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CLOSED SUBSPACES OF H-CLOSED SPACES

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The concept of an H-set (a generalization of an H-closed space) was introduced by N. V. Velicko. In this paper we obtain internal properties of H-sets in terms of the Iliadis absolute EX and the Hausdorff absolute PX. Some of the main results are:

- **An H-closed space X is Urysohn iff $P^{-1}(A)$ is an H-set in PX for every H-set $A \subset X$.**
- **If A is an H-closed subset of X then there exists a compact $B \subset EX$ such that $\pi(B) = A$ and $\pi: B \to A$ is θ-continuous.**
- **There exists a space X and an H-set $A \subset X$ which is not the image of a compact subset of EX.**
- **If $\{H_i\}_i$ is a chain of H-closed subspaces in X, then $\bigcap H_i$ is the image of a compact subset of EX.**

A question of A. Dow and J. Porter is answered and a question of R. G. Woods is answered partially.

1. Introduction. Throughout this paper all spaces are assumed to be Hausdorff. Our main interests in this paper are the H-closed subspaces and the H-sets of a given space X. Our goal is to characterize these subsets. Our first attempt toward such a characterization is to consider preimages of these subsets in the Iliadis absolute EX and the Hausdorff absolute PX. If X is an H-closed Urysohn space we obtain an answer, namely: $A \subset X$ is an H-set iff $P^{-1}(A) \subset PX$ is an H-set. For H-closed subspaces of an H-closed Urysohn space the situation is less clear. We give an example of such a space X and an H-closed subspace A such that $P^{-1}(A)$ is not H-closed.

For the case of non-Urysohn H-closed spaces the situation differs. We present an example of an H-set in a space X which is not the image of a compact subset of EX (or an H-set in PX).

Although our approach to these questions is quite technical, results 4.7, 4.8, 5.1 and 5.7(i) show that it is rather useful. However, many questions remain unanswered.

2. Preliminaries. In this section we collect some basic notions and notations.

2.0. We assume the reader is familiar with the class of H-closed spaces and the class of minimal Hausdorff spaces. Also the definition of notions such as regular open (closed) subset and the semiregularization of a space X are assumed to be known. The papers [5], [6] and [14] provide the necessary background.

For a space X, τ_X denotes its topology and τ_S denotes the topology of the semiregularization X_s of X. If $x \in X$, U_x [or V_x] always denotes an *open* neighborhood of x in X.

In particular, if X is H-closed then cl U_x is H-closed. It is well-known that X is H-closed iff X_s is minimal Hausdorff. In particular, a space is minimal Hausdorff iff it is semiregular and H-closed (see [5]).

2.1. A compact and closed map $f\colon X \to Y$ is called a *perfect* map. (Hence, a perfect map is not assumed to be continuous nor surjective.)

A surjection $f\colon X \to Y$ is called *irreducible*, provided $f(A) \neq Y$ for each proper closed subset A of X. A map $f\colon X \to Y$ is said to be θ-continuous, iff:

$$\forall x \in X, \forall U_{f(x)}, \exists V_x \text{ such that } f(\operatorname{cl} V_x) \subset \operatorname{cl} U_{f(x)}.$$

It is obvious that continuous maps are θ-continuous, and that a θ-continuous map $f\colon X \to Y$ is continuous if the space Y is regular. θ-continuous functions must be treated rather carefully, as they may cause unexpected problems. For example, if $f\colon X \to Y$ is θ-continuous, the map $f\colon X \to f(X)$ need not be θ-continuous, (even if $f(X)$ is a regular subspace of Y (see example 4.5)). However the following statements are valid for a θ-continuous map $f\colon X \to Y$.

(i) If $Y \subset Z$, then $f\colon X \to Z$ is θ-continuous.

(ii) If $A \subset X$, then $f|A\colon A \to Y$ must be θ-continuous.

(iii) If $f(X) \subset Z \subset Y$, and Z is dense in Y, then $f\colon X \to Z$ is θ-continuous.

(iv) The map id: $(X, \tau_X) \to (X, \tau_S)$ is θ-continuous.

(v) If X is H-closed and f is surjective, then Y is H-closed.

(vi) Compositions of θ-continuous maps are θ-continuous.

All these statements are easy to verify.

2.2. The *Iliadis absolute* EX of a space X is the subspace consisting of the fixed ultrafilters in the Stone space of the Boolean algebra of the regular open subsets of X. There is a natural map $\pi\colon EX \to X$ defined by: $\pi(\mathscr{F}) = \bigcap\{\operatorname{cl} F\colon F \in \mathscr{F}\}$. The following statements are well-known (see [4] and [14]).

(i) EX is extremally disconnected and Tychonoff.

(ii) EX is compact iff X is H-closed.

(iii) π is an irreducible perfect and θ-continuous surjection.

(iv) If $f\colon Y \to X$ is an irreducible, perfect and θ-continuous surjection, then there exists a map $\pi f\colon EX \to Y$ such that πf is θ-continuous, perfect and irreducible and $f \circ \pi f = \pi$.

The *Hausdorff absolute* PX can be constructed from EX in the following way. Consider the smallest topology τ on EX such that $\tau \supset \tau_{EX}$ and $\pi\colon (EX, \tau) \to X$ is continuous, thus the collection $\tau_{EX} \cup \{\pi^{-1}(U) \mid U$ open in $X\}$ is a subbase for τ.

The space (EX, τ) is denoted by PX, and is called the Hausdorff absolute of X.

The map $\pi\colon (EX, \tau) \to X$ we denote by $P\colon PX \to X$.

The following statements are well-known:

(i) $(PX)_s \cong EX$ (and so PX is extremally disconnected).

(ii) $P\colon PX \to X$ is an irreducible, perfect and continuous surjection.

For more information on EX and PX we refer to: **[6]**, **[7]** and **[14]**. The following theorem will be used in several places.

2.3. THEOREM **[12]**. *Let $f\colon X \to Y$ be a compact and irreducible surjection from a space X onto a set Y. Then the collection $\{f(A) \mid A$ a closed subset of $X\}$ is a closed base for a Hausdorff topology τ on Y. The map $f\colon X \to (Y, \tau)$ turns out to be θ-continuous. Furthermore, if X is compact then (Y, τ) is minimal Hausdorff.* □

COROLLARY. *Let $f\colon X \to Y$ be a compact and irreducible surjection. Then, if Y is H-closed, X is H-closed too and the map $f\colon X \to Y$ is θ-continuous.*

Proof. Assume the space X is not H-closed. Let $\mathscr{F}$ be an open filter on X without an accumulation point in X. As the map f is closed and irreducible, the collection: $\{Y - f(X - F) \mid F \in \mathscr{F}\}$ is a centered system of (non-empty) open subsets of Y.

