)}, e_2\right)$$

and

$$\sigma\left(e_1, g_{j_2 l_2}^{(2)}\right) = \left(e_1, g_{j_2, l_2+1}^{(2)}\right).$$

From this it is immediately clear that if (g_1, g_2) is an element of $\lambda(G_1 \times G_2)^*$ for which $g_1 \neq e_1$ and $g_2 \neq e_2$ then each of the $\lambda = \lambda_1 \cdot \lambda_2$ cycles in which (g_1, g_2) appears is of length $k = [k_1, k_2]$, where $[k_1, k_2]$ denotes lowest common multiple. If $g_1 = e_1$, $g_2 \neq e_2$, then each of the λ cycles in which (g_1, g_2) appears is of length k_2 and these cycles comprise λ_1 copies of a set of cycles isomorphic to those of the generalized complete mapping σ_2. A similar remark applies to the case $g_1 \neq e_1$, $g_2 = e_2$. Finally, it is easy to check that there are all together $(k_1 k_2/k) s_1 s_2$ cycles of length $k = [k_1, k_2]$, $\lambda_2 s_1$ cycles of length k_1 and $\lambda_1 s_2$ cycles of length k_2, whence the set $\{g^{-1}\sigma(g): g \in \lambda(G_1 \times G_2)^*\}$ has cardinal $s_1 s_2 k_1 k_2 + \lambda_2 s_1 k_1 + \lambda_1 s_2 k_2$. It is also easy to check that this set coincides with the set $\lambda(G_1 \times G_2)^*$ and so σ is a generalized complete mapping of $G_1 \times G_2$. [Note that ord $G_i = 1 + s_i k_i/\lambda_i$, whence

$$\text{ord } G^* = (1 + s_1 k_1/\lambda_1)(1 + s_2 k_2/\lambda_2) - 1$$

and so λG^* has cardinality $(\lambda_1 + s_1 k_1)(\lambda_2 + s_2 k_2) - \lambda_1 \lambda_2$.]

Since the direct product of two cyclic groups is cyclic if and only if the orders of the two cyclic groups are relatively prime, we can use Theorem 6.1 to construct generalized mappings of a cyclic group C_n from those of cyclic groups of smaller orders provided that these orders are relatively prime factors of n.

In the Appendix, we give two illustrations of the theorem which emphasize this (Examples 17 and 18).

The investigation of methods for constructing (K, λ) generalized complete mappings with $\lambda > 1$ was first initiated in [**9**], page 43, where certain $(K, 2)$ generalized complete mappings of a cyclic group C_n were constructed. It was shown (in Chapter 3 of [**9**]) that there exist $(6, 2)$ complete mappings of C_n for $n \equiv 1, 7$ or $13 \pmod{18}$; that there exist

$(K, 2)$ complete mappings of C_n for $n \equiv 3 \pmod{18}$ with $K = \{2, 2, 6, 6, \ldots, 6\}$ and that there exist $(K, 2)$ near complete mappings of C_n for $n \equiv 2, 8$ or $14 \pmod{18}$ with $K = \{2, 2; 6, 6, \ldots, 6\}$. It was also shown that the cyclic neofields associated with these generalized complete mappings (see Theorem I.46 in [**9**]) all have the XIP-property. (A cyclic neofield N is said to have the *crossed-inverse property* (XIP-property) if $(x + y) + (-x) = y$, for all $x, y \in N$). For illustration, see Examples 19 to 23.

Example 19 is different from Examples 20 and 22 in the sense that if we consider the cyclic sequences in each example as subsets of the respective groups then each of Examples 20 and 22 has two equal subsets. This suggests the following definition.

DEFINITION 6.1. A (K, λ) generalized complete mapping, $\lambda > 1$, of the group G is said to be *tight* if, when we ignore the cyclic order in each cyclic sequence (i.e. treat the sequences as subsets of the group G), all of the cyclic sequences are distinct.

The generalized complete mappings of Examples 20 and 22 are clearly not tight. The general problem of obtaining tight (and consequently non-patterned and not trivially decomposable) generalized complete mappings for a given group is difficult. So far, the only known family of tight generalized complete mappings are $(6, 2)$ complete mappings of the cyclic group C_n for $n \equiv 1, 13 \pmod{18}$ (see [**9**], page 80).

