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INVARIANTS OF THE HEAT EQUATION

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Let M be a compact Riemannian manifold without boundary and let P: $C^\infty V \to C^\infty V$ be a self-adjoint elliptic differential operator with positive definite leading symbol. The asymptotics of the heat equation $\operatorname{Tr}(\exp(-tP))$ as $t \to 0^+$ are spectral invariants given by local formulas in the jets of the total symbol of P. Let $A(x)$ and $B(x)$ be polynomials where the degree of B is positive and the leading coefficient is positive. The asymptotics of $\operatorname{Tr}(A(P)\exp(-tB(P)))$ can be expressed linearly in terms of the asymptotics of $\operatorname{Tr}(\exp(-tP))$. Thus no new spectral information is contained in these more general expressions. We also show the asymptotics of the heat equation are generically non-zero. If one relaxes the condition that the leading symbol of P be definite, the asymptotics of $\operatorname{Tr}(\exp(-tP^2))$ and $\operatorname{Tr}(P\exp(-tP^2))$ form a spanning set of invariants. These are related to the zeta and eta functions using the Mellin transform, and a similar non-vanishing result holds except for the single invariant giving the residue of eta at $s = 0$ which vanishes identically.

1. Introduction. Let M be a compact Riemannian manifold of dimension m without boundary. Let P: $C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic differential operator of order $u > 0$. Let $p(x, \xi)$ be the leading symbol of P for $x \in M$ and $\xi \in T^*M_x$. We suppose $p(x, \xi)$ is a positive definite matrix for $\xi \neq 0$; this implies u is even. Let $(\lambda_\nu, \theta_\nu)_{\nu=1}^\infty$ be a spectral resolution of P into a complete orthonormal basis θ_ν for $L^2(M)$ of eigensections so $P\theta_\nu = \lambda_\nu\theta_\nu$. The spectrum of P is bounded from below and the eigenvalues tend towards ∞ as $\nu \to \infty$. Order the eigenvalues so $\lambda_1 \leq \lambda_2 \leq \cdots$. If $t > 0$, $\exp(-tP)$ is an infinitely smoothing operator with smooth kernel

$$K(x, y, \exp(-tP)) = \sum_\nu \exp(-t\lambda_\nu)\theta_\nu(x) \otimes \theta_\nu(y)$$

where $\theta_\nu(x) \otimes \theta_\nu(y)$ is regarded as an endomorphism from the fiber of the bundle at y to the fiber at x.

On the diagonal, there is an asymptotic expansion of the form:

$$K(x, x, \exp(-tP)) \simeq \sum_{j=0}^{m} t^{(n-m)/u} e_n(x, \exp(-tP)), \quad t \to O^+$$

where $e_n = 0$ if n is odd. In the literature, this sum is often reindexed since $e_1 = e_3 = \cdots = e_{2n+1} = 0$. We shall not adopt this convention as it would lead to difficulties in what follows when considering more general invariants.

The $e_n(x, \exp(-tP))$ are smooth invariants which are endomorphism valued. The somewhat surprising fact is that they can be computed locally in terms of the jets of the total symbol of the operator P. They do not depend upon the global behavior of the operator P. Let

$$a_n(x, \exp(-tP)) = \mathrm{Tr}_V e_n(x, \exp(-tP))$$
$$a_n(\exp(-tP)) = \int_M a_n(x, \exp(-tP))\, d\,\mathrm{vol}(x)$$

where $d\,\mathrm{vol}(x)$ is the Riemannian measure on M. The local fiber trace $a_n(x, \exp(-tP))$ is a scalar invariant of P which is sometimes denoted by $a_n(x, P)$ in the literatire.

$\mathrm{Exp}(-tP)$ is an operator of trace class which is easy to compute:

$$\mathrm{Tr}_{L^2} \exp(-tP) \simeq \sum_{n=0}^{\infty} t^{(n-m)/u} a_n(\exp - tP) \quad \text{as } t \to O^+.$$

The invariants $a_n(\exp(-tP))$ are spectral invariants of the operator which can be computed by integrating local invariants in the symbol of P.

Let $\Delta_p = (dd^* + d^*d)_p$ on $C^\infty(\Lambda^p(M))$ be the Laplacian acting on the space of smooth p-forms. This operator is self-adjoint, elliptic, with positive definite leading symbol. A wide literature exists which relates the spectrum of the Laplacian to the geometry of the manifold M. Much of this literature uses the asymptotic invariants $a_n(\exp(-t\Delta_p))$ to derive information about the spectral geometry. We only mention a few such results to give the flavor which is involved. Let $\mathrm{spec}(\Delta_p) = \{\lambda_\nu^p\}$ be the spectrum of the Laplacian where each eigenvalue is repeated according to the multiplicity with which it appears. Then:

THEOREM 1.1 (*Patodi*). *Let M_1 and M_2 be Riemannian manifolds with* $\mathrm{spec}(\Delta_p, M_1) = \mathrm{spec}(\Delta_p, M_2)$ *for* $p = 0, 1, 2$. *Then*:

(a) $\dim(M_1) = \dim(M_2)$ and $\mathrm{vol}(M_1) = \mathrm{vol}(M_2)$.

(b) *If M_1 has constant scalar curvature c, then so does M_2.*

(b) *If M_1 is Einstein, then so is M_2.*

(c) *If M_1 has constant sectional curvature c, then so does M_2.*

(d) *If M_1 is isometric to the standard sphere of radius* 1 *in* R^{m+1}, *then so is M_2 (see* [**14**]).

THEOREM 1.2 (*Sakai*). *Let M_i be Einstein manifolds of dimension* 6 *with the same Euler characteristic and assume* $\mathrm{spec}(\Delta_0, M_1) = \mathrm{spec}(\Delta_0, M_2)$. *Then if M_1 is a local symmetric space, so is m_2 (see* (**15**]).

THEOREM 1.3 (*Berger*). *Let* $\operatorname{spec}(\Delta_0, M_1) = \operatorname{spec}(\Delta_0, M_2)$. *If* $\dim(M) \leq 4$ *and if* M_1 *is isometric to the standard sphere of radius* 1 *in* R^{m+1} *then so is* M_2 (*see* [**6**]).

In the special case where M is a simply connected semisimple Lie group, $\operatorname{spec}(\Delta_p)$ can be calculated completely and the asymptotic expansion of the heat equation obtained see [**7**, **8**]. There is a vast literature of such results and we refer to the excellent bibliography of Berard-Berger [**5**] for more examples.

In a sense, all these results can be made sharper since they only rely on knowledge of the invariants a_n for $n = 0, 2, 4, 6$ which is asymptotic knowledge of the spectrum. It has been an open question for a number of years as to whether by studying other natural operators constructed from the Laplacian it is possible to obtain spectral invariants which are given by different local formulas in the jets of the symbol of the operator. Such new invariants would, of course, enable one to improve many of these results and to obtain new results in spectral geometry.