Claim. $\bigcap\{\mathrm{cl}_Y(Y - f(X - F)) \mid F \in \mathscr{F}\} = \varnothing$.

Choose $y \in Y$. Then $f^{-1}(y)$ is compact, therefore, there is a $F \in \mathscr{F}$ such that $f^{-1}(y) \cap \mathrm{cl}_X F = \varnothing$. Define $U = X - \mathrm{cl}_X F$. Hence, $Y - f(X - U)$ is a neighborhood of $y \in Y$ with the property that

$$(Y - f(X - U)) \cap (Y - f(X - F)) = \varnothing.$$

It follows that $y \notin \bigcap\{\mathrm{cl}_Y(Y - f(X - F)) \mid F \in \mathscr{F}\}$ and our claim holds. This contradicts our assumption that Y is H-closed. It follows that X is H-closed. From the previous theorem we can conclude that the map

$f: X \to Y$ is θ-continuous. Indeed, consider the topology τ on the set Y determined by the closed base $\{f(A) | A$ a closed subset of $X\}$. The previous theorem implies that the map $f: X \to (Y, \tau)$ is θ-continuous, hence, the topology τ on Y is H-closed. But the map f is closed, therefore, $\tau \subset \tau_Y$. But τ_Y is H-closed too. Hence it follows from 2.1. (iv) and 2.1 (vi) that both the maps id: $(Y, \tau) \to (Y, \tau_Y)$ and $P: X \to (Y, \tau_Y)$ are θ-continuous. □

2.4. A subset $A \subset X$ is called an *H-set* in X [9], provided A satisfies one of the following equivalent statements:

(i) If $\mathscr{U}$ is a cover of A consisting of open subsets of X, then there is a finite subfamily $\{U_i\}_{i=1}^n \subset \mathscr{U}$ such that $A \subset \bigcup_{i=1}^n \mathrm{cl}_X U_i$.

(ii) If $\mathscr{F}$ is an open filter of X such that $F \cap A \neq \varnothing$, for each $F \in \mathscr{F}$, then $A \cap \bigcap\{\mathrm{cl}_X F | F \in \mathscr{F}\} \neq \varnothing$.

Note that the property of being an H-set in X is an embedding property, i.e. it is not determined by the topology of A. If $A \subset X$ and $A \subset Y$ then it is possible that A is an H-set in X but not in Y (see 4.2).

However, we will use the terminology: "Let A be an H-set" instead of let "A be an H-set in X" when no ambiguity is possible. In the following lemma we collect some properties of H-sets. All the statements in this lemma are easy to verify.

LEMMA. (i) *If A is an H-set in X, then A is closed.*

(ii) *If A is a H-set in X and $X \subset Y$, then A is an H-set in Y.*

(iii) *If $f: X \to Y$ is θ-continuous and $A \subset X$ is an H-set, then $f(A)$ is an H-set in Y.*

(iv) *A is an H-set in X iff $A \subset X_s$ is an H-set in X_s*

(v) *An H-set in a regular space is compact.*

(vi) *Let B be a regular closed subset of X. If $A \subset X$ is an H-set and $B \subset A$ then B is H-closed.* □

2.5. The following lemma shows how to obtain H-closed topologies from an underlying minimal Hausdorff topology.

LEMMA. (i) *Let (X, τ) be a minimal Hausdorff space and let $\mathscr{F}$ be a collection of subspaces of X with the property: If $F_i \in \mathscr{F} (i = 1, \dots, n)$, then $\bigcap_i F_i$ is dense in X. Then the topology $\tau_{\mathscr{F}}$ generated by the open subbase $\tau \cup \mathscr{F}$ is H-closed.*

(ii) *Let (X, τ) be an H-closed space. Then there exist a filter-base $\mathscr{F}$ of dense subspaces of the minimal Hausdorff space (X, τ_s) such that $\tau_s \cup \mathscr{F}$ is an open subbase for τ.*

Proof. (i) It is straightforward to check that the map id: $(X, \tau) \to (X, \tau_{\mathscr{F}})$ is θ-continuous. The conclusion follows.

(ii) Define $\mathscr{F} = \{F | F$ is an open and dense subset of $(X, \tau)\}$. □

3. H-closed Urysohn spaces. We start our investigations on preimages of closed subsets of X in EX and PX in the class of H-closed Urysohn spaces. Recall that a space X is *Urysohn*, if distinct points of X have disjoint closed neighborhoods. The following theorem is well-known.

3.1. THEOREM [5]. *A space X is H-closed and Urysohn iff X_s is compact.* □

We immediately obtain the following trivial result.

3.2. PROPOSITION. *Let X be an H-closed Urysohn space. Then*:

(i) *A is an H-set of X iff A is a compact subspace of X_S.*

(ii) *A is an H-closed subspace of X iff A is a compact subspace of X_S and the restriction of the map* id: $X_s \to X$ *to A, i.e. the map* id: $A (\subset X_s) \to A (\subset X)$, *is θ-continuous.*

(iii) *$A \subset X$ is an H-set of X iff A is the intersection of H-closed subsets of X.*

Proof. (i) Combine parts (iv) and (v) in Lemma 2.4.

(ii) See Corollary 2.3.

(iii) Let A be an H-set in X. Then $A \subset X_s$ is compact, hence, $A = \bigcap\{U_i\colon U_i$ a regular closed subset of X_s, $U_i \supset A\}$. However, as U_i is regular closed, U_i, considered as a subspace of X, is H-closed. The conclusion follows. The converse follows directly from (i) and (ii). □

3.3. COROLLARY. *Let X be an H-closed space. The following statements are equivalent*:

(i) *X is Urysohn.*

(ii) *Every centered system of H-sets has non-empty intersection.*

(iii) *Every centered system of H-closed subspaces has non-empty intersection.*

Proof. (i) $\to$ (ii) Follows from 3.2.(i).

(ii) $\to$ (iii) Trivial.