One method of obtaining $(\lambda K, \lambda)$ tight generalized complete mappings is to construct λ distinct (in the sense implied by Definition 6.1) $(K, 1)$ generalized complete mappings and then adjoin them. Here λK is obtained from K by repeating each element of K λ times. As an example, we have that

$$[e\, a^4\, a^3][e\, a^2\, a^3][e\, a\, a^3][e\, a^5\, a^3](a\, a^2\, a^5)(a\, a^5\, a^4)(a^2\, a^5\, a^4)(a\, a^2\, a^4)$$

is a tight $(4K, 4)$ near complete mapping of the cyclic group C_6 obtained by putting four $(K, 1)$ near complete mappings together, where $K = \{3; 3\}$ and $4K = \{3, 3, 3, 3; 3, 3, 3, 3\}$.

The following theorem gives a construction which always yields tight generalized complete mappings. We first need a definition. In part I we defined an R-sequencing of a group. The following generalization was first introduced in [**12**].

DEFINITION 6.2. A group $(G, \cdot)$ of order n is said to be *R_l-sequenceable* if $n - 1$ of its elements (which we may take to be the non-identity elements) can be arranged in a sequence $c_1, c_2, \ldots, c_{n-1}$ in such a way that

the set of elements $c_i^{-1}c_{i+1}$ for $i = 1,2,\ldots,n-1$, are all distinct (where arithmetic of suffices is modulo $n-1$) and likewise the sets of elements $c_i^{-1}c_{i+2}$, $c_i^{-1}c_{i+3},\ldots,c_i^{-1}c_{i+l}$. In particular, a group which is R_l-sequenceable is R-sequenceable with $a_0 = e$, and $a_1 = c_{n-1}^{-1}c_1$, $a_2 = c_1^{-1}c_2$, $a_3 = c_2^{-1}c_3,\ldots,a_{n-1} = c_{n-2}^{-1}c_{n-1}$. (Compare Definition 2.2 of [**10**].)

An R_l-sequencing of a group is the same as an l-fold perfect $(v-1,1)$ complete mapping of that group as we mentioned in the previous section.

THEOREM 6.2. *If a group $(G,\cdot)$ of order n is R_l-sequenceable then it possesses a tight (K,λ) complete mapping where λ is equal to the number of integers in the set $\{(n-1)/i\colon i = 1,2,\ldots,l\}$.*

Proof. In the notation of Definition 6.2, the mapping $\sigma = (c_1\ c_2 \cdots c_{n-1})$ is an $(n-1,1)$ complete mapping of G. (Compare Theorem 2.1 (iii) of [**10**].) Since G is R_l-sequenceable, σ^2, $\sigma^3,\ldots,\sigma^l$ are also complete mappings of G. The cyclic sequences of σ^i and σ^j are different from each other and from the cyclic sequences of σ provided that i and j are distinct divisors of $n-1$. By adjoining these complete mappings, we get a tight generalized complete mapping of the kind described in the theorem.

It was shown in [**12**] that an elementary abelian group of order p^n is R_l-sequenceable for $l = 1,2,\ldots,p^n-2$. Making use of this fact, we obtain Examples 24 and 25.

We remark that, for any elementary abelian group of order p^n, Theorem 6.2 also provides a simple method of constructing $(k,1)$ complete mappings where k is any integer which divides $n-1$.

DEFINITION 6.3. A (K,λ) complete mapping $(g_{11}\ g_{12}\cdots g_{1k_1})(g_{21}\ g_{22}\cdots g_{2k_2})\cdots(g_{s1}\ g_{s2}\cdots g_{sk_s})$ of a group $(G,\cdot)$ is said to be *strong* if the elements $g_{ij}\ g_{i,j+1}$ (where $i = 1,2,\ldots,s$; and the second suffix j is added modulo k_i) comprise the non-identity elements of G each counted λ times. Similarly, a (K,λ) near complete mapping

$$\left[g'_{11}\ g'_{12}\cdots g'_{1h_1}\right]\cdots\left[g'_{\lambda 1}\ g'_{\lambda 2}\cdots g'_{\lambda h_\lambda}\right](g_{11}\ g_{12}\cdots g_{1k_1})\cdots(g_{s1}\ g_{s2}\cdots g_{sk_s})$$

of the group is said to be *strong* if the elements $g'_{ij}\ g'_{i,j+1}$ and $g_{ij}\ g_{i,j+1}$ together with the elements $g_{ik_i}\ g_{i1}$ comprise the non-identity elements of G each counted λ times.