In addition to results in spectral geometry, the asymptotics of the heat equation can be used to prove the Atiyah-Singer index theorem [**1**]. For the DeRham, signature, and spin complexes, the asymptotics given by the heat equation yield a direct proof of the Gauss-Bonnet theorem, the Hirzebruch signature formula, and the $\hat{A}$ formula for the spin-index. For the Dolbeault complex, however, the local invariants for a non-Kaehler metric are not the formula of the Riemann-Roch theorem. Consequently, it is natural to look for new invariants which will permit a direct heat equation proof of the Riemann-Roch theorem for non-Kaehler metrics. This is given particular force since such an approach might permit one to study the Lefschetz fixed point formulas in the holomorphic context directly using the heat equation; so far the only approach known is that of Toledo-Tong [**17**] using entirely different methods.

Thus from a variety of vantage points, it is natural to look for new spectral invariants. The basic asymptotic convergence result needed is the following:

THEOREM 1.4 (*Seeley*). *Let* P *be a self-adjoint elliptic differential operator of order* $u > 0$ *with positive definite leading symbol. Let* Q *be an auxiliary differential operator of order* a. *Then* $Q\exp(-tP)$ *is an infinitely smoothing operator of trace class given by a kernel operator*

$K(x, y, Q\exp(-tP))$ for $t > 0$. On the diagonal, there is an asymptotic series:

$$K(x, x, Q\exp(-tP)) \simeq \sum_{n=0}^{\infty} t^{(n-m-a)/u} e_n(x, Q\exp(-tP)).$$

The $e_n(x, Q\exp(-tP))$ are local invariants which can be computed functorially in terms of the derivatives of the total symbol of Q and P.

This asymptotic result is implicit in the work of Seeley **[16]**. Although Seeley was working with the zeta function rather than with the heat equation, the two are formally equivalent using the Mellin transform as we shall see later. We refer to Gilkey-Smith **[13]** where a proof of this result is given in the case of manifolds with boundary.

In fact, more can be said about the local invariants involved. Decompose the total symbol of $P = p$ and of $Q = q$ into homogeneous parts: $p = p_0 + \cdots + p_u$ and $q = q_0 + \cdots + q_a$. Introduce local coordinates $x = (x_1, \ldots, x_m)$ and let d_x^α be the standard notation for partial derivatives in terms of the multi-index α. The p's and q's are matrices and we introduce formal variables q_i^* and $p_{j/\alpha}^*$ for the components of the matrices q_i and jets of the matrices $p_{j/\alpha}$. We define:

$$\operatorname{order}(q_i j) = a - i \quad \text{and} \quad \operatorname{order}(p_{j/\alpha}) = u - j + |\alpha|.$$

Then:

THEOREM 1.5. *Let $e_n(x, Q\exp(-tP))$ be as defined in Theorem 1.4, $e_n = 0$ if $n + a$ is odd. If $n + a$ is even, then e_n can be computed functorially in terms of the variables $\{q_i^*, p_{j/\alpha}^*\}$. It is a polynomial in the variables with $u - j + |\alpha| \neq 0$ which is homogeneous of order n with coefficients which depend smoothly on the components of the leading symbol $p_u \cdot e_n$ is linear in the q_i variables.*

This is proved using dimensional analysis. Since the methods are by now completely standard, we omit the details. We apply Theorems 1.4 and 1.5 to the following special case:

LEMMA 1.6. *Let $A(r)$ and $B(r)$ be constant coefficient polynomials in an indeterminate r. Let a be the order of A and b be the order of B. We suppose $b > 0$ and the leading coefficient of B is positive. Let P be a self-adjoint elliptic differential operator of order $u > 0$ with positive leading order symbol. Then $B(P)$ is another such operator of order bu. Expand*

$$K(x, x, A(P)\exp(-tB(P))) \simeq \sum_{n=0}^{\infty} t^{(n-m-au)/bu} e_n(x, A(P)\exp(-tB(P))).$$

The $e_n(x, A(P)\exp(-tB(P)))$ *are local invariants which can be computed functorially in terms of the derivatives of the total symbol of* P. *If we let*

$$a_n(A(P)\exp(-tB(P))) = \int_M \operatorname{Tr} e_n(x, A(P)\exp(-tB(P))j)\, d\mathrm{vol}(x),$$

then these are spectral invariants of the operator P.

The question we will treat in this paper is whether or not we have actually constructed new invariants in this manner. Some previous results suggest this is not the case. If $B(t) = t^b$ and $A(t) = 1$ then it was proved in **[10]** that

$$e_n(x, \exp(-tP^b)) = \frac{1}{b}\Gamma\left(\frac{m-n}{u}\right)^{-1}\Gamma\left(\frac{m-n}{bu}\right) e_n(x, \exp(-tP))$$

where Γ is the classical Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}\exp(-t)\, dt.$$

Thus no new information results and some can be lost if we study $\exp(-tP^b)$ instead of $\exp(-tP)$ since the coefficients involved can be zero. Similarly, if we let $B(x) = x - c$ then $\exp(-t(P-c)) = \exp(tc)\exp(-tP)$ and comparison of the powers of t in the asymptotic expansion yields the formula:

$$e_n(x, \exp(-t(P-c))) = \sum_{k+ul=n} e_k(x, \exp(-tP))c^l/l!$$

In this case, the spectral information is rearranged, but none is lost.

In this paper, we will answer the question of obtaining new spectral information using the invariants $e_n(x, A(P)\exp(-tB(P)))$ in the negative by showing:

THEOREM 1.7. *Let* P *be a self-adjoint elliptic partial differential operator of order* $u > 0$ *with positive definite leading symbol. Let* $A(r)$ *and* $B(r)$ *be constant coefficient polynomials of orders* a *and* b. *Let* $b > 0$ *and let the leading coefficient of* B *be positive. Then there exist constants* $c_k = c_k(n, m, u, A, B)$ *which are non-zero for only a finite number of* k *so that*

$$e_n(x, A(P)\exp(-tB(P))) = \sum_k c_k e_k(x, \exp(-tP)).$$

The zeta function is linked closely with the heat equation by the Mellin transform and similar statements can be made about invariants arising from generalized zeta functions. This strongly suggests that the answer to the general question of finding new invariants is no, but of course one must first make precise the context in which one is working to make such a question well posed mathematically.

If we drop the assumption that the leading symbol of P is positive definite, then u need no longer be even. The spectrum of P can contain infinitely many positive and negative values tending towards $\pm\infty$. It is still possible to obtain asymptotic information about the spectrum using the heat equation but there are some new phenomena involved. This is closely related to the measure of spectral asymmetry of the eta invariant discussed in the Atiyah-Patodi-Singer index theorem [**2**, **3**, **4**].

This paper is divided into two sections. In the first section, we will give the formal arguments necessary to derive Theorem 1.7 from Lemma 1.6. We will then use some standard asymptotic results to justify the convergence of the various series appearing in these formal arguments. We have separated the convergence results from the formal procedures to avoid cluttering the formal arguments which contain new conceptual ideas from the arguments in analysis which are by now standard.

In the second section, we will discuss the invariants which arise if the operator is not assumed to have positive definite leading symbol. This will lead us to a discussion of both the zeta and the eta functions. We will also discuss the extent to which the invariants we will define are generically non-zero and actually give asymptotic information about the spectrum. In particular this gives a counterexample to the theorem of Wodzicki [**18**] who had claimed previously the vanishing of certain residues at certain poles of the eta function in general.