(iii) $\to$ (i). Assume the space X is not Urysohn. Choose $x, y \in X$ such that x and y are not contained in disjoint closed neighborhoods. Obviously, the collection $\{\mathrm{cl}\, U_x | U_x\} \cup \{\mathrm{cl}\, U_y | U_y\}$ is a centered system of H-closed subspaces with empty intersection. □

3.4. Remark. Of course Proposition 3.2 can hardly be considered to be an intrinsically interesting statement. There is a similar statement for arbitrary Hausdorff spaces, namely:

(i) A is an H-set of X iff A is an H-set of X_s.

(ii) A is an H-closed subset of X iff A is an H-closed subset of X_s and the map id: A ($\subset X$) $\to A$ ($\subset X_s$) is θ-continuous.

The reason why we added Proposition 3.2, is in fact that the Hausdorff absolute PX is always an Urysohn space.

3.5. Theorem. *Let X be an H-closed Urysohn space. The following are equivalent*:

(i) *A is an H-set in X.*

(ii) *$\pi^{-1}(A)$ is compact.*

(iii) *$P^{-1}(A)$ is an H-set in PX.*

Proof. $A \subset X$ is an H-set iff $A \subset X_s$ is compact iff $\pi^{-1}(A) \subset E(X_s)$ ($= EX$) is compact iff $P^{-1}(A) \subset PX$ is an H-set. (see 3.2) (Note that $(PX)_s = EX$.) □

3.6. Theorem. *Let X be an H-closed Urysohn space. Then*:

(i) *$A \subset X$ is H-closed iff $\pi^{-1}(A)$ is compact and the restricted map π: $\pi^{-1}(A) \to A$ is θ-continuous.*

(ii) *If $A \subset X$ is H-closed then $P^{-1}(A) \subset PX$ is an H-set in PX.*

Proof. (i) This follows directly from 3.2(ii) and the fact that the map π: $E(X_s) \to X_s$ is continuous.

(ii) This follows from 3.5(ii). □

Comparing 3.6(i) and 3.6(ii) one might hope for the following statement to be true.

"If X is an H-closed Urysohn space then $A \subset X$ is H-closed iff $P^{-1}(A) \subset PX$ is H-closed."

However, the following example shows that such hope is in vain.

3.7. Example. Let X be the unit interval $[0,1]$ and let Y be the following subspace of $\mathbf{R}^2$.

$$Y = (\{0\} \times [0,1]) \cup \left\{[0,1] \times \left\{\frac{1}{n}\right\} : n \in \mathbf{N}\right\} \cup ([0,1] \times \{0\}).$$

Define f: $Y \to X$ by $f((x, y)) = y$.

Clearly, Y is compact and f is a continuous, perfect non-irreducible surjection.

It easily follows that there exist compactifications $\alpha\mathbf{N}$ and $\gamma\mathbf{N}$ of $\mathbf{N}$ and a continuous map $g: \gamma\mathbf{N} \to \alpha\mathbf{N}$ such that

(i) $\alpha\mathbf{N} - \mathbf{N} = X$ and $\gamma\mathbf{N} - \mathbf{N} = Y$

(ii) $g/(\gamma\mathbf{N} - \mathbf{N}) = f$

(iii) $g/\mathbf{N} = \mathrm{id}_{\mathbf{N}}$.

A possible construction of $\alpha\mathbf{N}$ is as follows. Let $\mathscr{N}$ be a countable dense subset of X. Clearly the subject $(X \times \{0\}) \cup (\mathscr{N} \times \{1\})$ of the Alexander double of X can be considered as a compactification $\alpha\mathbf{N}$ of the countable discrete space with $\alpha\mathbf{N} - \mathbf{N} \cong X$.

From (iii) we see that g is a continuous *irreducible* surjection. Let $\alpha_1\mathbf{N}$ be the space obtained from $\alpha\mathbf{N}$ by declaring the dense subset $\mathbf{N} \cup ([0,1] - \{1/n: n \in \mathbf{N}\})$ to be open.

Clearly, $\alpha_1\mathbf{N}$ is an H-closed Urysohn space. Furthermore, $\alpha_1\mathbf{N} - \mathbf{N} = X_1$ is H-closed, since it is homeomorphic to the space $[0,1]$ in which the dense set $[0,1] - \{1/n: n \in \mathbf{N}\}$ is declared to be open.

Let $Y_1\mathbf{N}$ be the space obtained from $\gamma\mathbf{N}$ by declaring the dense set $g^{-1}(\mathbf{N} \cup ([0,1] - \{1/n: n \in \mathbf{N}\}))$ to be open.

Since g is irreducible it follows that the corresponding map g_1: $\gamma_1\mathbf{N} \to \alpha_1\mathbf{N}$ is a continuous, perfect and irreducible map. Therefore, $P(\gamma_1\mathbf{N}) = P(\alpha_1\mathbf{N})$ and $P\alpha_1 = g_1 \circ P\gamma_1$.

$$\begin{array}{ccc} P(\alpha_1\mathbf{N}) & = & P(\gamma_1\mathbf{N}) \\ {\scriptstyle P_{\alpha_1}}\downarrow & & \downarrow {\scriptstyle P_{\gamma_1}} \\ \alpha_1\mathbf{N} & \overset{g_1}{\leftarrow} & \gamma_1\mathbf{N} \end{array}$$

But now it follows that $P_1^{-1}(\alpha_1\mathbf{N} - \mathbf{N})$ is not H-closed (while $\alpha_1\mathbf{N} - \mathbf{N}$ is H-closed). For if this was the case it would follow that

$$\gamma_1\mathbf{N} - \mathbf{N} = P\gamma_1\,(\alpha_1\mathbf{N} - \mathbf{N})$$

would be H-closed.