The concept of "strongness" of a complete mapping was first introduced in [**8**], where (among other things) a special case of the following theorem applicable only to cyclic groups and to the particular value $k = 3$ was proved.

THEOREM 6.3. *Let $(G, \cdot)$ be a finite group. If there exists a strong $(2k, \lambda)$ complete mapping of G which satisfies the condition that, for each cyclic sequence $(g_{i1}\ g_{i2} \cdots g_{i,2k})$, $g_{ij}^{-1}g_{i,j+1} = g_{i,j+2}$ where the second suffix is added modulo $2k$, then there exists a (k, λ) complete mapping of G.*

Proof. Let $\sigma = (g_{11}\ g_{12} \cdots g_{1,2k}) \cdots (g_{s1}\ g_{s2} \cdots g_{s,2k})$ be a strong $(2k, \lambda)$ complete mapping of the group G such that $g_{ij}^{-1}g_{i,j+1} = g_{i,j+2}$. Then the mapping

$$\sigma^2 = (g_{11}\ g_{13} \cdots g_{1,2k-1})(g_{12}\ g_{14} \cdots g_{1,2k}) \cdots (g_{s2}\ g_{s4} \cdots g_{s,2k})$$

is the desired (k, λ) complete mapping because we have that

$$g_{ij}^{-1}g_{i,j+2} = \left(g_{ij}^{-1}g_{i,j+1}\right)\left(g_{i,j+1}^{-1}g_{i,j+2}\right) = g_{i,j+2}g_{i,j+3}$$

and, because σ is strong, the products $g_{i,j+2}g_{i,j+3}$ comprise the non-identity elements of G each counted λ times. (Arithmetic of all second suffices is modulo $2k$.)

We remark that if $(G, +)$ is an elementary abelian group of order p^n, we may regard it as the additive group of the Galois field GF$[p^n]$. If ω is a primitive element of GF$[p^n]$ then $\sigma = (1\ \omega\ \omega^2 \cdots \omega^{p^n-2})$ is an R_l-sequencing of $(G, +)$, where $l = p^n - 2$. So, if $2k$ is any even divisor of $p^n - 1$, and $t = (p^n - 1)/2k$ we have that

$$\sigma^t = (1\ \omega^t\ \omega^{2t} \cdots \omega^{(2k-1)t})(\omega\ \omega^{t+1}\ \omega^{2t+1} \cdots \omega^{(2k-1)t+1}) \cdots (\omega^{t-1}\omega^{2t-1} \cdots \omega^{2kt-1})$$

is a $(2k, 1)$ complete mapping of $(G, +)$. Also σ^t is strong because the elements $1 + \omega^t$, $\omega + \omega^{t+1}, \ldots, \omega^{t-1} + \omega^{2t-1}$, $\omega^t + \omega^{2t}$, $\omega^{t+1} + \omega^{2t+1}, \ldots, \omega^{2t-1} + \omega^{3t-1}, \ldots, \omega^{(2k-1)t-1} + \omega^{2kt-1}$ are the non-identity elements of G. It satisfies the further condition required by Theorem 6.3 if and only if $\omega^t - 1 \equiv \omega^{2t} (\text{mod } p)$. However, in this special case, σ^{2t} is a $(k, 1)$ complete mapping of $(G, +)$ even when this extra condition is not satisfied.

Example 26 is constructed by the method just described and satisfies the additional condition $\omega^t - 1 \equiv \omega^{2t} \pmod{13}$ where $\omega = 2$ and $t = 2$. Example 27 is constructed from the Galois field GF[**9**] by the method just described but does not satisfy the additional condition $\omega^t - 1 \equiv \omega^{2t}$. That is, $\alpha^2 - 1 \not\equiv \alpha^4 \pmod 3$.