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2. Invariants of operators with positive definite leading symbol. In this section, we will assume P is a self-adjoint elliptic differential operator of order $u > 0$ with positive definite leading symbol. Let c be a large positive constant and let $P_c = P + c$. Suppose we could prove Theorem 1.7 for the operator P_c whenever c is sufficiently large. Since $e_n(x, A(P_c)\exp(-tP_c))$ depends *polynomially* on the constant c by Theorem 1.5, we can use analytic continuation to assert the identity of Theorem 1.7 for all constants c and in particular for $c = 0$. If we let c be large, we ensure that the spectrum of P_c is positive since the spectrum of P

is bounded from below. Thus we may assume the spectrum of P is positive to prove Theorem 1.7. In this section, we will also derive a number of other identities among the invariants e_n under the assumption P is positive: these identities continue to hold in the general case.

If c is a positive constant, then $\exp(-tB(P)) = \exp(-tc^{-1}(cB(P)))$ from which it follows immediately that:

$$e_n(x, A(P)\exp(-t(cB)(P))) = c^{(m+au-n)/bu} e_n(x, A(P)\exp(-tB(P))).$$

We have assumed that leading coefficient of B is positive. To prove Theorem 1.7 we may assume without loss of generality that B is monic since the relevant invariants are rescaled. Since these invariants are linear in A, we may assume $A(r) = r^a$ for some a.

The basic technical result we shall need is a fairly weak estimate of the growth of the eigenvalues and eigensections. A proof of this estimate using the Sobolev estimates and Garding's inequality is to be found on **[9]**.

LEMMA 2.1. *Let P: $C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic differential operator of order $u > 0$ with positive definite leading symbol. Let $\{\lambda_\nu, \theta_\nu\}_{\nu=1}^\infty$ be a spectral resolution of P with $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$.*

(a) *There exists $\varepsilon > 0$ and a constant $C > 0$ so that $\nu^\varepsilon < C\lambda_\nu$ for $\nu = 1, 2, \ldots$.*

(b) *There exists $\alpha > 0$ and a constant $C > 0$ so that $|\theta_\nu(x)| \le C\lambda_\nu^\alpha$ for all x and for $\nu = 1, 2, \ldots$.*

We first consider the special case $A(r) = r^a$ and $B(r) = r^b$. We define:

$$f_{a,b}(t) = K(x, x, P^a \exp(-tP^b)) = \sum_\nu \lambda_\nu^a \exp(-t\lambda_\nu^b)\theta_\nu(x) \otimes \theta_\nu(x)$$

where we suppress dependence upon x and upon P for notational convenience. By Lemma 2.1, this series converges absolutely for $t > 0$. We also define the generalized zeta function

$$\zeta(s) = \sum_\nu \lambda_\nu^{-s}\theta_\nu(x) \otimes \theta_\nu(x)$$

This converges absolutely for $\mathrm{Re}(x) \gg 0$.

By Lemma 2.1, $f_{a,b}(t)$ decays exponentially as $t \to \infty$. The Mellin transform links the heat equation and the zeta function. We compute

$$\begin{aligned} Mf_{a,b}(s) &= \Gamma(s)^{-1}\int_0^\infty t^{s-1} f_{a,b}(t)\,dt \\ &= \Gamma(s)^{-1}\int_0^\infty \sum_\nu t^{s-1}\lambda_\nu^a \exp(-t\lambda_\nu^b)\theta_\nu(x) \otimes \theta_\nu(x)\,dt. \end{aligned}$$

We interchange the order of integration and summation and make a change of variables replacing t by $\tau = t\lambda_\nu^b$ to write:

$$Mf_{a,b}(s) = \Gamma(s)^{-1}\sum_\nu \lambda_\nu^{a-bs}\int_0^\infty \exp(-\tau)\,d\tau\,\theta_\nu(x)\otimes\theta_\nu(x)$$

$$= \sum_\nu \lambda_\nu^{a-bs}\theta_\nu(x)\otimes\theta_\nu(x) = \zeta(bs-a).$$

This proves

LEMMA 2.2. *Let P be a positive definite self-adjoint elliptic differential operator of order $u > 0$ with positive definite leading symbol. If $f_{a,b}(t)$ and $\zeta(s)$ are as above, then the Mellin transform is given by $Mf_{a,b}(s) = \zeta(bs-a)$.*

Decompose:

$$\int_0^\infty t^{s-1}f_{a,b}(t)\,dt = \int_0^1 t^{s-1}f_{a,b}(t)\,dt + \int_1^\infty t^{s-1}f_{a,b}(t)\,dt.$$

Because $f_{a,b}$ decays exponentially at ∞ this second integral defines an entire function of s. We use Theorem 1.4 to express

$$f_{a,b}(t) = \sum_{n=0}^N t^{(n-m-au)/bu}e_{n,a,b} + \varepsilon_{N,a,b}(t)$$

where $\varepsilon_{N,a,b} = O(t^{(N-m-au)/bu})$. Therefore,

$$\int_0^1 t^{s-1}f_{a,b}(t)\,dt = \sum_{n=0}^N \int_0^1 t^{2-1+(n-m-au)/bu}e_{n,a,b}\,dt$$

$$+\int_0^1 t^{s-1}\varepsilon_{N,a,b}(t)\,dt.$$

This second integral defines a function of s which is holomorphic for $\mathrm{Re}(s) > (m+au-N)/bu$. We evaluate the first integrals to express finally:

$$\int_0^\infty t^{s-1}f_{a,b}(t)\,dt = \sum_{n=0}^N\left(s+\frac{n-m-au}{bu}\right)^{-1}e_{n,a,b} + \chi_{N,a,b}(s)$$

where $\chi_{N,a,b}$ is holomorphic for $\mathrm{Re}(s) > (m+au-N)/bu$. We combine this with Lemma 2.2 to see

$$\zeta(bs-a) = \Gamma(s)^{-1}\left\{\sum_{n=0}^N\left(s+\frac{n-m-au}{bu}\right)^{-1}e_{n,a,b} + \chi_{N,a,b}(s)\right\}.$$

We change variables to let $\tilde{s} = bs - a$ or $s = (\tilde{s} + a)/b$ to express

$$\xi(\tilde{s}) = \Gamma((\tilde{s} + a)/b)^{-1} \cdot b \cdot \left\{ \sum_{n=0}^{N} \left(\tilde{s} + \frac{n-m}{u} \right)^{-1} e_{n,a,b} + \tilde{\chi}_{N,a,b}(\tilde{s}) \right\}$$

where $\tilde{\chi}$ is holomorphic on the half-plane $\mathrm{Re}(\tilde{s}) > (m - N)/u$. We replace $\tilde{s}$ by s for notational convenience.

