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ON THE WALLMAN ORDER COMPACTIFICATION

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The Wallman order compactification $w_0 X$ of a topological ordered space X has been constructed by Choe and Park. This paper establishes necessary and sufficient conditions for their compactification to be T_2-ordered, in which case it coincides with the Nachbin (or Stone–Čech order) compactification.

Introduction. Let $(X, \leq)$ be a poset. For $x \in X$, let $i(x) = \{y \in X: x \leq y\}$ and let $d(x) = \{y \in X: y \leq x\}$. If $A \subseteq X$, let $i(A) = \bigcup\{i(x): x \in A\}$, and $d(A) = \bigcup\{d(x): x \in A\}$. If $A = iA$ (respectively, $A = d(A)$), then A is called an *increasing* (respectively, *decreasing*) set; a set which is either increasing or decreasing is said to be *monotone*.

A *topological ordered space* $(X, \leq, \tau)$ consists of a poset $(X, \leq)$ equipped with a topology τ. If τ has an open subbase consisting of monotone sets, then the topological ordered space is said to be *convex*. Since only convex topological ordered spaces can have order compactifications which are T_2-ordered (see below), we shall henceforth consider only spaces of this type. For brevity, a convex topological ordered space $(X, \leq, \tau)$ will be simply called a *space* and designated by "X".

Following McCartan [4], we define a space X to be T_1-*ordered* if $i(x)$ and $d(x)$ are both closed for all $x \in X$, and T_2-*ordered* if the partial order relation is a closed subset of $X \times X$. A T_1-ordered space is T_4-*ordered* (*normally ordered* in [5]) if, whenever A and B are closed disjoint subsets, the former decreasing and the latter increasing, there are disjoint open sets U and V, the former decreasing and the latter increasing, such that $A \subseteq U$ and $B \subseteq V$. The "T_3-ordered" property is defined in [4], and "$T_{3.5}$-ordered" can be taken to mean "completely regular ordered" as defined in [5], but it will not be necessary to repeat these latter definitions here.

Nachbin has constructed a Stone–Čech type order compactification $\beta_0 X$ of an arbitrary $T_{3.5}$-ordered space X with the property that any continuous, increasing function from X into a T_2-ordered, compact space can be lifted to $\beta_0 X$. For details of the Nachbin compactification, see [3]. More recently, Choe and Park showed that X is T_4-ordered whenever $w_0 X$ is T_2-ordered, but were unable to prove the converse. Our main result establishes that $w_0 X$ is T_2-ordered if and only if X is strongly T_4-ordered (this term is defined below), and consequently that w_0X and β_0X are equivalent compactifications of a strongly T_4-ordered space X.

Let X be a topological ordered spae. If $A \subseteq X$, let $I(A)$ (respectively, $D(A)$) be the smallest increasing (respectively, decreasing) closed set containing A, and let $A^{\wedge} = I(A) \cap D(A)$. Let $\mathscr{C}_X = \{A \subseteq X: A = A^{\wedge}\}$. Note that all members of $\mathscr{C}_X$ are closed and convex; we shall call the members of $\mathscr{C}_X$ *c-sets*. All monotone closed sets are c-sets, and thus $\mathscr{C}_X$ is a closed subbase for τ. One can easily verify that every set of the form $A^{\wedge}$, for $A \subseteq X$, is a c-set, and also that $\mathscr{C}_X$ is closed under finite intersections.

Let $F(X)$ be the set of all filters on X; the fixed ultrafilter generated by $\{x\}$ will be denoted by $\dot{x}$ for $x \in X$. If $\mathscr{F}, \mathscr{G} \in F(X)$, then $\mathscr{F} \vee \mathscr{G}$ will designate the filter generated by $\{F \cap G \ F \in \mathscr{G}, G \in \mathscr{G}\}$ (assuming that the latter collection does not include $\varnothing$).

For $\mathscr{F} \in F(X)$, we denote by $i(\mathscr{F})$ the filter generated by $\{i(F): F \in \mathscr{G}\}$; the filters $d(\mathscr{F})$, $I(\mathscr{F})$, and $D(\mathscr{F})$ are defined analogously. A filter $\mathscr{F}$ is a *c-filter* (respectively, a *convex filter*) if it has a filter base of c-sets (respectively, convex sets). Note that $\mathscr{F}$ is a c-filter (respectively, a convex filter) iff $\mathscr{F} = I(\mathscr{F}) \vee D(\mathscr{F})$ (respectively, $\mathscr{F} = i(\mathscr{F}) \vee d(\mathscr{F})$). A c-filter which is not property contained in any other c-filter will be called a *maximal c-filter*. A standard Zorn's Lemma argument establishes that every c-filter is contained in a maximal c-filter.