However, $\gamma_1\mathbf{N} - \mathbf{N}$ is not H-closed since it is homeomorphic to Y in which the (non-dense) set

$$g^{-1}([0,1] - \{1/n: n \in \mathbf{N}\})$$
$$= (\{0\} \times ([0,1] - \{1/n: n \in \mathbf{N}\})) \cup ([0,1] \times \{0\})$$

is declared to be open. Clearly this space is not H-closed. For example, the collection

$$\left\{(p,1] \times \left\{\frac{1}{n} : n \geq k\right\} : 0 < p < 1, k \in \mathbf{N}\right\}$$

is an open filterbase on $\gamma_1\mathbf{N} - \mathbf{N}$ without an accumulation point. □

3.8. REMARK. It is easy to see that the statements in 3.5 and 3.6 about H-sets and H-closed subspaces remain valid in spaces which admit an H-closed Urysohn extension. □

4. The non-Urysohn case. The situation for the class of non-Urysohn H-closed spaces differs completely from the previous class. If we consider Corollary 3.3 we already obtain the following:

4.1. THEOREM. *Let X be an H-closed space and assume X satisfies one of the following properties*:

(i) *$A \subset X$ is an H-set iff $\pi^{-1}(A) \subset EX$ is compact.*
(ii) *$A \subset X$ is an H-set iff $P^{-1}(A) \subset PX$ is an H-set.*
(iii) *If $A \subset X$ is H-closed then $\pi^{-1}(A)$ is compact.*
(iv) *If $A \subset X$ is H-closed then $P^{-1}(A)$ is an H-set in PX.*

Then X is an Urysohn space (and therefore X satisfies all these properties).

Proof. Follows directly from 3.3, 3.6 and the fact that PX is Urysohn. □

Let us consider the standard example of a non-compact minimal Hausdorff space.

4.2. EXAMPLE. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be two disjoint copies of $\mathbf{N}$, disjoint from $\mathbf{N}$. If $A \subset \mathbf{N}$, A_i denotes the corresponding subset in $\mathbf{N}_i$ $(i = 1,2)$. Define

$$X = \{\omega_1\} \cup (\mathbf{N}_1 \times \mathbf{N}_1) \cup \mathbf{N} \cup (\mathbf{N}_2 \times \mathbf{N}_2) \cup \{\omega_2\}$$

with the following topology.

- The points of $\mathbf{N}_i \times \mathbf{N}_i$ are isolated $(i = 1,2)$.
- A neighborhood base for $n \in \mathbf{N} \subset X$ is the collection: $\{\{n\} \cup (D_1 \times \{n_1\}) \cup (D_2 \times \{n_2\}) | D \text{ a cofinite subset of } \mathbf{N}\}$.
- A neighborhood base for $\omega_i \in X$ is the collection: $\{\{\omega_i\} \cup (\mathbf{N}_i \times D_i) | D \text{ a cofinite subset of } \mathbf{N}\}$ $(i = 1,2)$.

Then X is a non-compact minimal Hausdorff space. Note that ω_1 and ω_2 are not contained in disjoint closed neighborhoods.

Define $A_i \subset X$ by $A_i = \mathrm{cl}_X(\mathbf{N}_i \times \mathbf{N}_i)$ $(i = 1,2)$. The subsets A_1 are non-compact, H-closed Urysohn subspaces of X $(i = 1,2)$. Define $Y = (A_1)_s \oplus (A_2)_s$ and let $f\colon Y \to X$ be the natural map which corresponds with the inclusions $A_i \subset X$ $(i = 1,2)$. Observe that Y is compact and that $f\colon Y \to X$ is a perfect, θ-continuous and irreducible map.

Let $\pi(f)$: $EX \to Y$ be the unique perfect, irreducible and continuous map such that $\pi = f \circ \pi(f)$. We observe the following:

(i) A_1 and A_2 are H-closed subspaces but $A_1 \cap A_2$ is not an H-set in X. (compare 3.2(iii)).

(ii) $A = \mathbf{N} \cup \{\omega_1\}$ is an H-set in X, but $f^{-1}(A) \subset Y$ is not compact, hence $\pi^{-1}(A) \subset EX$ is not compact, and $P^{-1}(A)$ is not an H-set in PX. Moreover, A is a subset of the regular closed subset

$$\mathrm{cl}((\mathbf{N}_2 \times \mathbf{N}_2) \cup (\{n_1\} \times \mathbf{N}_1)) = K\,(\subset X).$$

However, A is not an H-set in K.

(iii) $A_1 \subset X$ is H-closed and f: $f^{-1}(A_1) \to A_1$ is θ-continuous. But $f^{-1}(A_1) \subset Y$ is not compact nor is $\pi^{-1}(A_1) \subset EX$, although the map π: $\pi^{-1}(A_1) \to A_1$ is θ-continuous.

(iv) $G = A_1 \cup \{\omega_2\}$ is an H-closed subspace of X. The preimages $f^{-1}(G)$ and $\pi^{-1}(G)$ are compact, but the maps f: $f^{-1}(G) \twoheadrightarrow G$ and π: $\pi^{-1}(G) \to G$ are not θ-continuous. Therefore, the map id: $\pi^{-1}(G)$ $(\subset EX) \to P^{-1}(G)$ $(\subset PX)$ is not θ-continuous, (otherwise, the map $\pi|\pi^{-1}(G) = P|P^{-1}(G) \circ \mathrm{id} \to G$ would be θ-continuous). It follows from 3.2(ii) that $P^{-1}(G)$ is not H-closed. Of course, $P^{-1}(G)$ is an H-set of PX. Moreover, A_1 is a regular closed set of G (even clopen) but $\pi^{-1}(A_1)$ is not compact. (Although A_1 is H-closed and *Urysohn*). □

The following theorem shows that there is an analogue for Theorem 3.6(i) for H-closed subsets of arbitrary Hausdorff spaces.

4.3. THEOREM. *Let X be a Hausdorff space and assume $A \subset X$. Then the following are equivalent*:

(i) *A is an H-closed subspace.*

(ii) *There exists a compact subset $B \subset EX$ such that $\pi(B) = A$ and such that π: $B \to A$ is θ-continuous.*

(iii) *There exists an H-closed subset $B \subset PX$ such that $P(B) = A$.*

Proof. (ii) $\to$ (i) and (iii) $\to$ (i) are trivial. (i) $\to$ (ii). The map π: $\pi^{-1}(A) \twoheadrightarrow A$ is closed and compact. Therefore we can choose a subset $B \subset \pi^{-1}(A)$ such that $\pi(B) = A$ and such that the map π: $B \to A$ is closed, compact and irreducible. But now we can use Corollary 2.3 to conclude that B is H-closed (hence compact) and that π: $B \to A$ is θ-continuous: (i) $\to$ (iii). can be proved in the same way. □

4.4. Corollary. *Let X be a Hausdorff space. Let $A \subset X$ be a subset such that $\pi^{-1}(A)$ is compact. Then A is an H-set in every H-closed subspace of X in which it is embedded.*

Proof. Assume $A \subset H \subset X$ and H is H-closed. According to 4.3, there is a compact $B \subset EX$ such that $\pi(B) = H$ and such that $\pi\colon B \to H$ is θ-continuous. Clearly $B \cap \pi^{-1}(A)$ is compact and $\pi(B \cap \pi^{-1}(A)) = A$. From the θ-continuity of $\pi\colon B \to H$ we conclude that A is an H-set in H. □

Remark (i) The converse of Corollary 4.4 is of course not true. For example, if A is an H-closed subset of X such that $\pi^{-1}(A)$ is not compact, then clearly A is an H-set in every subset in which it is embedded.