$\Gamma(s)$ has meromorphic extension to C with isolated simple poles at $s = 0, -1, -2, \ldots$ and non-vanishing residues. It is non-zero on the real axis minus the non-positive integers. The formula given above shows:

LEMMA 2.3. *ζ has a meromorphic extension to C with isolated simple poles at $s = (m - n)/u$, $n = 0, 2, 4, \ldots$.*

(a) *If $(s + a)/b$ is not a non-positive integer, then ζ has a simple pole with residue $\Gamma((s + a)/b)^{-1} \cdot b \cdot e_n(x, P^a \exp(-tP^b))$ at $s = (m - n)/u$.*

(b) *If $(s + a)/b$ is a non-positive integer, then ζ is regular and the value is $\{\mathrm{Re}\, s_{z=s} \Gamma((z + a)/b)\}^{-1} b \cdot e_n(x, P^a \exp(-tP^b))$ at $s = (m - n)/u$.*

This shows that the asymptotics of the heat equation are reflected in the behavior of the zeta function. Since the zeta function does not depend upon the choice of (a, b) we can use this lemma to compare $e_{n,a,b}$ for different values of (a, b) and thereby show:

THEOREM 2.4. *Let P be a positive definite elliptic self-adjoint differential operator of order $u > 0$ with positive definite leading symbol. Let $a, b \in Z$ with $a \geq 0$ and $b \geq 1$. Then*:

$$\begin{aligned} &e_n(x, P^a \exp(-tP^b)) \\ &\quad = b^{-1} \{\Gamma((s + a)/b) \Gamma(s)^{-1}\}_{s=(m-n)/u} e_n(x, \exp(-tP)). \end{aligned}$$

Proof. Suppose first $s = (m - n)/u$ is not a non-positive integer so Γ is regular at both s and $(s + a)/b$. Then this result follows from Lemma 2.3(a) by comparing the residue at this value for ζ for the pairs $(0, 1)$ and (a, b). Next suppose $s = (m - n)/u$ is a non-positive integer and that $(s + a)/b$ is not. By using $a = 0$, $b = 1$ in Lemma 2.3, we conclude ζ is regular at this value. Since the coefficient in Lemma 2.3 is non-zero for the pair (a, b) we see $e_{n,a,b} = 0$. The coefficient of Theorem 2.4 vanishes in this case which completes the proof. Finally, we suppose $s = (m - n)/u$ and $(s + a)/b$ are both non-positive integers. The result follows by (b) of Lemma 2.3. We note that Theorem 2.4 continues to be valid even if we drop the assumption that P is not positive definite by analytic continuation.

This result establishes Theorem 1.7 for $A(r) = r^a$ and $B(r) = r^b$. As already remarked, we may assume B is monic so B has the form:

$$B(r) = r^b + \sum_{j<b} c_i r^j.$$

We proceed formally for the moment and expand

$$\begin{aligned} P^a \exp(-tB(P)) &= P^a \Big\{ \prod_{j<b} \exp(-tc_i P^j) \Big\} \exp(-tP^b) \\ &= P^a \left\{ \prod_{j<b} \sum_{\mu=0}^{\infty} \frac{(-tc_i p^i)^{\mu}}{\mu!} \right\} \exp(-tP^b). \end{aligned}$$

To avoid formidable notational complexities, we suppose only one of the c's is different from zero; the general case uses exactly the same arguments. Therefore we shall suppose $B(x) = x^b + cx^j$ for some $j < b$. Then the kernel satisfies the identity:

$$(*) \qquad \begin{aligned} &K\big(x, x, P^a \exp(-t(P^b + cP^j))\big) \\ &\quad = \sum_{\mu=0}^{\infty} (-tc)^{\mu}/\mu! K\big(x, x, P^{a+i\mu} \exp(-tP^b)\big). \end{aligned}$$

We shall show that this series converges with uniformity in the parameter t. It is therefore permissible to expand each $K(x, x, P^{a+i\mu} \exp(-tP^b))$ in an asymptotic expansion in t to conclude:

$$\begin{aligned} &K\big(x, x, P^q \exp(-t(P^b + cP^j))\big) \\ &\quad \simeq \sum_{\mu, n=0}^{\infty} (-tc)^{\mu}/\mu! t^{(n-m-(a+i\mu)u)/bu} e_{n, a+i\mu, b}. \end{aligned}$$

By hypothesis, $i < b$ and therefore $i\mu/b < \mu$. The power of t is $\mu(1 - i/b) + (m - n)/bu$ and consequently only a finite number of terms in this double sum contain any given power of t. We rearrange this series to express the invariants $e_n(x, P^a(-t(P^b + cP^i)))$ in terms of the invariants $e_{k,\alpha,\beta}$. These in turn can be expressed in terms of the invariants $e_k(x, \exp(-tP))$. This will complete the proof of Theorem 1.7.

In what follows, we shall let C denote a generic constant which is allowed to change from inequality to inequality. It can depend upon other parameters, but is always to be independent of t, x, and of the subscript for the particular eigenvalue being considered. The following calculus lemma is helpful in various estimates:

LEMMA 2.5. *Let t, b, $r > 0$. Then* $r^b \exp(-tr) \leq t^{-b} b^b \exp(-b)$.

Proof. Let $f(r) = t^b \exp(-tr)$. This is continuous on $(0, \infty)$ and vanishes on both endpoints. It must have an interior maximum where

$f'(r) = r^{b-1}(b - tr)\exp(-tr) = 0$. The maximum value is attained at $r = b/t$ so $f(b/t) = t^{-b}b^b\exp(-b)$ is the maximum value.

We use Lemma 2.1 and 2.5 to obtain an asymptotic estimate for the tail in equation $(*)$ which is uniform in t. We define

$$T(\mu_0) = \sum_{\mu > \mu_0} \sum_{\nu} (-tc)^{\mu}/\mu!\lambda_\nu^{a+i\mu}\exp\left(-t\lambda_\nu^b\right)\theta_\nu(x) \otimes \theta_\nu(x)$$

as the kernel of the tail in the series. We must show $T(\mu_0)$ vanishes to arbitrarily high order in t as $\mu_0 \to \infty$ to complete the proof of Theorem 1.7.

By Lemma 2.1 we estimate $|\theta_\nu(x) \otimes \theta_\nu(x)| \le C|\lambda_\nu|^{2\alpha}$ so

$$|T(\mu_0)| \le C \sum_{\mu < \mu_0} \sum_{\nu} |tc|^{\mu}/\mu!\lambda_\nu^{a+i\mu+2\alpha}\exp\left(-t\lambda_\nu^b\right).$$

We save half the exponential for later use and use Lemma 2.5 to estimate

$$\begin{aligned}\lambda_\nu^{a+i\mu+2\alpha}\exp\left(-t\lambda_\nu^b/2\right) &= \left(\lambda_\nu^b\right)^{(a+i\mu+2\alpha)/b}\exp\left(-t\lambda_\nu^b/2\right)\\ &\le (t/2)^{-(a+i\mu+2\alpha)/b}\{(a+i\mu+2\alpha)/b\}^{(a+i\mu+2\alpha)/b}\\ &\quad\times\exp(-(a+i\mu+2\alpha)/b).\end{aligned}$$

Since $\exp(-(a + i\mu + 2\alpha)/b) \le 1$ we can ignore this term. Since $i < b$, for large values of μ we have $(a + i\mu + 2\alpha)/b < \mu$. We can estimate $\{(a + i\mu + 2\alpha)/b\}^{(a+i\mu+2\alpha)/b} \le \mu^\mu$ which shows:

$$\lambda_\nu^{a+i\mu+2\alpha}\exp\left(-t\lambda_\nu^b/2\right) \le (t/2)^{-(a+i\mu+2\alpha)/b}\mu^\mu.$$