A final observation concerning Theorem 6.3 is that, if the group G is abelian, the condition $g_{ij}^{-1}g_{i,j+1} = g_{i,j+2}$ can only hold when $k = 6$. Let a, b be two elements of such a group. Then the cyclic sequence $(a\ b \cdots)$ must continue as follows: a, b, $a^{-1}b$, $b^{-1}(a^{-1}b) = a^{-1}$, $(a^{-1}b)^{-1}a^{-1} = b^{-1}$, ab^{-1}, a, b. That is, it has length six.

The following is an analogue of Theorem 6.3 for near complete mappings.

THEOREM 6.4. *Let* $(G, \cdot)$ *be a finite group and let*

$$\sigma = [e\ g_1'][e\ g_2'] \cdots [e\ g_\lambda'](g_{11}\ g_{12} \cdots g_{16}) \cdots (g_{s1}\ g_{s2} \cdots g_{s6})$$

be a (K, λ) *near complete mapping of* $(G, \cdot)$, *where* $K = \{2, 2, \ldots, 2;$ $6, 6, \ldots, 6\}$. *If* σ *is strong and satisfies the condition that, for each cyclic sequence* $(g_{i1}\ g_{i2} \cdots g_{i6})$, $g_{ij}^{-1}\ g_{i,j+1} = g_{i,j+2}$ *where the second suffix is added modulo* 6, *then there exists a* $(3, \lambda)$ *near complete mapping of* G.

Proof. The mapping

$$\sigma^* = [e\ g_1'][e\ g_2'] \cdots [e\ g_\lambda'](g_{11}\ g_{13}\ g_{15})(g_{12}\ g_{14}\ g_{16})$$
$$\cdots (g_{s1}\ g_{s3}\ g_{s5})(g_{s2}\ g_{s4}\ g_{s6})$$

is the desired $(3, \lambda)$ near complete mapping because for the cyclic sequences we have

$$g_{ij}^{-1}g_{i,j+2} = \left(g_{ij}^{-1}g_{i,j+1}\right)\left(g_{i,j+1}^{-1}g_{i,j+2}\right) = g_{i,j+2} \cdot g_{i,j+3}$$

and because σ is strong the products $g_{i,j+2}g_{i,j+3}$ together with the products $eg_1', eg_2', \ldots, eg_\lambda'$ comprise the non-identity elements of G each counted λ times.

APPENDIX–EXAMPLES.

(1) Let $G = C_{14} \cup \{\infty\}$ and $B = \bigcup_{i=0}^{13}\{(i\ \ 7+i\ \ \infty), (1+i\ \ 2+i\ \ 6+i\ \ 11+i\ \ 3+i), (4+i\ \ 12+i\ \ 9+i\ \ 5+i), (8+i\ \ 10+i\ \ 13+i)\}$, where all addition is modulo 14. Then (G, B) is a $(15, K, 1)$-MD with $K = \{5, 4, 3\}$.

(2) Let $G = C_{13}$ and $B = \bigcup_{i=0}^{12}\{(1+i\ \ 4+i\ \ 3+i\ \ 12+i\ \ 9+i\ \ 10+i), (2+i\ \ 6+i\ \ 4+i\ \ 11+i\ \ 7+i\ \ 9+i), (5+i\ \ 2+i\ \ 11+i\ \ 7+i\ \ 9+i), (5+i\ \ 2+i\ \ 10+i\ \ 8+i\ \ 11+i\ \ 3+i), (6+i\ \ 1+i\ \ 8+i\ \ 7+i\ \ 12+i\ \ 5+i)\}$, where addition is modulo 13. Then (G, B) is a $(13, K, 2)$-MD with $K = \{6\}$. That is, it is a $(13, 6, 2)$-MD.

(3) Let $G = \{a, b, c, d\}$ and $B = \{(b\ c\ d), (a\ d\ c), (d\ a\ b), (c\ b\ a)\}$. Then (G, B) is a $(4, K, 1)$-MD with $K = \{3\}$. That is, it is a $(4, 3, 1)$-MD.

(4) Let G be the dihedral group D_4 of order 8 with generating relations $a^4 = b^2 = e$ and $ab = ba^{-1}$, and let $B = \bigcup_{x \in G}\{(xa\ xa^2\ xba^2\ xba\ xba^3\ xa^3\ xb)\}$. Then (G, B) is an $(8, 7, 1)$-MD.