(ii) What we proved in 4.3 is the following: if A is an H-closed subset of X, then *each* subset $B \subset \pi^{-1}(A)$ with the property $\pi(B) = A$ and $\pi\colon B \to A$ is perfect and irreducible, is a compact subset of EX. This need not be the case when A is an H-set of X. For example, consider the H-set $A = \mathbf{N} \cup \{\omega_1\}$ of the space X in Example 4.2. Then, $B_1 = (\mathbf{N} \cap A_1) \cup \{\omega_1\}$ is a compact subset of $f^{-1}(A)$ such that $f(B_1) = A$ and $f\colon B_1 \to A$ is perfect and irreducible. And $B_2 = (f^{-1}(\mathbf{N}) \cap (A_2)_s) \cup \{\omega_1\}$ is a non-compact closed subset of $f^{-1}(A)$ such that $f(B_2) = A$ and $f\colon B_2 \to A$ is perfect and irreducible. It is easy to translate this to statements about subsets in EX. □

If one compares the Theorems 3.5, 3.6 and 4.3 one might hope for the following to be true.

"$A \subset X$ is an H-set iff there exists a compact subset $B \subset EX$ with $\pi(B) = A$".

The following example shows that such hope is in vain.

4.5. Example. We use the example we constructed in [**12**] in the following way.

Define $Z = \{-1/n | n \in \mathbf{N}\} \cup \{0\} \cup \{1/n | n \in \mathbf{N}\}$ and put $X = Z \times [0, 1]$. On X the following topology is defined.

(i) Each point $(p, q) \in X$ with $p \neq 0$ is isolated.

(ii) A point $(0, p) \in X$ has the collection $\{U(A, p) | U$ an interval open in $[0, 1]$, $p \in U$, and A a cofinite subset of $\mathbf{N}\}$ as a local base, where the sets $U(A, p)$ are defined as follows:

For an open interval $U \subset [0,1]$, $p \in U$ and a cofinite subsete A of $\mathbf{N}$, put

$$U(A,p) = \{(0,p)\} \cup \left(\left\{-\frac{1}{n}\middle| n \in A\right\} \times ([p,1] \cap U)\right)$$
$$\cup \left(\left\{\frac{1}{n}\middle| n \in \mathbf{N}\right\} \times ([0,p] \cap U)\right);$$

Consider the subset $\{(0,p) | p \in [0,1]\} = B \subset X$.

(i) In **[12]** we showed that B is not the image of a compact subset of EX.

(ii) B is an H-set in X. To see this consider the map g: $[0,1] \to B$ defined by $g(p) = (0,p)$. Although the map g: $[0,1] \to B$ is not θ-continuous, the map g: $[0,1] \to X$ is. Since $[0,1]$ is compact, we can use 2.4(iii) to conclude that $g([0,1]) = B$ is an H-set in X. □

The previous example shows that it will be difficult to characterize H-sets in terms of closed subsets of EX, or in terms of compact spaces and θ-continuous maps. I conjecture the following:

4.6. *Conjecture*. A subset $A \subset X$ is an H-set iff there exists a compact space G and a θ-continuous map g: $G \to X$ with $g(G) = A$. □

(It is clear that if such a compact space G and θ-continuous map g exists, A must be an H-set in X.)

In **[5]** Katetov proved that an H-closed space in which every closed subset is H-closed must be compact. His proof was based on the first part of the following theorem.

4.7. THEOREM. *If $\{A_i\}_{i \in I}$ is a totally ordered collection (w.r.t. inclusion) of H-closed subspaces of a space X, then*:

(i) **[5]** $\bigcap A_i \neq \varnothing$.

(ii) *There exists a non-empty compact subset $B \subset EX$ such that $\pi(B) = \bigcap A_i$. In particular, $\bigcap A_i$ is an H-set in X.*

Proof. (ii) We may assume that the collection is well-ordered. Form $\{A_\alpha\}_{\alpha<\gamma}$ where $\alpha < \beta < (< \gamma) \Leftrightarrow A_\beta < A_\alpha$. Let B_1 be a compact subset of EX such that $\pi(B_1) = A_1$. (see 2.3) Assume for $\alpha < \beta(< \gamma)$ a compact subset $B_\alpha \subset EX$ is defined such that $\pi(B_\alpha) = A_\alpha$ and $\alpha_1 < \alpha_2 \Leftrightarrow B_{\alpha_2} \subset B_{\alpha_1}$ ($\alpha_i < \beta$ $(i = 1,2)$). Then, $\bigcap_{\alpha<\beta} B_\alpha$ is a compact subset of EX and, as the map π is compact, $\pi(\bigcap_{\alpha<\beta} B_\alpha) = \bigcap_{\alpha<\beta} A_\alpha$ $(\supset A_\beta)$. Choose a subset $B_\beta \subset \bigcap_{\alpha<\beta} B_\alpha$ such that the restricted map π: $B_\beta \to A_\beta$ is a perfect and irreducible surjection. According to Corollary 2.3, B_β is compact. But then

$B = \bigcap_\alpha B_\alpha$ is a compact subset of EX and $\pi(B) = \bigcap_\alpha A_\alpha$. By applying Lemma 2.4(iii) we see that $\pi(B) = A$ is an H-set. (Note that this proof also shows that $\bigcap A \neq \varnothing$.)

4.8. COROLLARY. *Let X be an H-closed space. If all H-sets of X are minimal Hausdorff, then X is compact.*

Proof. By the result of Katetov cited above it suffices to show that all closed subsets of X are H-closed. Let A be a closed subset of X. Well-order $X - A$, say $X - A = \{x_\alpha | \alpha < \gamma\}$. We construct a sequence $\{H_\alpha | \alpha < \gamma\}$ of H-closed subspaces such that

(i) $\alpha < \beta\, (< \gamma) \Rightarrow H_\beta \subset H_\alpha$.