We set $c_1 = 2c$ and estimate:

$$|T(\mu_0)| \le C \sum_{\mu > \mu_0} \sum_{\nu} (c_1 t)^{\mu-(a+i\mu+2\alpha)/b}\mu^\mu/\mu!\exp\left(-t/2\lambda_\nu^b\right).$$

We are interested in the behavior of this sum for small values of t. The series $\sum_\mu \mu^\mu r^\mu/\mu!$ converges for $|r \cdot e| < 1$ and consequently we can replace $\mu^\mu/\mu!$ by e^μ. By changing the value of c_1 appropriately, we can estimate the tail by

$$|T(\mu_0)| \le C \sum_{\mu > \mu_0} \sum_{\nu} (c_1 t)^{\mu(a+i\mu+2\alpha)/b}\exp\left(-t/2\lambda_\nu^b\right).$$

By hypothesis we have $(1 - i/b) > 0$ so the sum in μ is behaving like a convergent geometric series in t and for t small, we can estimate

$$|T(\mu_0)| \le Ct^{\mu_0-(a+i\mu_0+2\alpha)/b}\sum_{\nu}\exp\left(-t/2\lambda_\nu^b\right).$$

We use Lemma 2.1 to estimate $\lambda_\nu > C \cdot \nu^\varepsilon$ for some $\varepsilon > 0$. We choose β so $\varepsilon\beta > 1$ and apply Lemma 2.5:

$$\exp(-t/2\lambda_\nu^b) = \lambda_\nu^{-\beta}\lambda_\nu^\beta \exp(-t/2\lambda_\nu^\beta) \leq \lambda_\nu^{-\beta} \cdot C \cdot t^{-\beta/b}$$

where $C = C(\beta)$. Therefore

$$\sum_\nu \exp(-t/2\lambda_\nu^b) \leq Ct^{-\beta/b}\sum_\nu \lambda_\nu^{-\beta} \leq Ct^{-\beta/b}\sum_\nu V^{-\varepsilon b} \leq Ct^{-\beta/b}.$$

This estimates the tail uniformly by

$$|T(\mu_0)| \leq Ct^{\mu_0-(a+i\mu_0+2\alpha)/b-\beta/b}.$$

Since $i/b < 1$, the exponent tends towards ∞ as $\mu_0 \to \infty$ which yields the desired uniformity in t. This completes the proof of Theorem 1.7.

3. Invariants of operators with indefinite leading symbol. In section two, we assumed the leading symbol of P was positive definite. This implied that the spectrum was essentially positive. In this section, we relax that assumption and shall assume for the remainder of this paper that P is a self-adjoint elliptic differential operator of order $u > 0$ but make no assumption on the leading symbol of P. We shall assume for technical reasons that 0 is *not* an eigenvalue of P. This can always be arranged by replacing P by $P + \varepsilon$ for ε-small. As before, any identities proved using this assumption continue to hold true in the general case by analytic continuation. This assumption is invariant under small perturbations of P. In addition to the zeta function, there is a new invariant called the eta invariant. As before let $\{\lambda_\nu, \theta_\nu\}_{\nu=1}^\infty$ be a spectral resolution of P. The $|\lambda_\nu| \to \infty$ but there may be an infinite number of both positive and negative eigenvalues. The order u is not necessarily even.

We define:

$$\zeta(s, x, P) = \sum_\nu |\lambda_\nu|^{-s}\theta_\nu(x) \otimes \theta_\nu(x) \quad \text{for } \mathrm{Re}(s) \gg 0$$

$$\eta(s, x, P) = \sum_\nu \mathrm{sign}(\lambda_\nu)\, |\lambda_\nu|^{-s}\theta_\nu(x) \otimes \theta_\nu(x) \quad \text{for } \mathrm{Re}(s) \gg 0$$

$$f_{a,b}(t, x, P) = \sum_\nu \lambda_\nu^a \exp(-t\lambda_\nu^b)\theta_\nu(x) \otimes \theta_\nu(x) \quad \text{for } b \text{ even and } t > 0$$

where $a \geq 0$, $b > 0$ are integers. The operator P^2 has positive definite leading symbol so the estimates of Lemma 2.1 apply to show these series all converge.

If P is positive definite, then the zeta and eta functions agree with the zeta function for P^2 when we replace s by $s/2$. If, on the other hand, P is not positive definite, then the eta invariant is a new invariant not present previously. We shall show that both ζ and η admit meromorphic extensions to C with isolated simple poles. It is a deep theorem that $s = 0$ is a regular value of $\eta(s, P) = \int_M \operatorname{Tr} \eta(s, x, P)\, d\operatorname{vol}(x)$ since $s = 0$ is *not* in general a regular value of $\eta(s, x, P)$. The value $\eta(P) = \eta(0, P)$ is a measure of the spectral asymmetry of P and plays an important role in the Atiyah-Patodi-Singer Index theorem for manifolds with boundary [**2**, **3**, **4**]. It is a non-local invariant of the operator P which also plays a role in the $R \bmod Z$ index theorem with coefficients in a unitary representation of the fundamental group.

From the point of view taken in this paper, the importance of eta is that together with the zeta function, the residues at the poles (and occasionally the values) provide a complete list of invariants in the same sense that the zeta function provided in the positive definite case.

We must first generalize Lemma 2.3 to relate these invariants to the invariants of the heat equation. We suppress dependence upon (x, P) for notational convenience and just write $\zeta(s)$, $\eta(s)$, and $f_{a,b}(t)$. These are the kernel functions evaluated on the diagonal $x = y$ corresponding to the operators $(P^2)^{-s/2}$, $P(P^2)^{-(s-1)/2}$, and $P^a \exp(-tP^b)$. To ensure convergence, we let b be even so P^b is positive.

These are related by the Mellin transform. As in the proof of 2.3 we compute

$$\begin{aligned}
Mf_{a,b}(s) &= \Gamma(s)^{-1}\int_0^\infty t^{s-1} f_{a,b}(t)\, dt \\
&= \Gamma(s)^{-1}\sum_\nu \lambda_\nu^a \int_0^\infty t^{s-1}\exp(-t\lambda_\nu^b)\, dt \cdot \theta_\nu(x) \otimes \theta_\nu(x) \\
&= \Gamma(s)^{-1}\sum_\nu \lambda_\nu^a |\lambda_\nu|^{-bs}\int_0^\infty \tau^{s-1}\exp(-\tau)\, d\tau \cdot \theta_\nu(x) \otimes \theta_\nu(x) \\
&= \begin{cases} \zeta(bs-a) & \text{if } a \text{ is even} \\ \eta(bs-a) & \text{if } a \text{ is odd.} \end{cases}
\end{aligned}$$