(5) Let $G = C_8 \cup \{\infty\}$ and $B = \bigcup_{i=0}^{7}\{(i\ \ 4+i\ \ \infty), (i\ \ 4+i\ \ \infty), (1+i\ \ 2+i\ \ 7+i), (3+i\ \ 6+i\ \ 5+i), (1+i\ \ 7+i\ \ 6+i), (2+i\ \ 3+i\ \ 5+i)\}$, where addition is modulo 8. Then (G, B) is a $(9, 3, 2)$-MD.

(6) The R-sequencing $e\ a\ b\ a^3\ a^2\ ba^2\ ba^3\ ba$ of the dihedral group D_4 defines the $(8, 7, 1)$-MD whose blocks are given by left-multiplying the elements of the basis blocks by each of the eight elements of D_4 in turn. Here there is exactly one basis block $(a\ a^2\ ba^2\ ba\ ba^3\ a^3\ b)$ by Theorem 2.1(iii) of [**10**] and hence we get a design with eight blocks.

(7) The $(K, 2)$ near complete mapping of the dihedral group D_3 given in Example 2.6 of [**10**], where $K = \{2, 2; 4, 4\}$, defines a $(7, K^*, 2)$-MD with $K^* = \{3, 4\}$ whose blocks are

$$
\begin{array}{llll}
(e\ ba\ \infty), & (e\ ba^2\ \infty), & (a^2\ b\ ba^2\ a), & (a\ b\ ba\ a^2) \\
(a\ b\ \infty), & (a\ ba\ \infty), & (e\ ba^2\ ba\ a^2), & (a^2\ ba^2\ b\ e) \\
(a^2\ ba^2\ \infty), & (a^2\ b\ \infty), & (a\ ba\ b\ e), & (e\ ba\ ba^2\ a) \\
(b\ a\ \infty), & (b\ a^2\infty), & (ba^2\ e\ a^2\ ba), & (ba\ e\ a\ ba^2) \\
(ba\ e\ \infty), & (ba\ a\ \infty), & (b\ a^2\ a\ ba^2), & (ba^2\ a^2\ e\ b) \\
(ba^2\ a\ \infty), & (ba^2\ e\ \infty), & (ba\ a\ e\ b), & (b\ a\ a^2\ ba)
\end{array}
$$

The design admits a regular group of automorphisms isomorphic to D_3.

(8) The sequencing 0 3 2 1 of the cyclic group C_4 defines the $(5, 5, 1)$-MD whose blocks are $(0\ 3\ 1\ 2\ \infty)$, $(1\ 0\ 2\ 3\ \infty)$, $(2\ 1\ 3\ 0\ \infty)$ and $(3\ 2\ 0\ 1\ \infty)$. (See Theorem 2.2 of [**10**].)

(9) The $(14, 3)$ complete mapping of the cyclic group C_{15} given below is 2-fold perfect.

$$(1\ 2\ 13\ 4\ 11\ 10\ 3\ 6\ 8\ 5\ 9\ 14\ 12\ 7)\ (1\ 3\ 8\ 14\ 13\ 2\ 11\ 4\ 5\ 12\ 9\ 7\ 10\ 6)$$
$$(1\ 4\ 9\ 2\ 3\ 14\ 8\ 10\ 7\ 5\ 11\ 6\ 13\ 12).$$

(10) The $(6, 2)$ complete mapping $(1\ 3\ 2\ 6\ 4\ 5)\ (1\ 5\ 4\ 6\ 2\ 3)$ of the cyclic group C_7 is perfect.

(11) The $(7, 1)$ near complete mapping $[0\ 2\ 1\ 4\ 5\ 3]$ of the cyclic group C_6 is perfect.