(ii) $A \subset H_\alpha \subset X - \{x_\alpha\}$.

As follows: X is H-closed, hence an H-set. By assumption, X is minimal Hausdorff. Hence, there exists a regular closed subset $H_1 \subset X$ such that $A \subset H_1 \subset X - \{x_1\}$. Observe that H_1 is H-closed.

Assume for $\alpha < \beta\, (< \gamma)$ the H_α are defined.

Case 1. β is not a limit ordinal, say $\beta = \beta_0 + 1$. By assumption, H_{β_0} is minimal Hausdorff. There exists a regular closed (hence H-closed) subset H_β in H_{β_0} such that $A \subset H_\beta \subset H_{\beta_0} - \{x_\beta\}$.

Case 2. β is a limit ordinal. According to 4.7 $\bigcap_{\alpha<\beta} H_\alpha$ is an H-set in X, hence $\bigcap_{\alpha<\beta} H_\alpha$ is minimal Hausdorff. But then there exists an H-closed subset $H_\beta \subset \bigcap_{\alpha<\beta} H_\alpha$ such that $A \subset H_\beta \subset \bigcap_{\alpha<\beta} H_\alpha - \{x_\beta\}$.

We conclude from 4.7 that $A = \bigcap_{\alpha<\gamma} H_\alpha$ is an H-set, hence A is minimal Hausdorff. □

4.9. *Question.* Let X be an H-closed space. Does one of the following properties imply that X is compact?

(i) All H-sets are H-closed.

(ii) All H-closed subspaces of X are minimal Hausdorff. □

Next we establish the existence of a non-compact minimal Hausdorff space every regular closed subset of which is minimal Hausdorff. First we consider a special case.

4.10. PROPOSITION. *Let X be a countable H-closed space. If all regular closed subsets of X are minimal Hausdorff, then X is compact.*

Proof. Assume X is not compact. Since the space X is countable, X is not countably compact. Let $A = \{a_n | n \in \mathbf{N}\}$ be a countable closed discrete subset of X and write $X - A = \{x_n | n \in \mathbf{N}\}$. We construct a sequence $\{H_n | n \in \mathbf{N}\}$ of regular closed subsets such that:

(i) $H_{n+1} \subset H_n \subset X - \{x_n, a_{n-1}\}$.

(ii) $\{a_m | m \geq n\} \subset H_n$.

We proceed as follows: There is a regular closed subset $H_1 \subset X$ such that $A \subset H_1 \subset X - \{x_1\}$, as X is minimal Hausdorff. Assume $H_1, \ldots, H_n$ are defined. Consider H_n. By assumption, H_n is minimal Hausdorff and so, as $x_{n+1} \notin \mathrm{cl}_X A$, there exists a regular closed subset H'_{n+1} of H_n such that $\{a_m | m \geq n\} \subset H'_{n+1} \subset H_n - \{x_{n+1}\}$. Observe that H'_{n+1} is also a regular closed subset of X, hence H'_{n+1} is minimal Hausdorff. As $a_n \notin \mathrm{cl}_X\{a_m\colon m \geq n+1\}$, there exists a regular closed subset H_{n+1} of H'_{n+1}, (which is regular closed in X) such that $\{a_m | m \geq n+1\} \subset H_{n+1}$ and $a_n \notin H_{n+1}$. This ends the construction. We have constructed a decreasing sequence of H-closed subspaces with empty intersection. Contradiction. □

4.11. Remark. (i) In the construction of Proposition 4.10 of the subset H_{n+1}, one cannot conclude that there exists a regular closed subset $H_{n+1} \subset H_n$ with $\{a_m | m \geq n+1\} \subset H_{n+1} \subset H_n - \{a_n, x_{n+1}\}$, simply from the fact that H_n is minimal Hausdorff, as the following example shows. In Example 4.2, $\mathbf{N} \subset X$ is a closed subset. However, there does not exist a regular closed subset $A \subset X$ such that $\mathbf{N} \subset A \subset X - \{\omega_1, \omega_2\}$.

(ii) The following is easy to verify: If $A \subset X$ is an H-set and if $x_i \notin A$ $(i = 1, \ldots, n)$, then there is regular closed $G \subset X$ with $A \subset G \subset X - \{x_1, \ldots, x_n\}$. I do not know an example of an H-closed space and a subset $A \subset X$ which has this property, but which is not an H-set. □

4.12. Example. For the construction of a non-compact minimal Hausdorff space every regular closed subset of which is minimal Hausdorff we need the existence of a compact space Y with the following property:

If $\{U_i\}_i$ is a collection of at most c $(= 2^{\aleph_0})$ many open subsets with $\bigcap_i \{\mathrm{cl}\, U_i\} \neq \varnothing$, then

$$(*) \qquad \left|\bigcap_i \{\mathrm{cl}\, U_i\}\right| > 1.$$

The existence of such a space Y is beyond any doubt. For example, one can choose Y to be a compact extremally disconnected space every point of which has local character larger than c. Hence we merely state the existence of Y.

Let $\{Y_k^n | n, k \in \mathbf{N}\}$ be countably many pairwise disjoint copies of Y, all disjoint from Y. If $U \subset Y$, let U_k^n denote the corresponding copy of U

in Y_k^n. Put $X = Y \cup \bigcup\{Y_k^n | n, k \in \mathbf{N}\} \cup \mathbf{N}$, equipped with the following topology.

(i) Y_k^n is a clopen subset of X.

(ii) The collection $\{\{k\} \cup \bigcup\{Y_k^n | n \geq l\} | l \in \mathbf{N}\}$ is defined to be a local base at $k \in \mathbf{N} \subset X$.

(iii) The collection $\{U \cup \bigcup\{U_k^n | n \in \mathbf{N}, k \geq l\} | l \in \mathbf{N}, y \in U, U$ open in $Y\}$ is defined to be a local base at $y \in Y \subset X$.

(Observe that if $Y = \{1, 2\}$, then the space X is homeomorphic to the minimal Hausdorff space defined in 4.2.) It is easy to see that if $|Y| > 1$, then X is a non-compact minimal Hausdorff space.

Claim. If Y satisfies (∗), then every regular closed subset of X is minimal Hausdorff. Indeed, let V be a regular closed subset of X. To establish the claim we consider the following cases.