We can assume that X is a T_1-ordered space and define $w_0(X)$ to be the set of all maximal c-filters on X. Note that the only convergent maximal c-filters are the fixed ultrafilters. It will be convenient to write $w_0X = \{\dot{x}: x \in X\} \cup X'$, where X' is the set of all non-convergent maximal c-filters. An order relation "$\lesssim$" for w_0X is defined as follows: $\mathscr{F} \lesssim \mathscr{G}$ iff $I(\mathscr{F}) \subseteq \mathscr{G}$ and $D(\mathscr{G}) \subseteq \mathscr{F}$. It is a simple matter to verify that $(w_0X, \lesssim)$ is a poset and that the canonical map $\varphi: (X, \leq) \to (w_0X, \lesssim)$, defined by $\varphi(x) = \dot{x}$, is increasing.

We next introduce a topology on w_0X. For $A \subseteq X$, define $A^* = \{\mathscr{F} \in w_0X: A \in \mathscr{F}\}$. Then $\mathscr{C}^* = \{A^*: A \in \mathscr{C}_X\}$ is a closed subbase for a topology on w_0X which we shall denote by $w_0\tau$. Clearly, $(A \cap B)^* = A^* \cap B^*$ for all subsets A, B of X; from this one easily deduces that w_0X is a topological ordered space. It is obvious that $A = \varphi^{-1}(A^*)$ for any $A \subseteq X$; therefore $\varphi: X \to w_0X$ is a topological embedding, and both φ and $\varphi^{-1}|\varphi(x)$ are increasing functions.

Before proceeding further, it is desirable to compare our construction of w_0X with that of Choe and Park. They define a *bifilter* $(\mathscr{G}, \mathscr{H})$ on X to be a pair of filters such that $\mathscr{G}$ has a base of decreasing closed sets, $\mathscr{H}$ has a base of increasing closed sets, and $\mathscr{G} \vee \mathscr{H}$ exists; the set of all maximal bifilters forms the underlying set for their compactification, which is also denoted by w_0X. It is easy to see that, for any bifilter $(\mathscr{G}, \mathscr{H})$ on X, the filter $\mathscr{F} = \mathscr{G} \vee \mathscr{H}$ is a c-filter, and that, for any c-filter $\mathscr{F}$, $(D(\mathscr{F}), I(\mathscr{F}))$ is a corresponding bifilter. If $(\mathscr{G}, \mathscr{H})$ is a maximal bifilter, then $\mathscr{F} = \mathscr{G} \vee \mathscr{H}$ is a maximal c-filter, and $(D(\mathscr{F}), I(\mathscr{F})) = (\mathscr{G}, \mathscr{H})$; thus a bijection exists between the set of maximal bifilters on X and the set of maximal c-filters on X. A comparison of the order relation and topology defined for w_0X in [**2**] with our definitions given above reveals the equivalence of these spaces both as posets and as topological spaces. Thus the results obtained concerning w_0X in [**2**] are applicable here, albeit with appropriate terminological alterations. The next two results are obtained in this way.

PROPOSITION 1.1. *For any T_1-ordered space X, (w_0X, φ) is an order compactification of X, and w_0X is a T_1 topological space. If w_0X is T_2-ordered, then X is T_4-ordered.*

PROPOSITION 1.2. *Let X be a T_1-ordered space, Y a T_2-ordered compact space, and $f\colon X \to Y$ a continuous, increasing function. Then there is a unique, continuous, increasing function $\bar{f}\colon w_0X \to Y$ such that $\bar{f} \cdot \varphi = f$.*

We define a T_4-ordered space X to be *strongly* T_4-ordered if, whenever A and B are c-sets:

$$I(A) \cap B = \varnothing \quad \text{implies} \quad I(A) \cap D(B) = \varnothing$$

$$D(A) \cap B = \varnothing \quad \text{implies} \quad D(A) \cap I(B) = \varnothing$$

Note that a T_4-ordered space X is strongly T_4-ordered iff, for a c-set A and a decreasing open set U with $A \subseteq U$, $D(A) \subseteq U$ and dually.