We use the asymptotic expansion:

$$f_{a,b}(t) \simeq \sum_{n=0}^{N} t^{(n-m-au)/bu} e_{n,a,b}$$

where $e_{n,a,b} = e_n(x, P^a\, exp(-tP^b))$ to conclude:

$$Mf_{a,b}(s) = \Gamma(s)^{-1}\left\{ \sum_{n=0}^{N} \left(s + \frac{N-m-au}{bu} \right)^{-1} e_{n,a,b} + \chi_{N,a,b}(s) \right\}$$

where $\chi_{N,a,b}(s)$ is holomorphic on the half-plane

$$\mathrm{Re}(s) > (m + au - n)/bu.$$

After making a change of variables to replace $bs - a$ by s, we generalize Lemma 2.3 to be:

LEMMA 3.1. *Let P be a self-adjoint elliptic differential operator of order $u > 0$ without O spectrum and let ζ, η be as defined previously. Both have meromorphic extensions to C with isolated simple poles at $s = (m - n)/u$. Let $a, b \in Z$ with $a \geq 0$ and $b > 0$ even. Then*:

(a) *If a is even, ζ has a simple pole at $s = (m - n)/u$ with residue $\Gamma((s + a)/b)^{-1} \cdot b \cdot e_n(x, P^a \exp(-tP^b))$. If $((s + a)/b)$ is a non-positive integer, then ζ is regular at this value and the value is*

$$b \cdot e_n\big(x, P^a \exp(-tP^b)\big)/\Big\{\operatorname*{Res}_{z=s}\Gamma((z + a)/b)\Big\}.$$

$e_n = 0$ if n is odd.

(b) *If a is odd, η has a simple pole at $s = (m - n)/u$ with residue $\Gamma((s + a)/b)^{-1} \cdot b \cdot e_n(x, P^a \exp(-tP^b))$. If $((s + a)/b)$ is a non-positive integer, then η is regular at this value and the value is*:

$$b \cdot e_n\big(x, P^a \exp(-tP^b)\big)/\operatorname*{Res}_{z=s}\Gamma((Z + a)/b).$$

$e_n = 0$ if $u + n$ is odd.

This lemma implies

THEOREM 3.2. *Let P be a self-adjoint elliptic differential operator of order $u > 0$ without O spectrum. Let $a, b \in Z$ with $a \geq 0$ and $b > 0$ even.*

(a) *If a is even then*

$$e_n\big(x, P^a \exp(-tP^b)\big)$$
$$= \frac{2}{b}\left\{\Gamma\left(\frac{s+a}{b}\right)\Gamma\left(\frac{s}{2}\right)^{-1}\right\}_{s=(m-n)/u^e n} e_n\big(x, \exp(-tP^2)\big).$$

(b) *If a is odd then*

$$e_n\big(x, P^a \exp(-tP^b)\big)$$
$$= \frac{2}{b}\left\{\Gamma\left(\frac{s+a}{b}\right)\Gamma\left(\frac{s+1}{2}\right)^{-1}\right\}_{s=(m-n)/u} e_n\big(x, P\exp(-tP^2)\big).$$

Proof. This theorem follows from Lemma 3.1 in exactly the same way that Theorem 2.4 follows from Lemma 2.3 and we therefore omit the proof.

We use this result and argue exactly as in the proof of Theorem 1.7 to prove:

THEOREM 3.3. *Let P be a self-adjoint elliptic differential operator of order $u > 0$. Let $A(r)$ be a polynomial and let $B(r)$ be a polynomial of even degree with positive leading coefficient. There exist constants $c_k = c_k(n, m, u, A, B)$ and $d_k = d_k(n, m, u, A, B)$ which are non-zero for only a finite number of values k such that*

$$e_n(x, A(P)\exp(-tB(P))) = \sum_k c_k e_k(x, \exp(-tP^2)) + d_k e_k(x, P\exp(-tP^2)).$$

Theorem 1.7 shows that the invariants $\{e_n(x, \exp(tP))\}$ span the space of all invariants arising from the heat equation if the leading symbol of P is positive definite. Theorem 3.3 shows that the invariants $\{e_n(x, \exp(-tP^2)), e_n(x, P\exp(-tP^2))\}$ span the corresponding space if the leading symbol of P is indefinite. We form the spectral invariants

$$a_n(\exp(-tP)) = \int_M \operatorname{Tr} e_n(x, \exp(-tP))\, d\operatorname{vol}(x)$$

and similarly $a_n(\exp(-tP^2))$ and $a_n(P\exp(-tP^2))$. It is a natural question to ask to what extent these are linearly independent. For the case of positive definite leading symbols, the local question and the global question are the same, but for indefinite leading symbols, the local and the global question turn out to have quite different answers.

We first assume the leading symbol of P is positive definite. From the asymptotic series given in Lemma 1.6 it follows that:

$$e_n(x, \exp(-tcP)) = c^{(m-n)/u} e_n(x, \exp(-tP))$$

and similarly for the global integrated invariants. Consequently, if the invariants are non-zero, they are linearly independent. By Lemma 1.6, we know $e_n(x, \exp(-tP)) = 0$ if n is odd and we restrict henceforth to n even.

THEOREM 3.4. *Fix an order $u > 0$ and a dimension m. The invariants $\{a_n(\exp(-tP))\}$ for n even are linearly independent: i.e., given any finite sequence of nonzero constants c_k there exists a self-adjoint positive definite differential operator P of order u so $\sum_k c_k a_{2k}(\exp(-tP)) \neq 0$. The odd invariants vanish: $a_{2k+1} = e_{2n+1} = 0$.*

Of course, since the integrated invariants are linearly independent, the local invariants must also be so. Before beginning the proof, we will need a result concerning e_0. It follows directly from Seeley's work and we shall omit the proof:

LEMMA 3.5. *Let P be a self-adjoint elliptic differential operator with positive definite leading symbol p_u. Then*

$$e_0(x, \exp(-tP)) = (2\pi)^{-m} \int \exp(-p_u(x, \xi))\, d\xi \neq 0$$

$$a_0(\exp(-tP)) = (2\pi)^{-m} \int \operatorname{Tr} \exp(-p_u(x, \xi))\, d\xi\, d\operatorname{vol}(x) \neq 0.$$

We now begin the proof of Theorem 3.4. Assume first that $u = 2$. We must show $a_{2k}(\exp(-tP)) \neq 0$ for some P. For any operator P, $a_0(P) \neq 0$ by Lemma 3.5. Let c be a constant, then

$$\exp(-t(P - c)) = \exp(tc)\exp(-tP).$$

By comparing powers of t in the resulting asymptotic expansion, we conclude that

$$a_n(P - c) = \sum_{k+2j=n} a_k(P) c^j / j!.$$

We assume n is even so that $a_n(P - c) = a_0(P) c^{n/2} / (\frac{n}{2})! +$ lower order terms and hence $a_n(P - c) \neq 0$ for arbitrarily small values of c. This proves Theorem 3.4 if $u = 2$.

Next, let $u = 2b$ for $b > 0$. Let P have order 2, Theorem 2.4 implies:

$$a_n(\exp(-tP^b)) = b^{-1}\{\Gamma(s/b)\Gamma(s)^{-1}\}_{s=(m-n)/2} a_n(\exp(-tP)).$$

We have shown we can choose P so $a_n(\exp(-tP)) \neq 0$ which proves $a_n(\exp(-tP^b)) \neq 0$ if $(m - n)/2$ is not a non-positive integer. Since n is even, this completes the proof if m is odd.