Case 1. Assume V satisfies one of the following properties (a) $V \subset X - Y$; (b) $V \subset (X - \mathbf{N}) \cup A$ (A a finite subset of $\mathbf{N}$); (c) $V \cap Y \neq \varnothing$, $V \cap \mathbf{N} \neq \varnothing$, but $\forall p \in V \cap Y, \exists U_p$ (in V): $(\mathrm{cl}\, U_p) \cap \mathbf{N} = \varnothing$. In all these three cases it is easy to see that V is regular and therefore compact.

Case 2. Assume $V \cap Y \neq 0$, $V \cap \mathbf{N} \neq \varnothing$ and $\exists p \in V \cap Y$ such that $(\mathrm{cl}\, U_p) \cap \mathbf{N} \neq \varnothing$ (for every neighborhood U_p of p in V). We show that each point of $V \cap Y$ has a neighborhood base in V of in V regular open subsets. (Clearly, this implies that V is minimal Hausdorff.) Assume there exists $p \in V \cap Y$ lacking such a neighborhood base. Let U_p be a neighborhood of p in V such that U_p contains no regular open neighborhood of p in V. Therefore, $\forall V_p \subset U_p$: $\mathbf{N} \cap \mathrm{int}_V \mathrm{cl}\, V_p \neq \varnothing$. Denote $\mathbf{N} \cap \mathrm{int}_V \mathrm{cl}\, U_p$ by K, $V \cap Y_k^n$ by V_k^n, and $\{n \in \mathbf{N}: V_k^n \neq \varnothing\}$ by $\mathbf{N}_k$ for each $k \in K$. Note that $\mathbf{N}_k$ is not finite, since $k \in \mathrm{cl}\, U_p$.

Since $k \in \mathrm{int}_V \mathrm{cl}\, U_p$ $(k \in K)$ we conclude that:

$\forall k \in K$, $V_k^n \subset U_p$ for cofinitely many $n \in \mathbf{N}_K$(!) If $p \in V_p \subset U_p$, put $K(V_p) = \mathbf{N} \cap \mathrm{int}_V \mathrm{cl}\, V_p$. Clearly, $\mathscr{F} = \{K(V_p): p \in V_p \subset U_p\}$ is a centered system of subsets of $\mathbf{N}$, hence, card $\mathscr{F} \leq c$.

Observe that:

(a) $\forall V_p$: $K(V_p) \neq \varnothing$.

(b) $\forall k \in K(V_p)$, $V_k^n \subset V_p$ for cofinitely many $n \in \mathbf{N}_k$. Clearly, $\forall F \in \mathscr{F}$, $\forall k \in F$, $\forall n(k) \in \mathbf{N}_K$ we have:

$$p \in \mathrm{cl}_X\Big(\bigcup_{k \in F} \bigcup\{V_k^l: l \geq n(k)\}\Big).$$

Hence, if V_k^l is considered as a (regular closed!) subset $\hat{V}_k^l$ of Y, we get:

$$p \in \mathrm{cl}_Y\Big(\bigcup_{k \in F} \bigcup\{\hat{V}_k^l : l \geq n(k)\}\Big)$$

i.e.

$$p \in \bigcap\Big\{\mathrm{cl}_Y\Big(\bigcup_{k \in F} \bigcup_{l \geq n(k)} \{\hat{V}_k^l\} : F \in \mathscr{F}\Big)\Big\}.$$

This is an intersection of at most c-many regular closed sets, so, by assumption, $\exists\, q \neq p$ in this intersection. Clearly $q \in V$. Let V_p and V_q be disjoint neighborhoods of p and q in V and assume $V_p \subset U_p$. Consider $K(V_p)$.

$\forall\, k \in K(V_p) \exists\, n(k) \in \mathbf{N}_K$ such that $\bigcup\{V_k^l : l \geq n(k)\} \subset V_p$. But then

$$\bigcup_{k \in K(V_p)} \bigcup\{V_k^l : l \geq n(k), l \in \mathbf{N}_k\} \subset V_p.$$

However, this is not possible since V_q must intersect this set and $V_q \cap V_p = \varnothing$.

From this contradiction we conclude that V is minimal Hausdorff. □

4.13. Remark. (i) Observe that in the X space of Example 4.12 not every H-closed subspace of X is minimal Hausdorff. For example, if $y \in Y$ then $\{y\} \cup \bigcup\{\{y\}_k^n : n, k \in \mathbf{N}\} \cup \mathbf{N}$ is such a subspace.

(ii) Example 4.12 also shows that preimages of (regular closed) subsets of X which are minimal Hausdorff need not be H-sets in PX. (See the proof of 3.3((iii) $\to$ (i))). (Observe that if X is H-closed and Urysohn this is the case.) □

5. Some final remarks. In [3] A. Dow and J. Porter proved the following: If the topology τ_X of a space X contains an H-closed subtopology, then X is homeomorphic to the remainder of some H-closed extension of a discrete space. They asked whether the converse of this statement is true. This is the case.

5.1. Proposition. *The following are equivalent for a space X*

(i) *T_X contains a minimal Hausdorff subtopology.*

(ii) *$X \cong \alpha D - D$, for some H-closed extension αD of a discrete space D.*

Proof. (i) $\to$ (ii) see [3].

(ii) $\to$ (i). Consider αD. Then $E\alpha D \cong \beta D$ and $\pi(\beta D - D) = \alpha D - D$. Therefore, we can consider the map π as a perfect surjection

π: $\beta D - D \to X$. There exists a compact subset $C \to \beta D - D$ such that π: $G \to X$ is a perfect and irreducible surjection. Theorem 2.3 shows that the closed subbase $\{\pi(A)|A$ a closed subset of $C\}$ induces a minimal Hausdorff topology τ_0 on X. But $\tau_0 \subset T_X$, since the map π: $C \to X$ is closed. □

The proof of the following proposition is omitted, since it is straightforward.

5.2. PROPOSITION. *The following are equivalent for a space X.*

(i) τ_X *contains a compact subtopology.*

(ii) $X \cong \alpha D - D$, *for some H-closed Urysohn extension* αD *of a discrete space D.*

(iii) *X can be embedded as an H-set in some H-closed Urysohn space.* □

If one considers 5.2(iii), the following question suggests itself.

5.3. *Question*. Are the following two statements equivalent for a space X?

(i) τ_X contains a minimal Hausdorff subtopology.