Priestly [**6**] defines a C-space to be a topological ordered space X such that, for each closed subset A, $i(A)$ and $d(A)$ are also closed. The class of strongly T_4-ordered spaces includes the T_4 C-spaces, among which are the T_2-ordered compact spaces.

PROPOSITION 1.3. *A T_1-ordered space X is strongly T_4-ordered if and only w_0X is T_2-ordered*

Proof. In Proposition 1, page 26, [**5**], Nachbin shows that a space is T_2-ordered if, whenever $a \nleq b$, there is an increasing neighborhood V of a and a decreasing W of b such that $V \cap W = \varnothing$.

Assume that $\mathscr{F}$, $\mathscr{G}$ are elements of w_0X such that $\mathscr{F} \leq \mathscr{G}$ is false. Then either $I(\mathscr{F}) \subseteq \mathscr{F}$ or $D(\mathscr{G}) \subseteq \mathscr{F}$ is false. In the former case, since $\mathscr{G}$ is a maximal c-filter, there is $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $I(F) \cap G = \varnothing$. By the assumption that X is strongly T_4-ordered, $I(F) \cap D(G) = \varnothing$, and so there are disjoint open neighborhoods U and V of $I(F)$ and $D(G)$, respectively, such that U is increasing and V decreasing. Then U^* and V^* are disjoint, open neighborhoods of $\mathscr{F}$ and $\mathscr{G}$, respectively, in w_0X, the former increasing and the latter decreasing. This w_0X is T_2-ordered.

Conversely, assume that w_0X is T_2-ordered. Let A, B be c-sets and suppose $I(A) \cap B = \varnothing$. Then $I(A)^* \cap B^* = \varnothing$. $I(A)^*$ is a closed, increasing subset of w_0X and $B^* = D(B)^* \cap I(B)^*$ is a closed subset of w_0X. Let $d_w(B^*) = \{\mathscr{F} \in w_0X: \mathscr{F} \leq \mathscr{G}$ for some $\mathscr{G} \in B^*\}$. By Proposition 4, page 44, [5], $d_w(B^*)$ is a closed subset of w_0X, and it follows that $I(A)^* \cap d_w(B^*) = \varnothing$. Then $\varphi^{-1}(I(A)^* \cap d_w(B^*)) = \varphi^{-1}(I(A)^*) \cap \varphi^{-1}(d_w(B^*)) = \varnothing$. Since $\varphi^{-1}(I(A)^*) = I(A)$ and $D(B) \subseteq \varphi^{-1}(d(B^*))$, it follows that $I(A) \cap D(B) = \varnothing$. A similar argument shows that if $D(A) \cap B = \varnothing$, then $D(A) \cap I(B) = \varnothing$. This conclusion that X strongly T_4-ordered now follows with the help of Proposition 1.1 □

COROLLARY 1.4. *A T_4-ordered space X is strongly T_4-ordered if and only if, for any c-set A, $d(A)$ and $i(A)$ are both closed.*

Proof. The condition is obviously sufficient. Suppose that X is strongly T_4-ordered and $x \notin d(A)$. Then $i(x)^* \cap A^* = \varnothing$, and consequently $i(x)^* \cap d_w(A^*) = \varnothing$. It follows that $i(x) \cap \varphi^{-1}(d_w(A^*)) = \varnothing$. Since the closure of $d(A)$ in X is a subset of $\varphi^{-1}(d_w(A^*))$, x is not in the closure of $d(A)$. Thus $d(A)$ is closed. □

COROLLARY 1.5. *Let X be $T_{3.5}$-ordered. Then the compactifications w_0X and β_0X are equivalent if and only if X is strongly T_4-ordered.*

If the order relation of X is trivial, then the c-sets are simply the closed sets, and the compactification w_0X is identical with the ordinary Wallman compactification. In this case, Corollary 1.5 yields the well-known equivalence of the Wallman and Stone–Čech compactifications for T_4 topological space.

We conclude by considering the Wallman order compactification for a simple and familiar class of spaces. We define a *totally ordered space* to be a totally ordered set with its order topology. If X is a totally ordered space, then one can show that w_0X (and hence β_0X) is a totally ordered space and a complete lattice. If $X = R$ is the totally ordered space of real numbers, then w_0X can be identified with the extended real line $[-\infty, \infty]$.

If $X = Q$ is the space of rationals, then $w_0 X$ can also be regarded as the extended real line, but with each irrational "occurring twice"; by identifying these "irrational pairs", one obtains $w_0 R$ as a quotient space of $w_0 Q$.

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