(12) The $(3, 2)$ near complete mapping of the cyclic group C_{14} given below is perfect

$$[0\ 7]\ [0\ 7]\ (1\ 2\ 5)\ (3\ 9\ 4)\ (6\ 10\ 12)\ (8\ 13\ 11)\ (1\ 11\ 9)\ (2\ 13\ 12)$$
$$(3\ 6\ 8)\ (4\ 5\ 10).$$

(13) The existence of the following (3, 2) near complete mapping of the quaternion group $Q = \text{gp}\{a, b: a^4 = e, a^2 = b^2, ab = ba^{-1}\}$ allows the construction of a (9, 3, 2)-PMD admitting this group as a regular group of automorphisms.

$$[e\ a][e\ a^3](ba\ ba^3\ a)(b\ ba^2\ a^3)(ba\ a^2\ ba^2)(b\ a^2\ ba^3).$$

(14) $[e\ ba^4](a^2\ a\ a^4)(a^3\ ba^2\ ba^6)(a^6\ b\ ba)(a^5\ ba^3\ ba^5)$ is a (3, 1) near complete mapping of the dihedral group $D_7 = \text{gp}\{a, b: a^7 = b^2 = e, ab = ba^{-1}\}$. By reversing the cyclic order of the elements in each cyclic sequence, the following is also a (3, 1) near complete mapping of D_7:

$$[e\ ba^4](a^2\ a^4\ a)(a^3\ ba^6\ ba^2)(a^6\ ba\ b)(a^5\ ba^5\ ba^3).$$

(15) $(a\ a^4\ a^2)(a^3\ a^5\ a^6)$ is a (3, 1) complete mapping of the cyclic group C_7. By observation (ii) of §6, $(a\ a^2\ a^4)(a^3\ a^6\ a^5)$ is also a (3, 1) complete mapping of C_7. By Theorem 3.1 of [**10**], their associated neofields of order 8 are:

z	0	1	a	a^2	a^3	a^4	a^5	a^6
$\phi(z)$	1	0	a^4	a	a^5	a^2	a^6	a^3

and

z	0	1	a	a^2	a^3	a^4	a^5	a^6
$\phi(z)$	1	0	a^2	a^4	a^6	a	a^3	a^5

.

One can easily check that the former neofield hs only the commutativity property: $x + y = y + x$, while the latter has only the right inverse property: $(x + y)(-y) = x$ (see [**9**]).

(16) The following two cyclic neofields of order 9 (associated with two near complete mappings of the cyclic group C_8 related as in observation (ii) of §6):

z	0	1	a	a^2	a^3	a^4	a^5	a^6	a^7
$\phi(z)$	1	a^4	a^7	a^3	a^5	0	a^2	a	a^6

and

z	0	1	a	a^2	a^3	a^4	a^5	a^6	a^7
$\phi(z)$	1	a^4	a^6	a^5	a^2	0	a^3	a^7	a

.

both have the left inverse property: $(-x) + (x + y) = y$. However the first neofield also possesses the commutative property. In fact, it is the Galois field of order 9.

(17) From the (2, 1) complete mapping $\sigma_1 = (a\ a^2)$ of the cyclic group $C_3 = \text{gp}\{a\}$ and the (3, 1) complete mapping $\sigma_2 = (b\ b^2\ b^4)(b^3\ b^6\ b^5)$ of

the cyclic group $C_7 = \mathrm{gp}\{b\}$, we can construct the following $(K, 1)$ complete mapping of the direct product $C_3 \times C_7$, where $K = \{6, 6, 3, 3, 2\}$. We can conveniently express this in the form of a complete mapping of $C_{21} = \mathrm{gp}\{c\}$ by means of the mapping $(a^x, b^y) \to c^{7x+3y}$.

$$((a, b), (a^2, b^2), (a, b^4), (a^2, b), (a, b^2), (a^2, b^4))$$
$$((a, b^3), (a^2, b^6), (a, b^5), (a^2, b^3), (a, b^6), (a^2, b^5))$$
$$((e, b), (e, b^2), (e, b^4))((e, b^3), (e, b^6), (e, b^5))$$
$$((a, e), (a^2, e))$$

(18) From the $(2, 1)$ complete mapping $\sigma_1 = (a\ a^4)(a^2\ a^3)$ and the $(4, 2)$ patterned complete mapping $\sigma_2 = (b\ b^3\ b^4\ b^2)(b^2\ b^4\ b^3\ b)$ of the cyclic group C_5, we can construct the following $(K, 2)$ complete mapping of the non-cyclic group $C_5 \times C_5$, where $K = \{4, 4, \ldots, 4, 2, 2, 2, 2\}$.