(ii) X can be embedded as an H-set in some H-closed space. □

Observe that in 5.3 the statement (i) $\to$ (ii) is always valid. Indeed, if (i) holds, Theorem 5.1 shows that $X \cong \alpha D - D$ for some H-closed extension αD of a discrete space D. In particular, X is an H-set in the H-closed space αD.

On the other hand, if Conjecture 4.6 turns out to be correct, then one can construct for any H-set $A \subset Y$ a minimal Hausdorff subtopology of τ_A in the same way as is done in 5.1(ii) $\to$ (i). It is well known that the topology of the space $Q \subset \mathbf{R}$ does not contain a minimal Hausdorff subtopology. The following lemma shows that Q cannot be used as a counterexample against Conjecture 4.6.

5.4. LEMMA. *The space Q cannot be embedded as an H-set in any space X.*

Proof. Assume $Q \subset X$. Write $Q = \{q_n | n \in \mathbf{N}\}$. Let U_1 be an open subset of X such that $q_2 \in U$, and $q_1 \notin \mathrm{cl}_X U_1$. Assume we have constructed open subsets $U_1, \ldots, U_n$ in X such that:

$$U_{i+1} \subset U_i; \quad q_i \notin \mathrm{cl}_X U_i \quad \text{and} \quad U_i \cap Q \neq \varnothing .$$

Since $U_n \cap Q \neq \varnothing$, $U_n \cap Q$ is infinite and so we can choose $q \in U_n \cap Q$ with $q \neq q_{n+1}$. Let U_{n+1} be an open subset of X such that $q \in U_{n+1} \subset U_n$ and $q_{n+1} \notin \mathrm{cl}_X U_{n+1}$.

The collection $\{U_n | n \in \mathbf{N}\}$ is a base for an open filter $\mathscr{F}$ on X such that $F \cap Q \neq 0$ $(\forall F \in \mathscr{F})$, but $Q \cap \bigcap\{\mathrm{cl}_X F | F \in \mathscr{F}\} = \varnothing$. Hence, Q is not an H-set in X. □

In Corollary 3.3 we showed that if each centered system of H-closed subspaces of an H-closed space X has non-empty intersection, the space X must be Urysohn. The centered system constructed in 3.3 for non Urysohn sapces, is really only a centered system, i.e. the system is not closed under finite intersection. Of course, one cannot expect that such a system will be closed under finite intersection, since finite intersection of H-closed subspaces need not be H-closed. However, one might expect the system $\mathscr{F}$ to have the following property:

If $F_i \in \mathscr{F}(i = 1,\dots,n)$ then there is a $F \in \mathscr{F}$ with $F \subset \bigcap_{i=1}^n F_i$, i.e., $\mathscr{F}$ is a filterbase. I was unable to answer the following question.

5.5. *Question*. Does any filterbase of H-closed subspaces have non-empty intersection?

The reason for asking this question is the following. In 4.7 we constructed for a decreasing sequence of H-closed subspaces of X a corresponding decreasing sequence of compact subsets in EX. Maybe this construction can be generalized to filterbases of H-closed subsets. The question is related to a question of R. G. Woods. Recall that a continuous map f: $X \to Y$ is called a p-map, if f permits a continuous extension f: $\kappa X \to \kappa Y$ to the Katetov-extensions. In **[15]** R. G. Woods obtained the following:

5.6. THEOREM. (i) *A p-map $X \to Y$ satisfies the following property* $(*)$: *If A is a regular closed subset of X, then* $\mathrm{cl}_y f(A) = B \cup H$, *where B is a regular closed subset of Y and H an H-closed nowhere dense subset of Y.* (ii) *If Y is regular or Lindelöf and f: $X \to Y$ is a continuous map with property* $(*)$, *then f is a p-map.* □

The question of R. G. Woods was whether the converse of 5.6(i) is always the case. The following proposition provides a partial answer.

5.7. PROPOSITION. *Let f: $X \to Y$ be a continuous map with property* $(*)$. *Then*:

(i) *If Y is Urysohn, then f is a p-map.*

(ii) *If Y has the property that each filterbase of H-closed subspaces has non-empty intersection, then f is a p-map.*

Proof. Choose $\mathscr{F} \in \kappa X - X$, i.e. $\mathscr{F}$ is an open ultrafilter on X.

Case 1. $\forall F \in \mathscr{F}$: $\operatorname{int}_Y \operatorname{cl}_Y f(\operatorname{cl}_X F) \neq \varnothing$.

The collection $\{\operatorname{int}_Y \operatorname{cl}_Y f(\operatorname{cl}_X F) | F \in \mathscr{F}\}$ is a centered system of open subsets of Y, and is contained in a unique open ultrafilter $\mathscr{G}$ on Y.

- If $\mathscr{G}$ is fixed, define $\kappa f(\mathscr{F}) = \bigcap\{\operatorname{cl} G \colon G \in \mathscr{G}\}$.
- If $\mathscr{G} \in \kappa Y - Y$, define $\kappa f(\mathscr{F}) = G$.

Case 2. $\exists F_0 \in \mathscr{F}$ such that $\operatorname{int}_Y \operatorname{cl}_Y f(\operatorname{cl}_X F_0) = \varnothing$.

Property $*$ implies that, if $F \in \mathscr{F}$ and $F \subset F_0$, then $\operatorname{cl}(f(\operatorname{cl} F))$ is H-closed. The collection $\{\operatorname{cl} f(\operatorname{cl} F)\colon F \in \mathscr{F},\ F \subset F_0\}$ is a filterbase of H-closed subspaces of Y. Both in (i) and (ii) we can conclude that $\bigcap\{\operatorname{cl} f(\operatorname{cl} F) | F \in \mathscr{F}\} \neq \varnothing$.

Note that $|\{\bigcap \operatorname{cl} f(\operatorname{cl} F) | F \in \mathscr{F}\}| = 1$ and define $\kappa f(\mathscr{F})$ to be the unique point in this intersection.

It is easy to verify that $\kappa f\colon \kappa X \to \kappa Y$ is a continuous extension of f. □

Added in proof (Aug. 21, 1980). (i) J. Petty also constructed an example of a non-compact minimal Hausdorff space every regular closed subspace of which is minimal Hausdorff.

(ii) R. F. Dickman and J. Porter constructed examples of non-H-sets $A \subset X$ with the property described in 4.11(ii).

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