To study the case m even, we need the following product formula:

LEMMA 3.6. *Let M_i be Riemannian manifolds and let P_i: $C^\infty(V_i) \to C^\infty(V_i)$ be self-adjoint elliptic differential operators of order $u > 0$ with positive definite leading symbol. Let $P = P_1 \otimes 1 + 1 \otimes P_2$ on $C^\infty(V_1 \otimes V_2)$. P is a selfadjoint elliptic differential operator of order $u > 0$ with positive definite leading symbol. Then $a_n(P) = \sum_{j+k=n} a_j(P_1) a_k(P_2)$.*

Proof. We use the identity $\exp(-tP) = \exp(-tP_1) \otimes \exp(-tP_2)$ and collect corresponding terms in the asymptotic expansions.

Let m be even and let $M = M_1 \times S^1$ where the dimension of $M_1 = m - 1$ is odd. S^1 is the unit circle with periodic parameter θ. We showed that we can find P_1 of order u on M_1 so $a_n(P_1) \neq 0$. Let $P_2 = (i\beta/\partial\theta)^u$. By Lemma 3.5, $a_0(P_2) \neq 0$. Since P_2 is a constant coefficient operator and since $e_n(x, P_2)$ is homogeneous of order n in the jets of P_2, we conclude $e_n(x, P_2) = 0$ for $n > 0$. This implies $a_n(P) = a_n(P_1)a_0(P_2) \neq 0$ which completes the proof of Theorem 3.5.

The situation is somewhat more complicated if we do not assume P is positive definite. By Theorem 1.5, we know $e_n(x, \exp(-tP^2)) = 0$ if n is odd and that $e_n(x, P\exp(-tP^2)) = 0$ if $n + u$ is odd. For the remainder of this paper, let P be an elliptic self-adjoint partial differential operator of order $u > 0$ where we do not necessarily assume the leading symbol of P to be definite. The appropriate generalization of Theorem 3.4 is:

THEOREM 3.7. *Fix an order $u > 0$ and a dimension m.*

(a) *If u is odd, the invariants*

$$\left\{a_{2k}\left(\exp(-tP^2)\right), a_{2l+1}\left(P\exp(-tP^2)\right)\right\}_{k,l=0}^{\infty} \quad \text{for } 2l + 1 \neq m$$

are linearly independent. The remaining invariants vanish identically.

(b) *If u is even, the invariants*

$$\left\{a_{2k}\left(\exp(-tP^2)\right), a_{2l}\left(P\exp(-tP^2)\right)\right\}_{k,l=0}^{\infty} \quad \text{for } 2l \neq m$$

are linearly independent. The remaining invariants vanish identically.

This theorem is somewhat different in flavor from Theorem 3.4. In particular, there is the assertion that the invariant $a_n(P\exp(-tP^2)) = 0$ if $n = m$. This is nothing but the assertion that the global integrated ζ function is regular at $s = 0$ by Lemma 3.1. This is proved for m odd in Atiyah et al. [**3**] and for m even by the second author [**12**]. It was also proved by Wodzicki [**18**]. We also remark that if $m \geq 2$ and if $m - u$ is even, then the local invariant $a_m(x, P\exp(-tP^2))$ does not vanish identically (even though it integrates to 0). We refer to [**11**] for suitable examples. Since $a_n(P\exp(-tP^2))$ changes sign if we replace P by $-P$ while $a_n(\exp(-tP^2))$ does not, it suffices to prove each collection of invariants is linearly independent separately. Using the homogeneity argument used in the proof of Theorem 3.4, it suffices to establish that the relevant invariants do not vanish identically.

Before completing the proof of Theorem 3.7, we must establish some recursion relationships. Let c be a constant and replace P by $P + c$. We differentiate and set $c = 0$ to establish the identities:

$$\frac{d}{dc}\left\{\exp\left(-t(P+c)^2\right)\right\}_{c=0} = -2P\exp(-tP^2),$$

$$\frac{d}{dc}\left\{(P+c)\exp\left(-t(P+c)^2\right)\right\}_{c=0} = \exp(-tP^2) - 2tP^2\exp(-tP^2)$$

$$= \left(1 + 2t\frac{d}{dt}\right)\exp(-tP^2).$$

We substitute the asymptotic series of Lemma 1.6 to conclude:

$$\frac{d}{dc}\left\{\sum t^{(n-m)/2u} e_n\left(x, \exp\left(-t(P+c)^2\right)\right)\right\}_{c=0}$$

$$= \sum t^{(n-m+u)/2u}\left\{-2e_n(x, P\exp(-tP^2))\right\}$$

$$= \sum t^{(n-m)/2u}\left\{-23_{n-u}(x, P\exp(-tP^2))\right\}$$

and

$$\frac{d}{dc}\left\{\sum t^{(n-m-u)/2u} e_n\left(x, (P+c)\exp\left(-t(P+c)^2\right)\right)\right\}_{c=0}$$

$$= \left(1 + 2t\frac{d}{dt}\right)\sum t^{(n-m)/2u} e_n(x, \exp(-tP^2))$$

$$= \sum t^{(n-m)/2u}\left(1 + \frac{n-m}{u}\right) e_n(x, \exp(-tP^2))$$

$$= \sum t^{(n-m-u)/2u}\frac{n-m}{u} e_{n-u}(x, \exp(-tP^2)).$$

We compare equal powers of t in these two asymptotic expansions to see

LEMMA 3.8.
(a)

$$\frac{d}{dc}\left\{e_n\left(x, \exp\left(-t(P+c)^2\right)\right)\right\}_{c=0} = -2e_{n-u}e_n(x, P\exp(-tP^2)),$$

(b)

$$\frac{d}{dc}\left\{e_n\left(x, (P+c)\exp\left(-t(P+c)^2\right)\right)\right\}_{c=0} = \frac{n-m}{u} e_{n-u}(x, \exp(-tP^2)).$$

We now begin the proof of Theorem 3.7. There are a number of special cases which must be considered and each involves a different technical trick. Let $u = 1$. Let $M = T^m = S^1 \times \cdots S^1$ be the flat torus with periodic parameters $\{\theta_1, \ldots, \theta_m\}$. Let $\{\varepsilon_0, \ldots, \varepsilon_m\}$ be a collection of Clifford matrices. These are self-adjoint matrices satisfying the commutation relations $\varepsilon_{ij} + \varepsilon_{ji} = 2\delta_{ij}$ where δ is the Kronecker index. For example, if $m = 2$, we could take

$$\varepsilon_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \varepsilon_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

More generally, such matrices arise from the spin representations. Let P be the operator:

$$P = i \sum_{\nu=1}^{m} \varepsilon_\nu \frac{\partial}{\partial \theta_\nu} + c \cdot \varepsilon_0$$

where c is a real constant. From the commutation relations,

$$P^2 = P_0 + c^2 \quad \text{where} \quad P_0 = -\sum_\nu \frac{\partial^2}{\partial \theta_\nu^2}.$$

Therefore if n is even,

$$a_n\big(\exp(-tP^2)\big) = \sum_{i+j=n} a_i\big(\exp(-tP_0)\big) c^{2j}/j!$$

does not vanish identically since $a_0(\exp(-tP_0)) \neq 0$. The recursion relation of Lemma 3.8b implies $a_{n+1}((P + c')\exp(-t(P + c')^2))$ can't vanish identically if $n + 1 \neq m$ which completes the proof Theorem 3.7 if $u = 1$.