$$((a, b), (a^4, b^3), (a, b^4), (a^4, b^2))((a, b), (a^4, b^2), (a, b^4), (a^4, b^3))$$
$$((a^2, b), (a^3, b^3), (a^2, b^4), (a^3, b^2))((a^2, b), (a^3, b^2), (a^2, b^4), (a^3, b^3))$$

$$((a^3, b), (a^2, b^3), (a^3, b^4), (a^2, b^2))((a^3, b), (a^2, b^2), (a^3, b^4), (a^2, b^3))$$
$$((a^4, b), (a, b^3), (a^4, b^4), (a, b^2))((a^4, b), (a, b^2), (a^4, b^4), (a, b^3))$$

$$((e, b), (e, b^3), (e, b^4), (e, b^2))((e, b), (e, b^2), (e, b^4), (e, b^3))$$
$$((a, e), (a^4, e))((a^2, e), (a^3, e))((a, e), (a^4, e))((a^2, e), (a^3, e))$$

(19)

$$(a\ a^5\ a^4\ a^{18}\ a^{14}\ a^{15})(a^2\ a^7\ a^5\ a^{17}\ a^{12}\ a^{14})(a^3\ a^9\ a^6\ a^{16}\ a^{10}\ a^{13})$$
$$(a^9\ a\ a^{11}\ a^{10}\ a^{18}\ a^8)(a^8\ a^2\ a^{13}\ a^{11}\ a^{17}\ a^6)(a^7\ a^3\ a^{15}\ a^{12}\ a^{16}\ a^4)$$

is a (6.2) complete mapping of the cyclic group C_{19}.

(20)

$$(a^7\ a^{14})(a^7\ a^{14})(a^2\ a^{10}\ a^8\ a^{19}\ a^{11}\ a^{13})(a^4\ a^9\ a^5\ a^{17}\ a^{12}\ a^{16})(a^6\ a\ a^{16}\ a^{15}\ a^{20}\ a^5)$$
$$(a^9\ a^3\ a^{15}\ a^{12}\ a^{18}\ a^6)(a\ a^4\ a^3\ a^{20}\ a^{17}\ a^{18})(a^{10}\ a^2\ a^{13}\ a^{11}\ a^{19}\ a^8)$$

is a $(K, 2)$ complete mapping of the cyclic group C_{21} with $K = \{2, 2, 6, 6, 6, 6, 6, 6\}$.

(21)

$$[e\ a^7][e\ a^7](a\ a^4\ a^3\ a^{13}\ a^{10}\ a^{11})(a^6\ a\ a^9\ a^8\ a^{13}\ a^5)$$
$$(a^2\ a^6\ a^4\ a^{12}\ a^8\ a^{10})(a^5\ a^2\ a^{11}\ a^9\ a^{12}\ a^3)$$

is a $(K, 2)$ near complete mapping of the cyclic group C_{14} with $K = \{2, 2; 6, 6, 6, 6\}$.

(22)

$$(a\,a^6\,a^5\,a^{24}\,a^{19}\,a^{20})(a^2\,a^8\,a^6\,a^{23}\,a^{17}\,a^{19})(a^4\,a^{12}\,a^8\,a^{21}\,a^{13}\,a^{17})$$
$$(a^{12}\,a\,a^{14}\,a^{13}\,a^{24}\,a^{11})(a^{11}\,a^2\,a^{16}\,a^{14}\,a^{23}\,a^9)(a^9\,a^4\,a^{20}\,a^{16}\,a^{21}\,a^5)$$
$$(a^3\,a^{10}\,a^7\,a^{22}\,a^{15}\,a^{18})(a^{10}\,a^3\,a^{18}\,a^{15}\,a^{22}\,a^7)$$

is a $(6, 2)$ complete mapping of the cyclic group C_{25}.

(23)

$$[e\,ba^3][e\,ba^4](a^2\,a\,a^4)(a^2\,a\,a^4)(a^3\,ba\,ba^5)(a^3\,ba^2\,ba^6)(a^6\,ba^6\,b)$$
$$(a^6\,b\,ba)(a^5\,ba^2\,ba^4)(a^5\,ba^3\,ba^5)$$

is a $(3, 2)$ near complete mapping of the dihedral group D_7.