We wish to take powers and use Lemma 3.2 to pass from the first order case to the general case. To do this, we must verify that certain coefficients involving the Γ function do not vanish. This is true if either $n < m$ or if m is odd. Let P_1 be a first order operator so that $a_n(\exp(-tP^2)) \neq 0$. Let $P = P_1^u$. Theorem 3.2(a) implies:

$$a_n\big(\exp(tP^{2u})\big) = \frac{1}{u}\left\{\Gamma\left(\frac{s}{2u}\right)\Gamma\left(\frac{s}{2}\right)^{-1}\right\}_{s=(m-n)} a_n\big(\exp(-tP_1^2)\big).$$

If $n < m$, then s is positive so the coefficient $\Gamma(s/2u)\Gamma(s/2)^{-1}$ is non-zero. If m is odd then since n is even, $s/2$ and $s/2u$ is not an integer and again this coefficient must be non-zero.

We apply Theorem 3.2(b) to compute $a_n(P^u \exp(-tP^{2u}))$ in terms of invariants relating to P_1:

$$a_n\big(P^u \exp(-tP^{2u})\big) = \frac{1}{u}\begin{cases} \left(\Gamma\left(\frac{s+u}{2u}\right)\Gamma\left(\frac{s}{2}\right)^{-1}\right)_{s=m-n} a_n\big(\exp(-tP_1^2)\big) & \text{if } u \text{ is even} \\ \left(\Gamma\left(\frac{s+u}{2u}\right)\Gamma\left(\frac{s+1}{2}\right)^{-1}\right)_{s=m-n} a_n\big(P_1\big(\exp(-tP_1^2)\big)\big) & \text{if } u \text{ is odd} \end{cases}$$

If $n < m$, then $\{s, s+u, s+1\}$ are all positive and these coefficients are all non-zero. Let m be odd and suppose first u is even. As n is even,

$m \neq n$. Choose P_1 so $a_n(\exp(-tP_1^2)) \neq 0$. Since m is odd, $s = m - n$ and $s + u = m - n + u$ are odd integers so the coefficient $\Gamma((s+u)/2u)\Gamma(s/2)^{-1}$ is non-zero and $a_n(P^u \exp(-tP^{2u})) \neq 0$. Next we suppose u is odd so n is odd. Choose P_1 so $a_n(P_1 \exp(-tP_1^2)) = 0$ if $n \neq m$. Then $s = m - n$ is an even integer so the coefficient $\Gamma((s+u)/2u)\Gamma((s+1)/2)^{-1}$ is non-zero and again $a_n(P^u \exp(-tP^{2u})) \neq 0$. This completes the proof of Theorem 3.7 if the dimension m is *odd*, or if $n < m$.

We now let m be even and assume $n \geq m$. We use twisted products to handle this case. Let $M = M_1 \times S^1$ where the dimension of $M_1 = m - 1$ is odd. S^1 is the unit circle with periodic parameter θ. Let j be given and let n be even. Since $(m-1)$ is odd, we can find P_1 of order j on M_1 so $a_n(\exp(-tP_1^2)) \neq 0$. Let $P_2 = (i\partial/\partial\theta)^u$ on S^1 then $a_0(\exp(-tP_2^2)) \neq 0$ while $a_j(-tP_2^2)) = 0$ for $j > 0$. Over M we define:

$$P = P_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + P_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_2 & -P_1 \end{pmatrix}$$

and compute immediately that:

$$P^2 = \left(P_1^2 + P_2^2\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that P is elliptic and self-adjoint. Lemma 3.6 shows

$$\begin{aligned} a_n\left(\exp(-tP^2)\right) &= 2 \sum_{i+j=n} a_i\left(\exp\left(-tP_1^2\right)\right) a_1\left(\exp\left(-tP_2^2\right)\right) \\ &= 2a_n\left(\exp\left(-tP_1^2\right)\right) a_0\left(\exp\left(-tP_2^2\right)\right) \neq 0. \end{aligned}$$

This completes the part of Theorem 3.7 which deals with the invariants $a_n(\exp(-tP^2))$.

Before studying the invariants $a_n(P\exp(-tP^2))$ if m is even, we must derive another recursion relationship. Let P be self-adjoint elliptic of order $u > 0$ and let Q be self-adjoint of order j so that $PQ = QP$. Let $P(c) = P + cQ$ for c small and positive and compute:

$$\begin{aligned} \frac{d}{dc}\left(P(c)\exp\left(-tP(c)^2\right)\right)\Big|_{c=0} &= Q\exp(-tP^2) - 2tQP^2\exp(-tP^2) \\ &= \left(1 + 2t\frac{d}{dt}\right) Q\exp(-tP^2). \end{aligned}$$

When the resulting asymptotic series are expanded, we conclude:

$$\frac{d}{dc} a_n\left(P(c)\exp\left(-tP(c)^2\right)\right)\Big|_{c=0} = \frac{m-n}{u} a_n\left(Q\exp(-tP^2)\right).$$

We apply this relationship to the operators:

$$P = P_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + P_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q + P_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then P and Q are self-adjoint and commute and

$$a_n\left(Q \exp(-tP^2)\right) = 2a_n\left(P_1 \exp\left(-tP_1^2\right)\right) a_0\left(\exp\left(-tP_2^2\right)\right).$$

We suppose $n \neq m$. Since $n \geq m$, $n \neq m - 1$. Therefore, we can choose P_1 on M_1 so $a_n(P_1 \exp(-tP_1^2)) \neq 0$ since $m - 1$ is odd. This shows $a_n(P(c) \exp(-tP(c)^2))$ does not vanish identically and completes the proof of Theorem 3.8.

There are some obvious generalizations of these results. It is only necessary to assume that the spectrum of the leading symbol $p(x, \xi)$ is contained in the cone $\mathrm{Re}(\lambda) > 0$ for $\xi \neq 0$ to define $\exp(-tP)$. Similarly to have $\exp(-tP^2)$ be well defined, we need only assume that the spectrum of the leading symbol lies in the cone $|\mathrm{Im}(\lambda)| < |\mathrm{Re}(\lambda)|$ for $\xi \neq 0$. The zeta and eta invariants are well defined under even weaker hypothesis and the proofs go through without change.

By studying $\frac{1}{2}\{\zeta \pm \eta\}$ we can construct invariants which are concentrated either on the positive or negative spectrum of P. Let $k \geq 2$ be a positive integer and let P be an elliptic operator such that the spectrum of the leading symbol p is contained on the rays $\lambda^k \in R^+$ for $\xi \neq 0$. We can construct invariants which are concentrated on the spectrum corresponding to each of the k such distinct rays and express such invariants in terms of $\mathrm{Tr}(P^j \exp(-tP^k))$, $0 \leq j \leq k$. The theorems in this case are similar to the cases already considered. As we know of no natural operators arising in geometry corresponding to the case $k > 2$, we shall not bother to discuss such generalizations in specific detail. We also refer to a recent paper of Wodzicki **[19]**.

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