(24) Since 2 is a primitive root of 13, the following sequence provides an R_{11}-sequencing of the cyclic group C_{13} when written additively. (See **[12]** for a proof.)

$$\sigma = (1\ 2\ 4\ 8\ 3\ 6\ 12\ 11\ 9\ 5\ 10\ 7)$$

We find that

$$\sigma^2 = (1\ 4\ 3\ 12\ 9\ 10)(2\ 8\ 6\ 11\ 5\ 7)$$
$$\sigma^3 = (1\ 8\ 12\ 5)(2\ 3\ 11\ 10)(4\ 6\ 9\ 7)$$
$$\sigma^4 = (1\ 3\ 9)(2\ 6\ 5)(4\ 12\ 10)(8\ 11\ 7)$$
$$\sigma^6 = (1\ 12)(2\ 11)(4\ 9)(8\ 5)(3\ 10)(6\ 7).$$

When these five mappings are adjoined they define a $(K, 5)$ tight generalized complete mapping of the cyclic group C_{13}, where $K = \{12, 6, 6, 4, 4, 4, 3, 3, 3, 3, 2, 2, \ldots, 2\}$.

(25) Since a root α of the equation $\alpha^2 = \alpha + 1$ is a primitive element of the Galois field GF[**9**], the following sequence provides an R_7-sequencing of the additive group of the field.

$$\sigma = (1\quad \alpha\quad \alpha + 1\quad -\alpha + 1\quad -1\quad -\alpha\quad -\alpha - 1\quad \alpha - 1)$$

We find that

$$\sigma^2 = (1\quad \alpha + 1\quad -1\quad -\alpha - 1)(\alpha\quad -\alpha + 1\quad -\alpha\quad \alpha - 1)$$
$$\sigma^4 = (1\quad -1)(\alpha + 1\quad -\alpha - 1)(\alpha\quad -\alpha)(-\alpha + 1\quad \alpha - 1).$$

When these three mappings are adjoined they define a $(K, 3)$ tight generalized complete mapping of the group $G = \mathrm{gp}\{a\colon a^3 = e\} \times \mathrm{gp}\{b\colon b^3 = e\}$ which we can write in the form

$$(a\ b\ ab\ ab^2\ a^2\ b^2\ a^2b^2\ a^2b)(a\ ab\ a^2\ a^2b^2)$$
$$(b\ ab^2\ b^2\ a^2b)(a\ a^2)(ab\ a^2b^2)(b\ b^2)(ab^2\ a^2b).$$

(26) $(a\ a^4\ a^3\ a^{12}\ a^9\ a^{10})(a^2\ a^8\ a^6\ a^{11}\ a^5\ a^7)$ is a strong $(6, 1)$ complete mapping of the cyclic group C_{13} which satisfies the additional condition required by Theorem 6.3.

(27) $(a\ ab\ a^2\ a^2b^2)(b\ ab^2\ b^2\ a^2b)$ is a strong $(4, 1)$ complete mapping of the group $C_3 \times C_3 \cong \mathrm{gp}\{a\colon a^3 = e\} \times \mathrm{gp}\{b\colon b^3 = e\}$.

(28) $(a\ a^4\ a^3\ a^{12}\ a^9\ a^{10})(a^2\ a^6\ a^4\ a^{11}\ a^7\ a^9)(a^5\ a^2\ a^{10}\ a^8\ a^{11}\ a^3)(a^6\ a\ a^8\ a^7\ a^{12}\ a^5)$ is a strong tight $(6, 2)$ complete mapping of the cyclic group C_{13} which satisfies the additional condition required by Theorem 6.3.

(29) $(a\ a^5\ a^4\ a^6\ a^2\ a^3)(a\ a^2\ a^4)(a^3\ a^6\ a^5)$ is a stong tight $(K, 2)$ complete mapping of the cyclic group C_7 constructed by the methods of Theorem 6.2, where $K = \{6, 3, 3\}$.

(30) $[e\ ba^3](a^2\ a\ a^4)(a^3\ ba\ ba^5)(a^6\ ba^6\ b)(a^5\ ba^2\ ba^4)$ is a strong $(K, 1)$ near complete mapping of the dihedral group $D_7 = \mathrm{gp}\{a, b\colon a^7 = b^2 = e, ab = ba^{-1}\}$, where $K = \{2; 3, 3, 3, 3\}$.

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